

MODULE 1

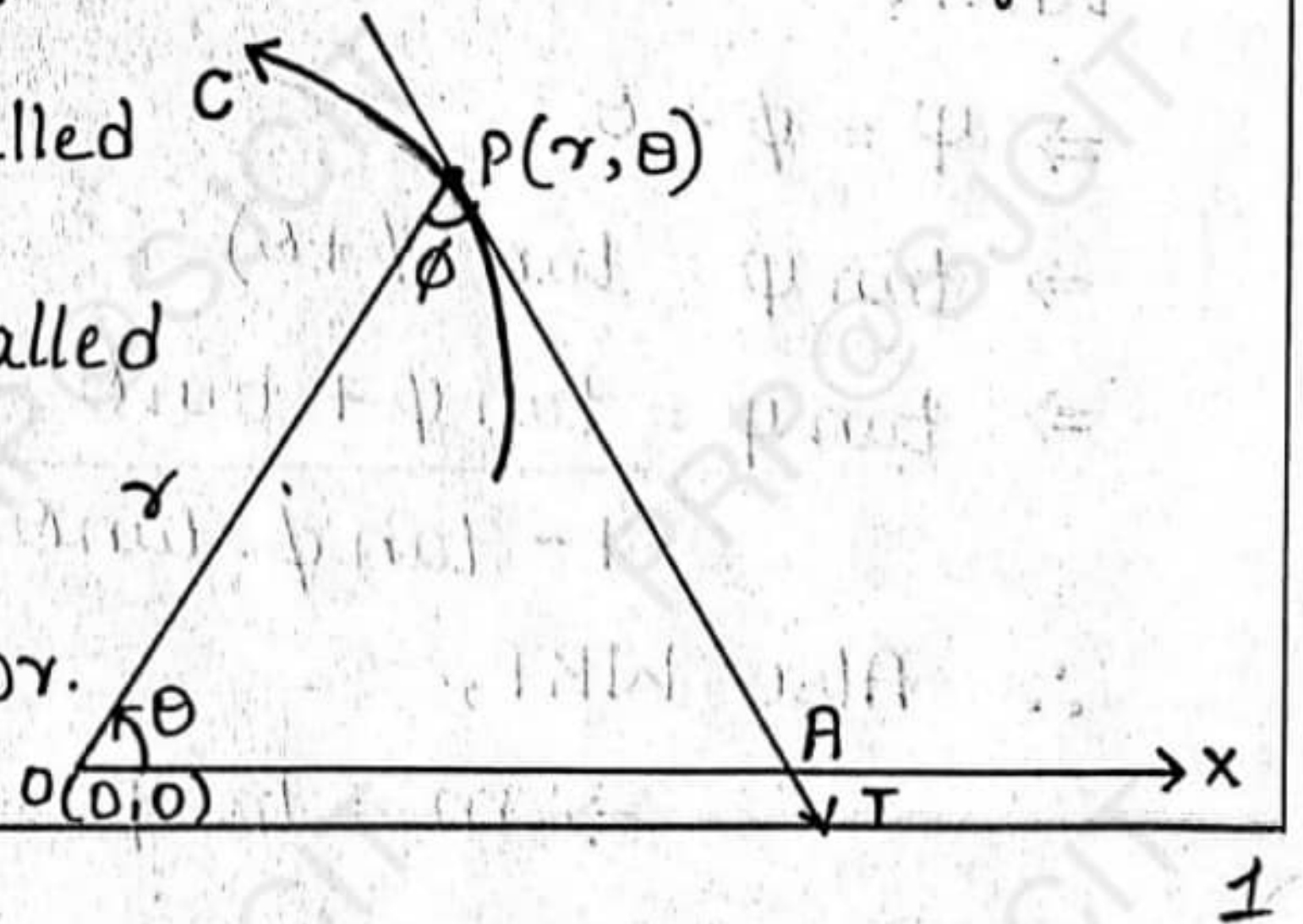
Introduction :-

Let 'P' be any point on the plane. $O(0,0)$ be an origin, which extends the $\overrightarrow{OX}$ and $OP = r$ and 'C' be a curve at the point 'P' with the tangent 'T'.

Here, the point $O(0,0)$ is called pole,

the point $P(r, \theta)$ is called polar co-ordinates,

the distance $OP = r$ is called radius of vector.



' θ ' be the angle between radius of vector and initial line,

' ϕ ' is called the angle between the radius of vector and tangent to the polar curve 'C' and the polar curve which is of the form either $r = f(\theta)$ or $f(r, \theta) = C$.

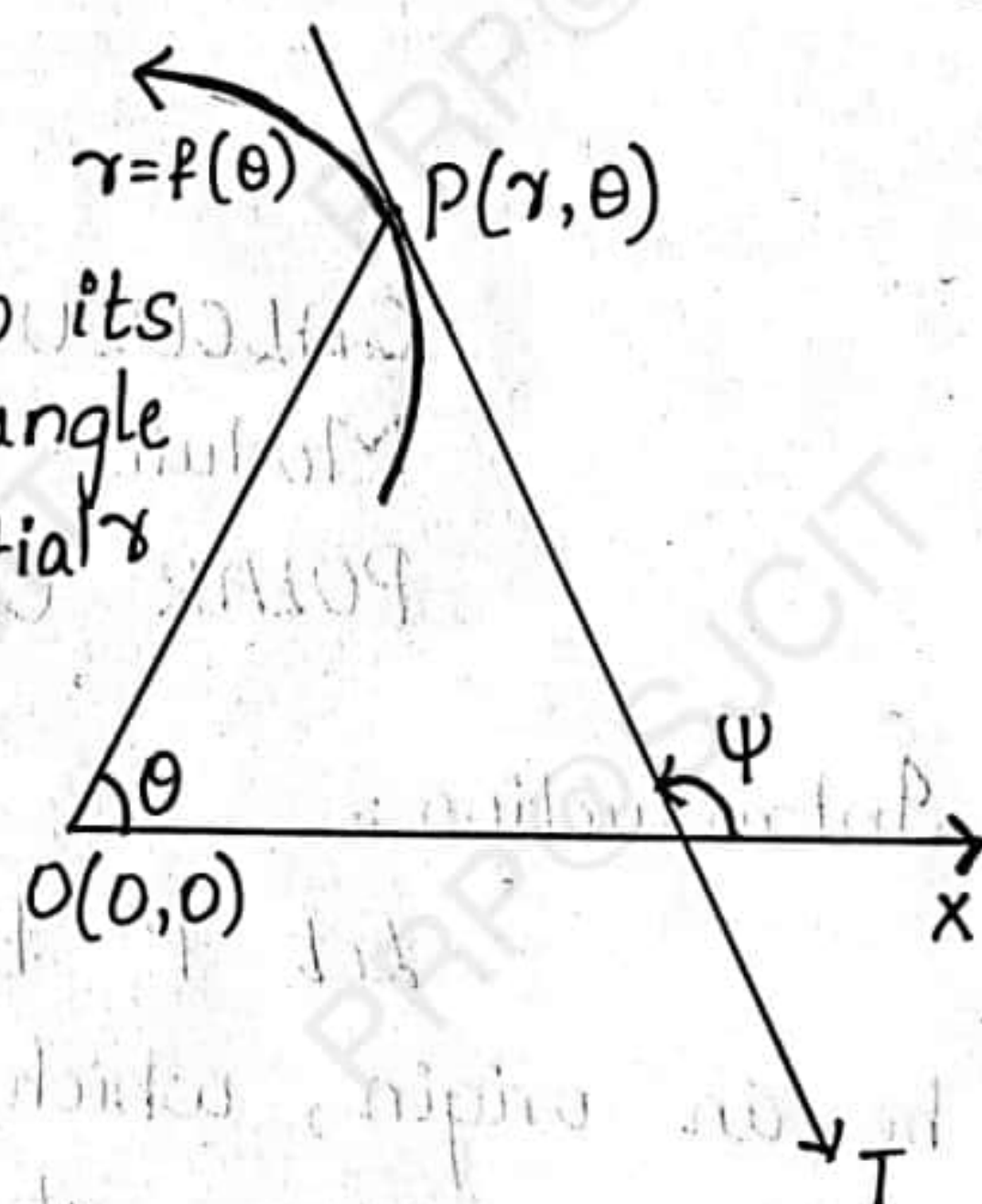
The entire above system is called "Polar System".

ANGLE BETWEEN THE RADIUS OF VECTOR AND TANGENT TO THE POLAR CURVE :-

Given,

$P(r, \theta)$ be any point on the plane, let, $OP = r$ be the radius of a vector, which makes the angle to its polar curve ' ϕ ' and ' ψ ' be the angle made by the tangent to the initial line.

Finally, ' θ ' be the angle between radius of vector and initial line $\vec{OX}$.



WKT,

$$\Rightarrow \psi = \phi + \theta$$

$$\Rightarrow \tan \psi = \tan(\phi + \theta)$$

$$\Rightarrow \tan \psi = \frac{\tan \phi + \tan \theta}{1 - \tan \phi \cdot \tan \theta} \quad \text{--- (1)}$$

$\therefore$ Also WKT,

$$\Rightarrow m = \tan \psi = \frac{dy}{dx}$$

$$\Rightarrow \tan \psi = \frac{dy}{dx}$$

$$\Rightarrow r \cos \theta = x \quad \text{--- (3)}$$

diff w.r to 'θ'

$$y = r \sin \theta \quad \text{--- (4)}$$

$$\textcircled{3} \Rightarrow \cos \theta \frac{dr}{d\theta} + r(-\sin \theta) = \frac{dx}{d\theta}$$

$$\Rightarrow \cos \theta \frac{dr}{d\theta} - r \sin \theta = \frac{dx}{d\theta}$$

$$\textcircled{4} \Rightarrow \frac{dy}{d\theta} = \sin \theta \frac{dr}{d\theta} + r \cos \theta$$

From equation (2)

$$\Rightarrow \tan \psi = \frac{dy/d\theta}{dx/d\theta}$$

$$\Rightarrow \tan \psi = \frac{\sin \theta \cdot \frac{dr}{d\theta} + r \cos \theta}{\cos \theta \cdot \frac{dr}{d\theta} - r \sin \theta}$$

$$\Rightarrow \tan \psi = \frac{\frac{\sin \theta \cdot \cancel{\frac{dr}{d\theta}}}{\cancel{\cos \theta \cdot \frac{dr}{d\theta}}} + \frac{r \cancel{\cos \theta}}{\cancel{\cos \theta \cdot \frac{dr}{d\theta}}}{\frac{\cancel{\cos \theta \cdot \frac{dr}{d\theta}}}{\cancel{\cos \theta \cdot \frac{dr}{d\theta}}} - \frac{r \sin \theta}{\cancel{\cos \theta \cdot \frac{dr}{d\theta}}}$$

$$\Rightarrow \tan \psi = \frac{\tan \theta + r \left[\frac{d\theta}{dr} \right]}{1 - (\tan \theta) \left[r \frac{d\theta}{dr} \right]}$$

$$\Rightarrow \tan \theta + \tan \phi = \tan \theta + r \frac{d\theta}{dr}$$

$$\Rightarrow \frac{1 - \tan \theta \cdot \tan \phi}{1 - \tan \theta \left[r \frac{d\theta}{dr} \right]}$$

$$\Rightarrow \tan \phi = r \frac{d\theta}{dr}$$

$$\Rightarrow \cot \phi = \frac{1}{r} \frac{dr}{d\theta}$$

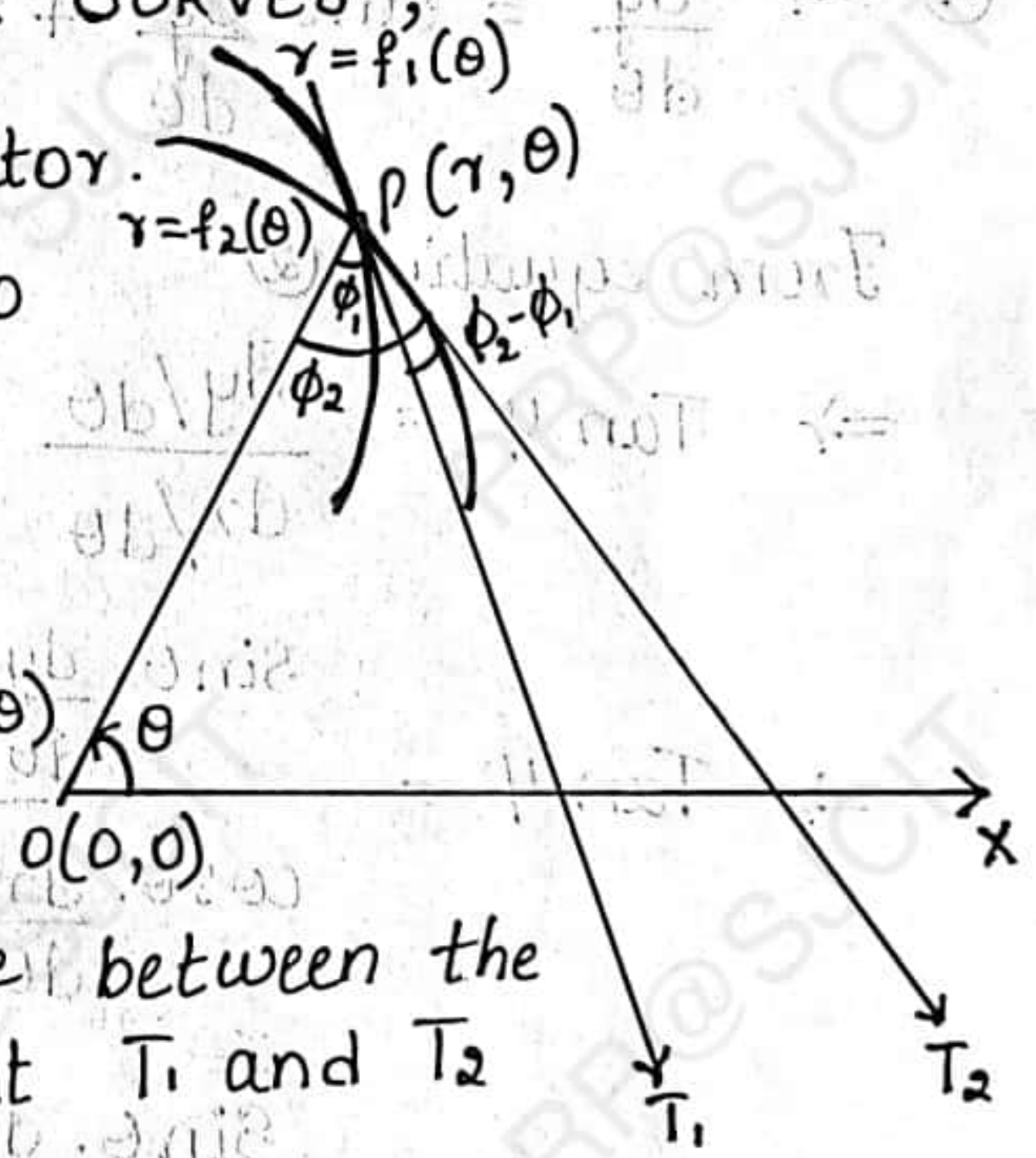
ANGLE BETWEEN TWO POLAR CURVES;

Let, $OP=r$ be the radius of vector.

$r=f_1(\theta)$, $r=f_2(\theta)$ be the two polar curves intersecting at 'P'.

Let, T_1 and T_2 be the two tangents to the curves, $r=f_1(\theta)$ and $r=f_2(\theta)$ at 'P'.

Let, ϕ_1 and ϕ_2 be the angle between the radius of vector and tangent T_1 and T_2 respectively.



$\therefore$ The angle between the given two polar curves can be evaluated as ϕ_1 and ϕ_2 .

If $|\phi_1 - \phi_2| = \frac{\pi}{2}$, then we can say that the given curves are intersecting perpendicular (or) Orthogonally.

PROVE THAT:

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} \left[\frac{dr}{d\theta} \right]^2$$

using usual notations.

Let $OP = r$, be the radius of vector,

Let $r = f(\theta)$ be a polar curve,

Let 'T' be the tangent to the polar curve at 'P',

Let ' ϕ ' be the angle between radius of vector and tangent to the polar curve,

Let 'M' be the foot of perpendicular,

Let $OM = p$,

$\therefore$ From, ΔOPM ;

$$\Rightarrow \sin \phi = \frac{OM}{OP}$$

$$\Rightarrow \sin \phi = \frac{p}{r}$$

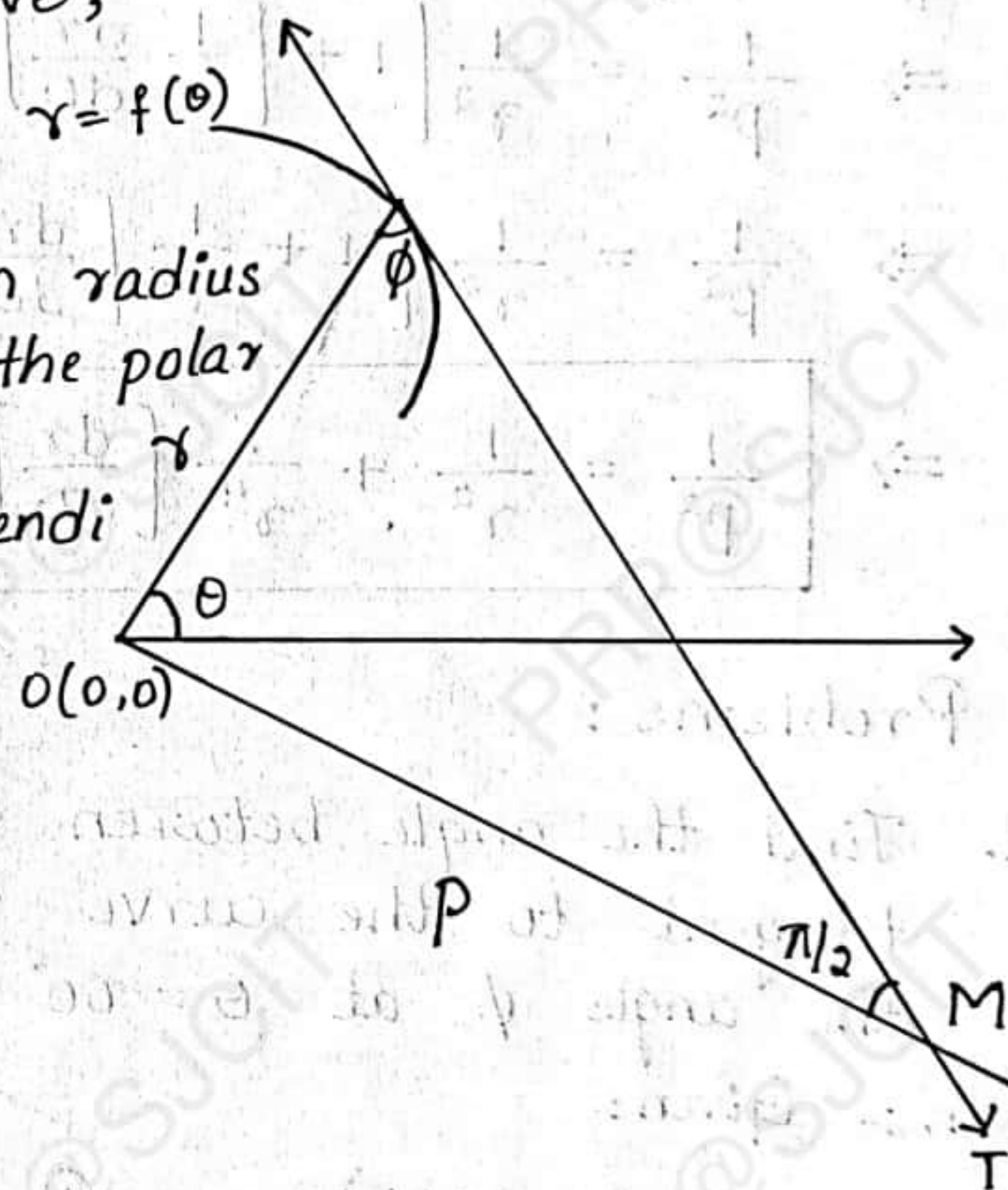
$$\Rightarrow \boxed{p = r \sin \phi} \text{ --- ① [SOBS]}$$

$$\Rightarrow p^2 = r^2 \sin^2 \phi \quad [\text{reciprocal}]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2 \sin^2 \phi}$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} \operatorname{cosec}^2 \phi$$

$$\Rightarrow \boxed{\frac{1}{p^2} = \frac{1}{r^2} (1 + \cot^2 \phi)} \text{ --- ②}$$



WKT,

$$\Rightarrow \tan \phi = r \frac{d\theta}{dr}$$

$$\Rightarrow \cot \phi = \frac{1}{r} \frac{dr}{d\theta}$$

from equation ②

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} \left[1 + \left[\frac{1}{r} \frac{dr}{d\theta} \right]^2 \right]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} \left[1 + \left[\frac{dr}{d\theta} \right]^2 \right]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} \left[\frac{dr}{d\theta} \right]^2$$

Hence proved.

Problems :

1. Find the angle between the radius of vector and tangent to the curve $r = a(1 - \cos \theta)$, also find the angle ϕ at $\theta = 60^\circ$

Soln :- Given :-

$$\Rightarrow r = a(1 - \cos \theta) \quad \text{--- ①}$$

diff w.r to θ

$$\Rightarrow \frac{dr}{d\theta} = a(0 - (-\sin \theta))$$

$$\Rightarrow \frac{dr}{d\theta} = a \sin \theta$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{a \sin \theta}{r}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{a \sin \theta}{a(1 - \cos \theta)}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{a \sin \frac{\theta}{2} \cos \frac{\theta}{2}}{a \sin^2 \frac{\theta}{2}}$$

$$\Rightarrow \cot \phi = \cot \left[\frac{\theta}{2} \right]$$

$$\Rightarrow \phi = \frac{\theta}{2}$$

$$\therefore \text{where, } \theta = \pi/3$$

$$\Rightarrow \theta = \frac{\pi/3}{2} \Rightarrow \boxed{\phi = \frac{\pi}{6}}$$

2. Find the angle b/w the radius of vector and tangent to the curve $r = a(1 + \cos \theta)$, also find the angle ϕ at $\theta = \pi/6$.

Soln:-

$$\Rightarrow r = a(1 + \cos \theta) \text{ --- ①}$$

diff w.r to θ

$$\Rightarrow \frac{dr}{d\theta} = -a \sin \theta$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = -\frac{a \sin \theta}{r}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = -\frac{a \sin \theta}{a(1 + \cos \theta)}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = -\frac{2 \sin \theta/2 \cos \theta/2}{2 \cos^2 \theta/2}$$

$$\Rightarrow \cot \phi = -\tan \left[\frac{\theta}{2} \right]$$

$$\Rightarrow \cot \phi = \tan \left[\frac{\pi}{2} + \frac{\theta}{2} \right]$$

$$\Rightarrow \cot \phi = \cot \left[\frac{\pi}{2} + \frac{\theta}{2} \right]$$

$$\Rightarrow \phi = \frac{\pi}{2} + \frac{\theta}{2}$$

$$\text{Where, } \theta = \frac{\pi}{6} \Rightarrow \phi = \frac{\pi}{2} + \frac{\pi/6}{2}$$

$$\Rightarrow \phi = \frac{\pi}{2} + \frac{\pi}{12}$$

$$\Rightarrow \phi = \frac{6\pi + \pi}{12} \Rightarrow \boxed{\phi = \frac{7\pi}{12}}$$

3. Show that the given pairs of polar curves can intersect orthogonally.

$$\text{if } r = a(1 - \cos \theta), \quad r = b(1 + \cos \theta)$$

$$\Rightarrow r = a(1 - \cos \theta) \text{ --- ①}$$

$$\Rightarrow \text{diff w.r to } \theta$$

$$\Rightarrow \frac{dr}{d\theta} = a(0 - (-\sin \theta))$$

$$\Rightarrow \frac{dr}{d\theta} = a \sin \theta$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{a \sin \theta}{r}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{a \sin \theta}{a(1 - \cos \theta)}$$

$$\Rightarrow \cot \phi_1 = \frac{a \sin \frac{\theta}{2} \cos \frac{\theta}{2}}{a \sin^2 \frac{\theta}{2}}$$

$$\Rightarrow \cot \phi_1 = \cot \frac{\theta}{2}$$

$$\Rightarrow \phi_1 = \frac{\theta}{2}$$

$$\Rightarrow |\phi_2 - \phi_1| = \frac{\pi}{2} + \frac{\theta}{2} - \frac{\theta}{2}$$

$$\Rightarrow |\phi_2 - \phi_1| = \frac{\pi}{2}$$

$$\Rightarrow r = b(1 + \cos \theta) \text{ --- ②}$$

$$\Rightarrow \text{diff. w.r to } \theta$$

$$\Rightarrow \frac{dr}{d\theta} = b(0 - \sin \theta)$$

$$\Rightarrow \frac{dr}{d\theta} = -b \sin \theta$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = -\frac{b \sin \theta}{r}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = -\frac{b \sin \theta}{b(1 + \cos \theta)}$$

$$\Rightarrow \cot \phi_2 = \frac{-b \sin \frac{\theta}{2} \cos \frac{\theta}{2}}{b \cos^2 \frac{\theta}{2}}$$

$$\Rightarrow \cot \phi_2 = -\tan \frac{\theta}{2}$$

$$\Rightarrow \cot \phi_2 = \cot \left[\frac{\theta}{2} + \frac{\pi}{2} \right]$$

$$\Rightarrow \phi_2 = \frac{\theta}{2} + \frac{\pi}{2}$$

$$\text{ii) } r = a(1 + \sin \theta)$$

$$r = b(1 - \sin \theta)$$

$$\Rightarrow r = a(1 + \sin \theta) \text{ --- ①}$$

diff w.r to θ

$$\Rightarrow \frac{dr}{d\theta} = a \cos \theta$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{a \cos \theta}{r}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{a \cos \theta}{a(1 + \sin \theta)}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{\cos^2 \theta/2 \sin^2 \theta/2}{\cos^2 \theta/2 + \sin^2 \theta/2 + 2 \sin \theta/2 \cos \theta/2}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{[\cancel{\cos \theta/2} + \sin \theta/2] [\cos \theta/2 - \cancel{\sin \theta/2}]}{[\cos \theta/2 + \sin \theta/2]^2}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{\cos \theta/2 - \sin \theta/2}{\cos \theta/2 + \sin \theta/2}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{\frac{\cos \theta/2}{\cos \theta/2} - \frac{\sin \theta/2}{\cos \theta/2}}{\frac{\cos \theta/2}{\cos \theta/2} + \frac{\sin \theta/2}{\cos \theta/2}}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{1 - \tan \theta/2}{1 + \tan \theta/2}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{\tan(\pi/4) - \tan(\theta/2)}{1 + \tan(\pi/4) \tan(\theta/2)}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \tan\left[\frac{\pi}{4} - \frac{\theta}{2}\right]$$

$$\Rightarrow \cot \phi_1 = \cot \left[\frac{\pi}{2} - \left[\frac{\pi}{4} - \frac{\theta}{2} \right] \right]$$

$$\Rightarrow \phi_1 = \frac{\pi}{2} - \frac{\pi}{4} + \frac{\theta}{2}$$

$$\Rightarrow \phi_1 = \frac{\pi}{4} + \frac{\theta}{2}$$

$$r = b(1 - \sin \theta) \text{ --- ②}$$

$$\Rightarrow \frac{dr}{d\theta} = -b \cos \theta$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = -\frac{b \cos \theta}{r}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{-b \cos \theta}{b(1 - \sin \theta)}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{-\cos^2 \left[\frac{\theta}{2} \right] \sin^2 \left[\frac{\theta}{2} \right]}{\cos^2 \left[\frac{\theta}{2} \right] + \sin^2 \left[\frac{\theta}{2} \right] - 2 \sin \left[\frac{\theta}{2} \right] \cos \left[\frac{\theta}{2} \right]}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{-\left[\cos \frac{\theta}{2} + \sin \frac{\theta}{2} \right] \left[\cos \frac{\theta}{2} - \sin \frac{\theta}{2} \right]}{\left[\cos \frac{\theta}{2} - \sin \frac{\theta}{2} \right]^2}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{-\cos \frac{\theta}{2} + \sin \frac{\theta}{2}}{\cos \frac{\theta}{2} - \sin \frac{\theta}{2}}$$

$$\Rightarrow \cot \phi_2 = -\left| \frac{1 + \tan \frac{\theta}{2}}{1 - \tan \frac{\theta}{2}} \right|$$

$$\Rightarrow \cot \phi_2 = -\tan \left[\frac{\pi}{4} + \frac{\theta}{2} \right]$$

$$\Rightarrow \cot \phi_2 = \cot \left[\frac{\pi}{2} + \left[\frac{\pi}{4} + \frac{\theta}{2} \right] \right]$$

$$\Rightarrow \phi_2 = \frac{\pi}{2} + \frac{\pi}{4} + \frac{\theta}{2}$$

$$\Rightarrow |\phi_2 - \phi_1| = \left| \frac{\pi}{2} + \frac{\pi}{4} + \frac{\theta}{2} - \frac{\pi}{4} - \frac{\theta}{2} \right|$$

$$\Rightarrow |\phi_2 - \phi_1| = \frac{\pi}{2}$$

$$\text{iii) } r = \frac{a}{1 + \cos \theta}$$

$$\Rightarrow r(1 + \cos \theta) = a \quad \text{--- ①}$$

$\Rightarrow$ diff w.r to θ

$$\Rightarrow (1 + \cos \theta) \frac{dr}{d\theta} + r(-\sin \theta) = 0$$

$$\Rightarrow (1 + \cos \theta) \frac{dr}{d\theta} = r \sin \theta$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{\sin \theta}{1 + \cos \theta}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{\sin \theta/2 \cos \theta/2}{\cos^2 \theta/2}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{\sin \theta/2}{\cos \theta/2}$$

$$\Rightarrow \cot \phi_1 = \tan \frac{\theta}{2}$$

$$\Rightarrow \cot \phi_1 = \cot \left[\frac{\pi}{2} - \frac{\theta}{2} \right]$$

$$\Rightarrow \phi_1 = \frac{\pi}{2} - \frac{\theta}{2}$$

$$\Rightarrow |\phi_2 - \phi_1| = -\frac{\theta}{2} - \left[\frac{\pi}{2} - \frac{\theta}{2} \right]$$

$$\Rightarrow |\phi_2 - \phi_1| = -\frac{\theta}{2} - \frac{\pi}{2} + \frac{\theta}{2}$$

$$\Rightarrow |\phi_2 - \phi_1| = \frac{\pi}{2}$$

$$r = \frac{b}{1 - \cos \theta}$$

$$\Rightarrow r(1 - \cos \theta) = b \quad \text{--- ②}$$

$\Rightarrow$ diff w.r to θ

$$\Rightarrow (1 - \cos \theta) \frac{dr}{d\theta} + r(\sin \theta) = 0$$

$$\Rightarrow (1 - \cos \theta) \frac{dr}{d\theta} = -r \sin \theta$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{-\sin \theta}{1 - \cos \theta}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{-2 \sin \theta/2 \cos \theta/2}{2 \sin^2 \theta/2}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = -\cot \frac{\theta}{2}$$

$$\Rightarrow \cot \phi_2 = \cot \left[-\frac{\theta}{2} \right]$$

$$\Rightarrow \phi_2 = -\frac{\theta}{2}$$

$$\forall \quad r^n = a^n \cos n\theta$$

$$\Rightarrow r^n = a^n \cos n\theta \quad \text{--- ①}$$

diff w.r to θ

$$\Rightarrow n r^{n-1} \frac{dr}{d\theta} = a^n (-\sin n\theta) (n)$$

$$\Rightarrow \frac{r^n}{r} \frac{dr}{d\theta} = -a^n \sin n\theta$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = -\frac{a^n \sin n\theta}{r^n}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = -\frac{a^n \sin n\theta}{a^n \cos n\theta}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = -\tan n\theta$$

$$\Rightarrow \cot \phi_1 = \cot \left[\frac{\pi}{2} + n\theta \right]$$

$$\Rightarrow \phi_1 = \frac{\pi}{2} + n\theta$$

$$\Rightarrow |\phi_2 - \phi_1| = n\theta - \frac{\pi}{2} - n\theta$$

$$\Rightarrow |\phi_2 - \phi_1| = \frac{\pi}{2}$$

$$\forall \quad r^n \cos n\theta = a^n$$

$$\Rightarrow r^n = \frac{a^n}{\cos n\theta} \quad \text{--- ①}$$

diff w.r to θ

$$\Rightarrow n r^{n-1} \frac{dr}{d\theta} = \frac{0(\cos n\theta) - a^n (-\sin n\theta)(n)}{(\cos n\theta)^2}$$

$$\Rightarrow \frac{r^n}{r} \frac{dr}{d\theta} = \frac{a^n \sin n\theta}{(\cos n\theta)^2}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{a^n \sin n\theta}{(\cos n\theta)^2 r^n}$$

$$r^n = b^n \sin n\theta$$

$$\Rightarrow r^n = b^n \sin n\theta \quad \text{--- ②}$$

diff w.r to θ

$$\Rightarrow n r^{n-1} \frac{dr}{d\theta} = b^n (\cos n\theta) (n)$$

$$\Rightarrow \frac{r^n}{r} \frac{dr}{d\theta} = b^n \cos n\theta$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{b^n \cos n\theta}{r^n}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{b^n \cos n\theta}{b^n \sin n\theta}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \cot n\theta$$

$$\Rightarrow \cot \phi_2 = \cot n\theta$$

$$\Rightarrow \phi_2 = n\theta$$

$$r^n \sin n\theta = b^n$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{a^n \sin n\theta}{(\cos n\theta)^2 \left[\frac{a^n}{\cos n\theta} \right]}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{\sin n\theta}{\cos n\theta}$$

$$\Rightarrow \cot \phi_1 = \tan n\theta$$

$$\Rightarrow \cot \phi_1 = \cot \left[\frac{\pi}{2} - n\theta \right]$$

$$\Rightarrow \phi_1 = \frac{\pi}{2} - n\theta$$

$$r^n \sin n\theta = b^n \quad \text{--- (2)}$$

$$\Rightarrow r^n = \frac{b^n}{\sin n\theta}$$

diff. w.r to θ

$$\Rightarrow n r^{n-1} \frac{dr}{d\theta} = \frac{0(\sin n\theta) - b^n (\cos n\theta)(n)}{(\sin n\theta)^2}$$

$$\Rightarrow \frac{r^n}{r} \frac{dr}{d\theta} = \frac{-b^n \cos n\theta}{(\sin n\theta)^2}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{-b^n \cos n\theta}{(\sin n\theta)^2 r^n}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{-b^n \cos n\theta}{(\sin n\theta)^2 \left[\frac{b^n}{\sin n\theta} \right]}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{-\cos n\theta}{\sin n\theta}$$

$$\Rightarrow \cot \phi_2 = -\cot n\theta$$

$$\Rightarrow \cot \phi_2 = \cot (-n\theta)$$

$$\Rightarrow \phi_2 = -n\theta$$

$$\Rightarrow |\phi_2 - \phi_1| = -n\theta - \frac{\pi}{2} + n\theta$$

$$\Rightarrow |\phi_2 - \phi_1| = \frac{\pi}{2}$$

$$\text{viz } r = 4 \sec^2\left(\frac{\theta}{2}\right)$$

$$\Rightarrow r = \frac{4}{\cos^2\left(\frac{\theta}{2}\right)}$$

$$\Rightarrow r \cos^2\left(\frac{\theta}{2}\right) = 4$$

$$\Rightarrow \frac{r(1+\cos\theta)}{2} = 4$$

$$\Rightarrow r(1+\cos\theta) = 8 \text{ --- ①}$$

diff w.r to θ

$$\Rightarrow (1+\cos\theta) \frac{dr}{d\theta} + r(-\sin\theta) = 0$$

$$\Rightarrow (1+\cos\theta) \frac{dr}{d\theta} = r \sin\theta$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{\sin\theta}{1+\cos\theta}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}}{2 \cos^2 \frac{\theta}{2}}$$

$$\Rightarrow \cot \phi_1 = \tan\left[\frac{\theta}{2}\right]$$

$$\Rightarrow \cot \phi_1 = \cot\left[\frac{\pi}{2} - \frac{\theta}{2}\right]$$

$$\Rightarrow \phi_1 = \frac{\pi}{2} - \frac{\theta}{2}$$

$$\Rightarrow |\phi_2 - \phi_1| = -\frac{\theta}{2} - \frac{\pi}{2} + \frac{\theta}{2}$$

$$\Rightarrow |\phi_2 - \phi_1| = \frac{\pi}{2}$$

$$r = 9 \operatorname{cosec}^2\left[\frac{\theta}{2}\right]$$

$$\Rightarrow r = \frac{9}{\sin^2\left(\frac{\theta}{2}\right)}$$

$$\Rightarrow r \sin^2\left[\frac{\theta}{2}\right] = 9$$

$$\Rightarrow \frac{r(1-\cos\theta)}{2} = 9$$

$$\Rightarrow r(1-\cos\theta) = 18 \text{ --- ②}$$

diff. w.r to θ

$$\Rightarrow (1-\cos\theta) \frac{dr}{d\theta} + r \sin\theta = 0$$

$$\Rightarrow (1-\cos\theta) \frac{dr}{d\theta} = -r \sin\theta$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = -\frac{\sin\theta}{1-\cos\theta}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = -\frac{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}}{2 \sin^2 \frac{\theta}{2}}$$

$$\Rightarrow \cot \phi_2 = -\cot\left[\frac{\theta}{2}\right]$$

$$\Rightarrow \cot \phi_2 = \cot\left[-\frac{\theta}{2}\right]$$

$$\Rightarrow \phi_2 = -\frac{\theta}{2}$$

4. Find the angle between the given polar curves;

i) $r = 2 \sin \theta$

$\Rightarrow r = 2 \sin \theta$ — (1)

diff w.r to θ

$\Rightarrow \frac{dr}{d\theta} = 2 \cos \theta$

$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{2 \cos \theta}{r}$

$\Rightarrow \cot \phi_1 = \frac{2 \cos \theta}{2 \sin \theta}$

$\Rightarrow \cot \phi_1 = \cot \theta$

$\Rightarrow \phi_1 = \theta$

$r = 2 \cos \theta$

$\Rightarrow r = 2 \cos \theta$ — (2)

diff w.r to θ

$\Rightarrow \frac{dr}{d\theta} = -2 \sin \theta$

$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = -\frac{2 \sin \theta}{r}$

$\Rightarrow \cot \phi_2 = -\frac{2 \sin \theta}{2 \cos \theta}$

$\Rightarrow \cot \phi_2 = -\tan \theta$

$\Rightarrow \phi_2 = \frac{\pi}{2} + \theta$

$\Rightarrow |\phi_2 - \phi_1| = \frac{\pi}{2} + \theta - \theta$

$\Rightarrow |\phi_2 - \phi_1| = \frac{\pi}{2}$

ii) $r = a \log \theta$

$\Rightarrow r = a \log \theta$ — (1)

diff w.r to ' θ '

$\Rightarrow \frac{dr}{d\theta} = a \cdot \frac{1}{\theta}$

$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{a}{r\theta}$

$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{a}{a \log \theta(\theta)}$

$\Rightarrow r \frac{d\theta}{dr} = \theta \log \theta$

$\Rightarrow \tan \phi_1 = \theta \log \theta$ — (3)

$r = \frac{a}{\log \theta}$

$\Rightarrow r \log \theta = a$ — (2)

diff w.r to ' θ '

$\Rightarrow (\log \theta) \frac{dr}{d\theta} + r \left[\frac{1}{\theta} \right] = 0$

$\Rightarrow \log \theta \frac{dr}{d\theta} = -\frac{r}{\theta}$

$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{1}{-\theta \log \theta}$

$\Rightarrow r \frac{d\theta}{dr} = -\theta \log \theta$

$\Rightarrow \tan \phi_2 = -\theta \log \theta$ — (4)

WKT,

$$\Rightarrow \tan(\phi_1 - \phi_2) = \frac{\tan \phi_1 - \tan \phi_2}{1 + \tan \phi_1 \cdot \tan \phi_2}$$

$$\Rightarrow \tan(\phi_1 - \phi_2) = \frac{\theta \log \theta - (-\theta \log \theta)}{1 + \theta \log \theta (-\theta \log \theta)}$$

$$\Rightarrow \tan(\phi_1 - \phi_2) = \frac{\theta \log \theta + \theta \log \theta}{1 + \theta^2 (\log \theta)^2}$$

$$\Rightarrow \tan(\phi_1 - \phi_2) = \frac{2\theta \log \theta}{1 - \theta^2 (\log \theta)^2} \quad \text{--- (4)}$$

From equation ① & ②

$$\Rightarrow x \log \theta = \frac{x}{\log \theta}$$

$$\Rightarrow \log \theta = \frac{1}{\log \theta}$$

$$\Rightarrow (\log \theta)^2 = 1$$

$$\Rightarrow \log_e \theta = 1$$

$$\Rightarrow \theta = e^1$$

$$\Rightarrow \theta = e$$

from equation ④

$$\Rightarrow \tan(\phi_1 - \phi_2) = \frac{2e \cdot 1}{1 - e^2(1)^2}$$

$$\Rightarrow \tan(\phi_1 - \phi_2) = \frac{2e}{1 - e^2}$$

$$\Rightarrow |\phi_1 - \phi_2| = \tan^{-1} \left[\frac{2e}{1 - e^2} \right] = 2 \tan^{-1} e.$$

$$\text{iii) } r^2 \sin 2\theta = 4$$

$$\Rightarrow r^2 \sin 2\theta = 4 \text{ --- (1)}$$

diff wr to θ

$$\Rightarrow 2r \frac{dr}{d\theta} \sin 2\theta + r^2 \cos 2\theta (2) = 0$$

$$\Rightarrow r \frac{dr}{d\theta} \sin 2\theta = -r^2 \cos 2\theta$$

$$\Rightarrow \frac{r}{r^2} \frac{dr}{d\theta} = -\frac{\cos 2\theta}{\sin 2\theta}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = -\cot 2\theta$$

$$\Rightarrow \cot \phi_1 = \cot (-2\theta)$$

$$\Rightarrow \phi_1 = -2\theta$$

$$\Rightarrow |\phi_2 - \phi_1| = |2\theta - (-2\theta)|$$

$$\Rightarrow |\phi_2 - \phi_1| = 4\theta \text{ --- (3)}$$

From (1) & (2)

$$\Rightarrow \frac{4}{\sin 2\theta} = 4 \sin 2\theta$$

$$\Rightarrow 1 = 4 \sin^2 2\theta$$

$$\Rightarrow \sin^2 2\theta = \frac{1}{4} \quad (\div 2)$$

$$\Rightarrow \sin 2\theta = \frac{1}{2}$$

$$\Rightarrow 2\theta = \sin^{-1} \left[\frac{1}{2} \right]$$

$$\Rightarrow 2\theta = \pi/6$$

$$\Rightarrow \theta = \pi/12$$

$$\therefore |\phi_2 - \phi_1| = 4 \left[\frac{\pi}{12} \right] = \frac{\pi}{3}$$

$$r^2 = 16 \sin 2\theta$$

$$\Rightarrow r^2 = 16 \sin 2\theta \text{ --- (2)}$$

diff wr to θ

$$\Rightarrow 2r \frac{dr}{d\theta} = 16 \cos 2\theta (2)$$

$$\Rightarrow r \frac{dr}{d\theta} = 16 \cos 2\theta$$

$$\Rightarrow \frac{r}{r^2} \frac{dr}{d\theta} = \frac{16 \cos 2\theta}{r^2}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{16 \cos 2\theta}{16 \sin 2\theta}$$

$$\Rightarrow \cot \phi_2 = \cot 2\theta$$

$$\Rightarrow \phi_2 = 2\theta$$

$$\text{iv) } r = a(1 - \cos \theta)$$

$$\Rightarrow r = a(1 - \cos \theta) \text{ ——— ①}$$

diff w.r to θ

$$\Rightarrow \frac{dr}{d\theta} = a \sin \theta$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{a \sin \theta}{r}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{a \sin \theta}{a(1 - \cos \theta)}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{2 \sin \theta/2 \cos \theta/2}{2 \sin^2 \theta/2}$$

$$\Rightarrow \cot \phi_1 = \cot \theta/2$$

$$\Rightarrow \phi_1 = \frac{\theta}{2}$$

$$\Rightarrow |\phi_2 - \phi_1| = \frac{\pi}{2} + \theta - \frac{\theta}{2}$$

$$\Rightarrow |\phi_2 - \phi_1| = \frac{\pi}{2} + \frac{\theta}{2} \text{ ——— ③}$$

From, ① & ②

$$\Rightarrow a(1 - \cos \theta) = 2a \cos \theta$$

$$\Rightarrow 1 - \cos \theta = 2 \cos \theta$$

$$\Rightarrow 1 = 2 \cos \theta + \cos \theta$$

$$\Rightarrow 1 = 3 \cos \theta$$

$$\Rightarrow \frac{1}{3} = \cos \theta$$

$$\Rightarrow \theta = \cos^{-1} \left[\frac{1}{3} \right]$$

$$\therefore |\phi_2 - \phi_1| = \frac{\pi}{2} + \frac{\cos^{-1} [1/3]}{2}$$

$$r = 2a \cos \theta$$

$$\Rightarrow r = 2a \cos \theta \text{ ——— ②}$$

diff w.r to θ

$$\Rightarrow \frac{dr}{d\theta} = -2a \sin \theta$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = -\frac{2a \sin \theta}{r}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = -\frac{2a \sin \theta}{2a \cos \theta}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = -\frac{\sin \theta}{\cos \theta}$$

$$\Rightarrow \cot \phi_2 = -\tan \theta$$

$$\Rightarrow \cot \phi_2 = \cot \left[\frac{\pi}{2} + \theta \right]$$

$$\Rightarrow \phi_2 = \frac{\pi}{2} + \theta$$

PEDAL EQUATION OF POLAR CURVES :-

Let ' r ' = $f(\theta)$ be a polar curve,

Let ' ϕ ' be the angle between radius of vector and tangent to the curve $f(\theta)$ and

' p ' be the perpendicular distance from pole to the tangent ' T ', then

$$p = r \sin \phi \quad (\text{or}) \quad \frac{1}{p^2} = \frac{1}{r^2} (1 + \cot^2 \phi) \quad (\text{or})$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} \left[\frac{dr}{d\theta} \right]^2$$

is called the pedal equation of a polar curve $r = f(\theta)$, it is called p - r equation excluding ' θ '.

Problems :-

1. Find the pedal equation of a polar curve.

if $r = a(1 + \cos \theta)$

$$\Rightarrow r = a(1 + \cos \theta) \quad \text{--- ①}$$

diff w.r to ' θ '

$$\Rightarrow \frac{dr}{d\theta} = -a \sin \theta$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = -\frac{a \sin \theta}{r}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = -\frac{a \sin \theta}{a(1 + \cos \theta)}$$

$$\Rightarrow \cot \phi = \frac{-\cancel{a} \sin^{\theta/2} \cos^{\theta/2}}{\cancel{a} \cos^{\theta/2}}$$

$$\Rightarrow \cot \phi = -\tan \theta/2$$

WKT,

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} [1 + \cot^2 \phi]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} [1 + \tan^2 \frac{\theta}{2}]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} \sec^2 \frac{\theta}{2}$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2 \cos^2 \frac{\theta}{2}}$$

$$\Rightarrow p^2 = r^2 \cos^2 \frac{\theta}{2}$$

$$\Rightarrow p^2 = r^2 \left[\frac{1 + \cos \theta}{2} \right]$$

$$\Rightarrow p^2 = \frac{r^2}{2} [1 + \cos \theta]$$

$$\Rightarrow p^2 = \frac{r^2}{2} \left[\frac{r}{a} \right]$$

$$\Rightarrow p^2 = \frac{r^3}{2a}$$

$$\Rightarrow 2ap^2 = r^3$$

if $r = a(1 - \cos \theta)$ — (2)

diff w.r to θ

$$\Rightarrow \frac{dr}{d\theta} = a \sin \theta$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{a \sin \theta}{r}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{a \sin \theta}{a(1 - \cos \theta)}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}}{2 \sin^2 \frac{\theta}{2}}$$

$$\Rightarrow \cot \phi = \cot \frac{\theta}{2}$$

WKT,

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^3} [1 + \cot^2 \phi]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^3} [1 + \cot^2 \theta/2]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^3} [\operatorname{cosec}^2 \theta/2]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^3 \sin^2 \theta/2}$$

$$\Rightarrow p^2 = r^3 \sin^2 \theta/2$$

$$\Rightarrow p^2 = r^3 \left[\frac{1 - \cos \theta}{2} \right]$$

$$\Rightarrow p^2 = \frac{r^3}{2} [1 - \cos \theta]$$

$$\Rightarrow p^2 = \frac{r^3}{2} \left[\frac{r}{a} \right]$$

$$\Rightarrow p^2 = \frac{r^3}{2a}$$

$$\Rightarrow 2ap^2 = r^3$$

iii) $r^m = a^m [\cos m\theta + \sin m\theta]$

$\Rightarrow$ diff w.r to θ

$$\Rightarrow m r^{m-1} \frac{dr}{d\theta} = a^m [-m \sin m\theta + m \cos m\theta]$$

$$\Rightarrow \frac{m r^m}{r} \frac{dr}{d\theta} = m [\cos m\theta - \sin m\theta] a^m$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{(\cos m\theta - \sin m\theta) a^m}{r^m}$$

$$\Rightarrow \cot \phi = \frac{(\cos m\theta - \sin m\theta) a^m}{a^m [\cos m\theta + \sin m\theta]}$$

$$\Rightarrow 1 + \cot^2 \phi = \frac{(\cos m\theta - \sin m\theta)^2}{(\cos m\theta + \sin m\theta)^2}$$

$$\Rightarrow 1 + \cot^2 \phi = \frac{(\cos m\theta + \sin m\theta)^2 + (\cos m\theta - \sin m\theta)^2}{(\cos m\theta + \sin m\theta)^2}$$

$$\Rightarrow 1 + \cot^2 \phi = \frac{2(\cos^2 m\theta + \sin^2 m\theta)}{\left[\frac{r^m}{a^m}\right]^2}$$

$$\Rightarrow 1 + \cot^2 \phi = \frac{2}{\frac{r^{2m}}{a^{2m}}}$$

$$\Rightarrow 1 + \cot^2 \phi = \frac{2a^{2m}}{r^{2m}}$$

$\therefore$ WKT,

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} [1 + \cot^2 \phi]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} \frac{2a^{2m}}{r^{2m}}$$

$$\Rightarrow \frac{1}{p^2} = \frac{2a^{2m}}{r^{2m+2}}$$

$$\Rightarrow 2a^{2m} p^2 = r^{2(m+1)}$$

iv) $r = a(e^{\theta \cot \alpha})$

$$\Rightarrow r = ae^{\theta \cot \alpha} \quad \text{--- (1)}$$

diff w.r to θ

$$\Rightarrow \frac{dr}{d\theta} = a \cdot e^{\theta \cot \alpha} \cdot \cot \alpha$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{(a \cot \alpha) e^{\theta \cot \alpha}}{a e^{\theta \cot \alpha}}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{(\alpha \cot \alpha) e^{\theta \cot \alpha}}{\alpha e^{\theta \cot \alpha}}$$

$$\Rightarrow \cot \phi = \cot \alpha$$

$$\Rightarrow \phi = \alpha$$

WKT,

$$p = r \sin \phi$$

$$\boxed{p = r \sin \alpha}$$

$$\forall \frac{l}{r} = 1 + e \cos \theta$$

$$\Rightarrow r(1 + e \cos \theta) = l \quad \text{--- (2)}$$

diff w.r to 'θ'

$$\Rightarrow (1 + e \cos \theta) \frac{dr}{d\theta} + r(0 - e \sin \theta) = 0$$

$$\Rightarrow (1 + e \cos \theta) \frac{dr}{d\theta} = r e (\sin \theta)$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{e \sin \theta}{1 + e \cos \theta}$$

$$\Rightarrow 1 + \cot^2 \phi = \frac{1 + (e \sin \theta)^2}{(1 + e \cos \theta)^2}$$

$$\Rightarrow 1 + \cot^2 \phi = \frac{(1 + e \cos \theta)^2 + (e^2 \sin^2 \theta)}{(1 + e \cos \theta)^2}$$

$$\Rightarrow 1 + \cot^2 \phi = \frac{1 + 2e \cos \theta + e^2 \cos^2 \theta + e^2 \sin^2 \theta}{(1 + e \cos \theta)^2}$$

$$\Rightarrow 1 + \cot^2 \phi = \frac{1 + 2e \cos \theta + e^2 (\cos^2 \theta + \sin^2 \theta)}{(1 + e \cos \theta)^2}$$

$$\Rightarrow 1 + \cot^2 \phi = \frac{1 + 2e \cos \theta + e^2}{(1 + e \cos \theta)^2}$$

$$\Rightarrow 1 + \cot^2 \phi = \frac{1 + e^2 + 2 \left[\frac{l}{r} - 1 \right]}{(l/r)^2}$$

$$\Rightarrow 1 + \cot^2 \phi = \frac{1 + e^2 + 2 \frac{l}{r} - 2}{l^2/r^2}$$

$$\Rightarrow 1 + \cot^2 \phi = \frac{r^2}{l^2} \left[e^2 + \frac{2l}{r} - 1 \right]$$

$\therefore$ WKT

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} (1 + \cot^2 \phi)$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} \frac{r^2}{l^2} \left[e^2 + \frac{2l}{r} - 1 \right]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{l^2} \left[e^2 + \frac{2l}{r} - 1 \right]$$

Let $r^m = a^m \cos m\theta + b^m \sin m\theta$

$\Rightarrow$ diff w.r to ' θ '

$$\Rightarrow m r^{m-1} \frac{dr}{d\theta} = -m a^m \sin m\theta + m b^m \cos m\theta$$

$$\Rightarrow m \frac{r^m}{r} \frac{dr}{d\theta} = a^m [b^m \cos m\theta - a^m \sin m\theta]$$

$$\Rightarrow \frac{r^m}{r} \frac{dr}{d\theta} = \frac{b^m \cos m\theta - a^m \sin m\theta}{r^m}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{b^m \cos m\theta - a^m \sin m\theta}{a^m \cos m\theta + b^m \sin m\theta}$$

$$\Rightarrow \cot^2 \phi = \frac{(b^m \cos m\theta - a^m \sin m\theta)^2}{(a^m \cos m\theta + b^m \sin m\theta)^2}$$

$$\Rightarrow 1 + \cot^2 \phi = \frac{1 + (b^m \cos m\theta - a^m \sin m\theta)^2}{(a^m \cos m\theta + b^m \sin m\theta)^2}$$

$$\Rightarrow 1 + \cot^2 \phi = \frac{(a^m \cos m\theta + b^m \sin m\theta)^2 + (b^m \cos m\theta - a^m \sin m\theta)^2}{(a^m \cos m\theta + b^m \sin m\theta)^2}$$

$$\Rightarrow 1 + \cot^2 \phi = \frac{a^{2m} \cos^2 n\theta + b^{2m} \sin^2 n\theta + 2a^m b^m \cos n\theta \sin n\theta + b^{2m} \cos^2 n\theta + a^{2m} \sin^2 n\theta - 2a^m b^m \cos n\theta \sin n\theta}{(\gamma^m)^2}$$

$$\Rightarrow 1 + \cot^2 \phi = \frac{(a^{2m} + b^{2m}) \cos^2 n\theta + (a^{2m} + b^{2m}) \sin^2 n\theta}{\gamma^{2m}} \quad (8)$$

$$\Rightarrow 1 + \cot^2 \phi = \frac{(a^{2m} + b^{2m}) (\cos^2 n\theta + \sin^2 n\theta)}{\gamma^{2m}}$$

$$\Rightarrow 1 + \cot^2 \phi = \frac{a^{2m} + b^{2m}}{\gamma^{2m}}$$

WKT

$$\Rightarrow \frac{1}{\rho^2} = \frac{1}{\gamma^2} [1 + \cot^2 \phi]$$

$$\Rightarrow \frac{1}{\rho^2} = \frac{1}{\gamma^2} \left[\frac{a^{2m} + b^{2m}}{\gamma^{2m}} \right]$$

ARC LENGTH OF DIFFERENT CURVES :-

(i) Let $f(x, y) = c$ be a cartesian curve, then the arc length can be evaluated by the given formula.

$$\frac{ds}{dx} = \sqrt{1 + \left[\frac{dy}{dx} \right]^2} = \sqrt{1 + y'^2} = \sqrt{1 + y_1'^2}$$

$$\frac{ds}{dy} = \sqrt{1 + \left[\frac{dx}{dy} \right]^2} = \sqrt{1 + x'^2} = \sqrt{1 + x_1'^2}$$

(2) Let $x = x(t)$, $y = y(t)$ be the parametric equation for x and y , then

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$\frac{ds}{dt} = \sqrt{[x'(t)]^2 + [y'(t)]^2}$$

(3) Let, $f(r, \theta) = c$ be a polar curve, then

$$\frac{ds}{d\theta} = \sqrt{r^2 + \left[\frac{dr}{d\theta}\right]^2}$$

$$\frac{ds}{d\theta} = \sqrt{1 + r^2 \left[\frac{d\theta}{dr}\right]^2}$$

CURVATURE AND RADIUS OF CURVATURE :-

The rate of change of direction of a curve with respect to distance along the curve is called curvature.

It can be denoted by 'k'.

The reciprocal of the curvature is called the radius of curvature.

It can be denoted by 'p'.

$$\therefore \text{The radius of curvature 'p'} = \frac{1}{k} = \left| \frac{ds}{d\psi} \right|$$

Derivation of Radius of Curvature in Cartesian form;

Let $f(x, y) = c$, curve in Cartesian form,

Let ' τ ' be a tangent to the curve $f(x, y) = c$, having the slope $m = \tan \psi$.

$$\Rightarrow f(x, y) = c$$

$$\Rightarrow m = \tan \psi = y_1$$

$$\Rightarrow \psi = \tan^{-1}(y_1) \quad \text{--- (1)}$$

diff. w.r to θ

$$\Rightarrow \frac{d\psi}{d\theta} = \frac{1}{1+y_1^2} \frac{d(y_1)}{dx}$$

$$\Rightarrow \frac{d\psi}{d\theta} = \frac{y_2}{1+y_1^2}$$

WKT

$$\frac{ds}{dx} = \sqrt{1+y_1^2}$$

also, WKT

$$\Rightarrow \rho = \frac{ds}{d\psi}$$

$$\Rightarrow \rho = \frac{ds/dx}{d\psi/dx}$$

$$\Rightarrow \rho = \frac{\sqrt{1+y_1^2}}{y_2/(1+y_1^2)}$$

$$\Rightarrow \rho = \frac{(1+y_1^2)^{1/2}(1+y_1^2)}{y_2}$$

$$\Rightarrow \rho = \frac{(1+y_1^2)^{3/2}}{y_2}, \quad y_2 \neq 0$$

Derivative of Radius of Curvature in Polar form :-

Let, $r = f(\theta)$ be a polar curves

at $p(r, \theta)$ with $OP = r$

radius of a curve having the tangent 'T'

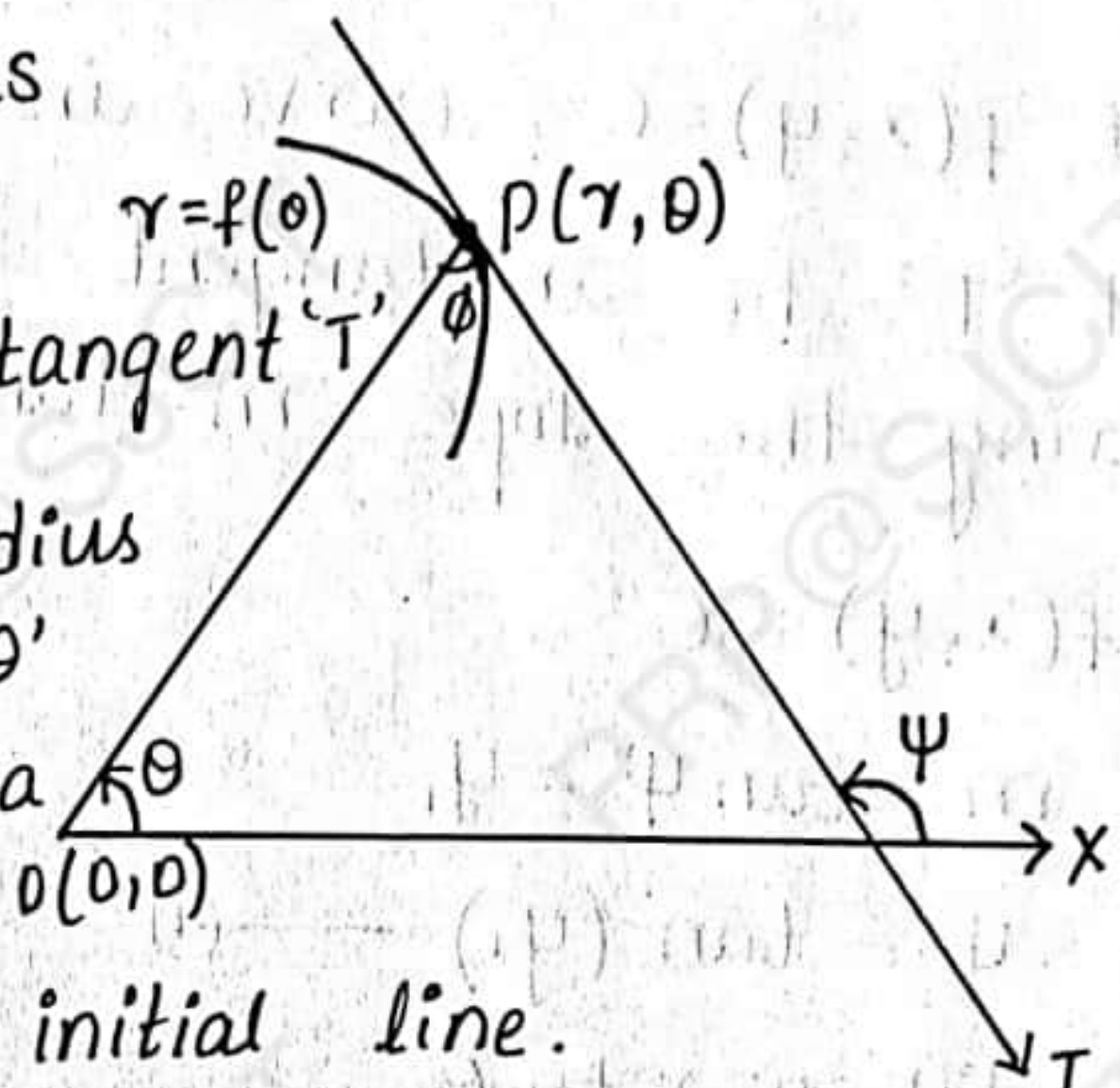
Let ' ϕ ' be the angle between radius

of a curve and tangent, let ' θ '

be the angle between radius of a

curve and initial line.

' ψ ' be the angle made by the initial line.



WKT,

$$\Rightarrow \psi = \phi + \theta \quad \text{--- ①}$$

diff w.r to 's'

$$\Rightarrow \frac{d\psi}{ds} = \frac{d\phi}{ds} + \frac{d\theta}{ds}$$

$$\Rightarrow \frac{d\psi}{ds} = \frac{d\phi/d\theta}{ds/d\theta} + \frac{d\theta}{ds}$$

$$\Rightarrow \frac{d\psi}{ds} = \frac{d\phi}{d\theta} \cdot \frac{d\theta}{ds} + \frac{d\theta}{ds}$$

$$\Rightarrow \frac{d\psi}{ds} = \frac{d\theta}{ds} \left[\frac{d\phi}{d\theta} + 1 \right] \quad \text{--- ②}$$

Also, WKT

$$\Rightarrow \tan \phi = r \frac{d\theta}{dr}$$

$$\Rightarrow \tan \phi = \frac{r}{(dr/d\theta)}$$

$$\Rightarrow \tan \phi = \frac{r}{r_1} \quad \text{where, } r_1 = \frac{dr}{d\theta}$$

$$\Rightarrow \phi = \tan^{-1} [r/r_1] \quad \text{--- ③}$$

using diff ③ w. r to θ

$$\Rightarrow \frac{d\phi}{d\theta} = \frac{1}{1 + \left(\frac{r}{r'}\right)^2} \frac{d}{d\theta} \left[\frac{r}{r'} \right]$$

$$\Rightarrow \frac{d\phi}{d\theta} = \left[\frac{1}{1 + \frac{r^2}{r'^2}} \right] \left[r' \frac{d}{d\theta} \left(\frac{r}{r'} \right) - r \frac{d}{d\theta} \left(\frac{1}{r'} \right) \right]$$

$$\Rightarrow \frac{d\phi}{d\theta} = \frac{r'}{r^2 + r'^2} \left[r' r_1 - r r_2 \right]$$

$$\Rightarrow \frac{d\phi}{d\theta} = \frac{r' r_1 - r r_2}{r^2 + r'^2}$$

$$\therefore \textcircled{2} \Rightarrow \frac{d\psi}{ds} = \frac{d\theta}{ds} \left[1 + \frac{r' r_1 - r r_2}{r^2 + r'^2} \right]$$

$$\Rightarrow \frac{1}{ds/d\psi} = \frac{1}{ds/d\theta} \left[\frac{r^2 + r'^2 + r' r_1 - r r_2}{r^2 + r'^2} \right]$$

$$\Rightarrow \frac{1}{\rho} = \frac{1}{\sqrt{r^2 + r'^2}} \left[\frac{r^2 + r'^2 + r' r_1 - r r_2}{r^2 + r'^2} \right]$$

$$\Rightarrow \frac{1}{\rho} = \frac{r^2 + r'^2 + r' r_1 - r r_2}{(r^2 + r'^2)^{3/2}}$$

$$\Rightarrow \rho = \frac{(r^2 + r'^2)^{3/2}}{r^2 + r'^2 + r' r_1 - r r_2}$$

RADIUS OF CURVATURE IN PARAMETRIC FORM:-

Let $x = x(t)$, $y = y(t)$ with a two parametric equations, then the radius of curvature can be evaluated from the given formula.

$$\Rightarrow \rho = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - x''y'}$$

where,

$$x' = \frac{dx}{dt}, \quad y' = \frac{dy}{dt}$$

$$x'' = \frac{d^2x}{dt^2}, \quad y'' = \frac{d^2y}{dt^2}$$

The Radius of Curvature of a polar curve in pedal form;

$$\rho = r \frac{dr}{dp}$$

Problems:-

1. Find the radius of curvature of :-

$$y = a \log(\sec x/a)$$

$\Rightarrow$ diff w.r to θ 'x'

$$\Rightarrow \frac{dy}{dx} = \frac{a}{\sec\left[\frac{x}{a}\right]} \cdot \tan\left[\frac{x}{a}\right] \sec\left[\frac{x}{a}\right] \left[\frac{1}{a}\right]$$

$$\Rightarrow \frac{dy}{dx} = \tan\left[\frac{x}{a}\right]$$

diff w.r to 'x'

$$\Rightarrow \frac{d^2y}{dx^2} = \sec^2\left(\frac{x}{a}\right) \left(\frac{1}{a}\right)$$

$$\Rightarrow y_2 = \frac{1}{a} \sec^2\left[\frac{x}{a}\right]$$

WKT

$$\Rightarrow f = \frac{(1 + y_1^2)^{3/2}}{y_2}$$

$$\Rightarrow f = \frac{\left[1 + \tan^2 \left[\frac{x}{a}\right]^{3/2}\right]}{\frac{1}{a} \sec^2 \left[\frac{x}{a}\right]}$$

$$\Rightarrow f = \frac{a \left[\sec^2 \left(\frac{x}{a}\right)\right]^{3/2}}{\sec^2 \left[\frac{x}{a}\right]}$$

$$\Rightarrow f = \frac{a \sec^3 \left(\frac{x}{a}\right)}{\sec^2 \left(\frac{x}{a}\right)}$$

$$\Rightarrow f = a \sec \left[\frac{x}{a}\right]$$

2. Find the radius of curvature at any point of the parabola $y^2 = 4ax$

$$\Rightarrow y^2 = 4ax \text{ ——— ①}$$

diff w.r to 'x'

$$\text{①} \Rightarrow 2y \cdot \frac{dy}{dx} = 4a \cdot 1 \quad (\div 2)$$

$$\Rightarrow y \frac{dy}{dx} = 2a$$

$$\Rightarrow \frac{dy}{dx} = y_1 = \frac{2a}{y} \text{ ——— ②}$$

diff w.r to 'x'

$$\text{②} \Rightarrow \frac{d}{dx} \left[\frac{dy}{dx} \right] = 2a \cdot \frac{d}{dx} \left[\frac{1}{y} \right]$$

$$\Rightarrow \frac{d^2y}{dx^2} = 2a \cdot \left[-\frac{1}{y^2} \right] \frac{dy}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\frac{2a}{y^2} \left[\frac{2a}{y} \right]$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\frac{4a^2}{y^3} = -\frac{4a^2}{y^3}$$

WKT,

$$\Rightarrow \rho = \frac{(1+y'^2)^{3/2}}{y^2}$$

$$\Rightarrow \rho = \frac{\left[1 + \left(\frac{2a}{y}\right)^2\right]^{3/2}}{-\frac{4a^2}{y^3}}$$

$$\Rightarrow \rho = \frac{\left[1 + \frac{4a^2}{y^2}\right]^{3/2} \times y^3}{-4a^2}$$

$$\Rightarrow \rho = \frac{(y^2 + 4a^2)^{3/2} \times y^3}{4a^2}$$

$$\Rightarrow \rho = \frac{(y^2 + 4a^2)^{3/2}}{4a^2}$$

3. Find the radius of curvature at any point of the curve, $y = c \cosh(x/c)$

$$\Rightarrow y = c \cosh\left[\frac{x}{c}\right] \text{ --- (1)}$$

diff w.r to x

$$\Rightarrow \textcircled{1} \frac{dy}{dx} = c \sinh\left[\frac{x}{c}\right] \left[\frac{1}{c}\right]$$

$$\Rightarrow y_1 = \sinh\left[\frac{x}{c}\right] \text{ --- (2)}$$

diff w.r to x

$$\textcircled{2} \Rightarrow \frac{d}{dx}(y_1) = \cosh\left[\frac{x}{c}\right] \cdot \frac{1}{c}$$

$$\Rightarrow y_2 = \frac{1}{c} \cosh\left[\frac{x}{c}\right]$$

WKT,

$$\Rightarrow \rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$\Rightarrow \rho = \frac{\left[1 + \sinh^2\left(\frac{x}{c}\right)\right]^{3/2}}{\frac{1}{c} \cosh\left(\frac{x}{c}\right)}$$

$$\Rightarrow \rho = \frac{c \left[\cosh^2\left(\frac{x}{c}\right)\right]^{3/2}}{\cosh\left(\frac{x}{c}\right)}$$

$$\Rightarrow \rho = \frac{c \cosh^3\left(\frac{x}{c}\right)}{\cosh\left(\frac{x}{c}\right)}$$

$$\Rightarrow \rho = c \cosh^2\left(\frac{x}{c}\right)$$

$$\Rightarrow \rho = c \left(\frac{y}{c}\right)^2$$

$$\Rightarrow \rho = \frac{c \cdot y^2}{c^2}$$

$$\Rightarrow \rho = \frac{y^2}{c}$$

4. Find the radius of curvature of $x^3 + y^3 = 3axy$ at the point $\left[\frac{3a}{2}, \frac{3a}{2}\right]$

$$\Rightarrow x^3 + y^3 = 3axy \quad \text{--- (1)}$$

$$\text{let, } p = \left[\frac{3a}{2}, \frac{3a}{2}\right]$$

diff w.r to 'x'

$$\Rightarrow 3x^2 + 3y^2 y_1 = 3a[1 \cdot y + x y_1] \quad (+3)$$

$$\Rightarrow x^2 + y^2 y_1 = ay + axy,$$

$$\Rightarrow y^2 y_1 - axy_1 = ay - x^2$$

$$\Rightarrow y_1 (y^2 - ax) = ay - x^2$$

$$\Rightarrow y_1 = \frac{ay - x^2}{y^2 - ax} \quad \text{--- (2)}$$

$$\Rightarrow (y_1)_p = \frac{a\left(\frac{3a}{2}\right) - \left(\frac{3a}{2}\right)^2}{\left(\frac{3a}{2}\right)^2 - a\left[\frac{3a}{2}\right]}$$

$$\Rightarrow (y_1)_p = \frac{\left[\frac{3a}{2}\right]^2 - a\left[\frac{3a}{2}\right]}{\left[\frac{3a}{2}\right]^2 - a\left[\frac{3a}{2}\right]}$$

$$\Rightarrow (y_1)_p = -1$$

diff w.r to 'x'

$$\textcircled{2} \Rightarrow y_2 = \frac{(y^2 - ax)(ay_1 - 2x) - (ay - x^2)(2yy_1 - a)}{y^2 - ax^2}$$

$$\Rightarrow y_2 p = \frac{\left[\left(\frac{3a}{2}\right)^2 - a\left(\frac{3a}{2}\right)\right] \left[a(-1) - 2\left[\frac{3a}{2}\right] - \left[a\left(\frac{3a}{2} - \frac{3a}{2}\right)^2\right]}{2\left[\frac{3a}{2}(-1) - a\right]}$$

$$\Rightarrow y_2 p = \frac{\left[\left(\frac{3a}{2}\right)^2 - 4\left(\frac{3a}{2}\right)\right]^2}{\left[\left(\frac{3a}{2}\right)^2 - 4\left(\frac{3a}{2}\right)\right]^2}$$

$$\Rightarrow (y_2)_p = \frac{\left[\frac{9a^2}{4} - \frac{3a^2}{2}\right] [-4a] - \left[-\frac{3a^2}{2} + \frac{9a^2}{4}\right] [4a]}{\left[\frac{9a^2}{4} - \frac{3a^2}{2}\right]^2}$$

$$\Rightarrow (y_2)_p = \frac{\left[\frac{9a^2}{4} - \frac{3a^2}{4}\right] [-4a - 4a]}{\left[\frac{9a^2}{4} - \frac{3a^2}{4}\right]^2}$$

$$\Rightarrow y_2 = \frac{-8a}{\frac{9a^2 - 6a^2}{4}}$$

$$\Rightarrow y^2 = \left| \frac{8a \times 4}{3a^2} \right|$$

$$\Rightarrow y^2 = \frac{32}{3a}$$

WKT,

$$\Rightarrow \rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$\Rightarrow \rho = \frac{[1+(-1)^2]^{3/2}}{32/3a}$$

$$\Rightarrow \rho = \frac{2^{3/2} \times 3a}{3a}$$

$$\Rightarrow \rho = \frac{2\sqrt{2} \times 3a}{8 \times 2\sqrt{2} \times \sqrt{2}}$$

$$\Rightarrow \rho = \frac{3a}{8\sqrt{2}}$$

5. Find the radius of the curvature of a curve

$x^2y = a(x^2+y^2)$ at the point $P(-2a, 2a)$

$$\Rightarrow x^2y = a(x^2+y^2) \text{ --- ①}$$

$$\Rightarrow \text{where, } P = (-2a, 2a)$$

diff w.r to 'x'

$$\text{①} \Rightarrow x^2(1) + y \cdot 2xx_1 = a(2x \cdot x_1 + 2y)$$

$$\Rightarrow x^2 + 2xy \cdot x_1 = 2ax \cdot x_1 + 2ay$$

$$\Rightarrow 2xyx_1 - 2axx_1 = 2ay - x^2$$

$$\Rightarrow x_1 (2xy - 2ax) = 2ay - x^2$$

$$\Rightarrow x_1 = \frac{2ay - x^2}{2xy - 2ax} \quad \text{--- (2)}$$

$$\therefore (x_1)_p = \frac{2a(2a) - (-2a^2)}{2(-2a)(2a) - 2a(-2a)}$$

$$(x_1)_p = \frac{4a^2 - 4a^2}{-8a^2 + 4a^2}$$

$$(x_1)_p = \frac{0}{-4a^2}$$

$$\Rightarrow (x_1)_p = 0$$

diff w.r to 'y'

$$\textcircled{2} \Rightarrow x_2 = \frac{(2xy - 2ax)(2a - 2xx_1) - (2ay - x^2)(2x + 2xy - 2ax_1)}{(2xy - 2ax)^2}$$

$$\Rightarrow (x_2)_p = \frac{[2(-2a)(2a) - 2a(-2a)] [2a - 2(-2a)(0)] - [2a(2a) - (-2a)^2] [2(-2a) - 2(0) - 0]}{[2(-2a)[2a] - 2a(-2a)]^2}$$

$$\Rightarrow (x_2)_p = \frac{(-8a^2 + 4a^2) - 0}{(-8a^2 + 4a^2)^2}$$

$$\Rightarrow (x_2)_p = \frac{(2a) - (4a^2)}{(-4a^2)^2}$$

$$\Rightarrow x_2 = \frac{2a}{-4a^2} = \frac{1}{-2a}$$

$$\Rightarrow f = \frac{(1 + x_1^2)^{3/2}}{x_2} = \frac{(1 + (0)^2)^{3/2}}{1/-2a} = \frac{(1+0)^{3/2}}{1/-2a} = \frac{(1)^{3/2} \times (-2a)}{1}$$

$$\Rightarrow \rho = \frac{x\sqrt{x} \times (-3a)}{r}$$

$$\Rightarrow \rho = \sqrt{x}(-3a)$$

6. Find the radius of a curvature the curve $a^2y = x^3 - a^3$ at the point where the curve cuts 'x' axis.

$\Rightarrow$ Given:-

$$a^2y = x^3 - a^3 \text{ --- (1)}$$

Let, ρ be a point cuts the x-axis ($y=0$)

$$\textcircled{1} \Rightarrow 0 = x^3 - a^3$$

$$\Rightarrow x^3 = a^3$$

$$\Rightarrow x = a$$

$$\therefore \rho(a, 0)$$

diff (1) w.r to 'x'

$$\textcircled{1} \Rightarrow a^2y_1 = 3x^2 - 0$$

$$\Rightarrow y_1 = \frac{3x^2}{a^2} \text{ --- (2)}$$

$$\Rightarrow (y_1)\rho = \frac{3a^2}{a^2}$$

$$\Rightarrow (y_1) = 3$$

$$\Rightarrow y_2 = \frac{3}{a^2}(2x)$$

$$\Rightarrow y_2 = \frac{6x}{a^2}$$

$$\Rightarrow (y_2)\rho = \frac{6a}{a^2}$$

$$\Rightarrow y_2 = \frac{6}{a}$$

$$\therefore \rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$\Rightarrow \rho = \frac{(1+9)^{3/2}}{6/a}$$

$$\Rightarrow \rho = \frac{a(10)^{3/2}}{6}$$

$$\Rightarrow \rho = \frac{a \times 10^5 \times \sqrt{10}}{6 \times 3}$$

$$\Rightarrow \rho = \frac{5\sqrt{10} \cdot a}{3}$$

7. For the curve $y = \frac{ax}{a+x}$, prove that

$\left(\frac{2\rho}{a}\right)^{2/3} = \left(\frac{y}{x}\right)^2 + \left(\frac{x}{y}\right)^2$, where, ρ is the radius of curvature at its point $p(x, y)$

$$\Rightarrow y = \frac{ax}{a+x} \text{ --- (1)}$$

diff w.r to 'x'

$$\Rightarrow y_1 = \frac{(a+x)(a) - ax(1)}{(a+x)^2}$$

$$\Rightarrow y_1 = \frac{a^2 + ax - ax}{(a+x)^2}$$

$$\Rightarrow y_1 = \frac{a^2}{(a+x)^2} \text{ --- (2)}$$

diff w.r to 'x'

$$\Rightarrow y_2 = \frac{(a+x)^2(0) - a^2 \cdot 2(a+x)(1)}{(a+x)^4}$$

$$\Rightarrow y_2 = \frac{-2a^2(a+x)}{(a+x)^4}$$

$$\Rightarrow y_2 = \frac{-2a^2}{(a+x)^3}$$

$\therefore$ WKT ;

$$\Rightarrow \rho = \left(\frac{(1+y_1^2)^{3/2}}{y_2} \right)$$

$$\Rightarrow \rho = \left(\frac{\left[1 + \frac{a^4}{(a+x)^4} \right]^{3/2}}{\frac{-2a^2}{(a+x)^3}} \right)$$

$$\Rightarrow \rho = \frac{(a+x)^3}{2a^2} \left[1 + \frac{a^4}{(a+x)^4} \right]^{3/2}$$

$$\Rightarrow \frac{\partial \rho}{\partial a} = \frac{\partial}{\partial a} \frac{(a+x)^3}{2a^2} \left[1 + \frac{a^4}{(a+x)^4} \right]^{3/2}$$

$$\Rightarrow \frac{\partial \rho}{\partial a} = \frac{(a+x)^3}{a^3} \left[1 + \frac{a^4}{(a+x)^4} \right]^{3/2}$$

$$\Rightarrow \left(\frac{\partial \rho}{\partial a} \right)^{2/3} = \frac{[(a+x)^3]^{2/3}}{(a^3)^{2/3}} \left[1 + \frac{a^4}{(a+x)^4} \right]^{3/2 \times 2/3}$$

$$\Rightarrow \left(\frac{\partial \rho}{\partial a} \right)^{2/3} = \frac{(a+x)^2}{a^2} \left[1 + \frac{a^4}{(a+x)^4} \right]$$

$$\Rightarrow \left(\frac{\partial \rho}{\partial a} \right)^{2/3} = \frac{(a+x)^2}{a^2} + \frac{a^2}{(a+x)^2}$$

$$\Rightarrow \left(\frac{\partial \rho}{\partial a} \right)^{2/3} = \left(\frac{a+x}{a} \right)^2 + \left(\frac{a}{a+x} \right)^2$$

$$\Rightarrow \left[\frac{2f}{a} \right]^{2/3} = \left[\frac{x}{y} \right]^2 + \left[\frac{y}{x} \right]^2$$

8. Find the radius of curvature at any point of the cycloid.

$$x = a(\theta + \sin\theta) \quad y = a(1 - \cos\theta)$$

$$\Rightarrow x = a(\theta + \sin\theta) \text{ --- (1)}$$

diff w.r to ' θ '

$$\Rightarrow \frac{dx}{d\theta} = a[1 + \cos\theta]$$

$$\Rightarrow y = a(1 - \cos\theta) \text{ --- (2)}$$

diff w.r to θ

$$\Rightarrow \frac{dy}{d\theta} = a[0 + \sin\theta]$$

$$\Rightarrow \frac{dy}{d\theta} = a \sin\theta$$

$$\therefore y_1 = \frac{dy/d\theta}{dx/d\theta}$$

$$\Rightarrow y_1 = \frac{a \sin\theta}{a(1 + \cos\theta)} = \frac{\sin\theta}{1 + \cos\theta} \text{ --- (3)}$$

diff eqn (3) w.r to x

$$\textcircled{3} \Rightarrow \frac{d}{dx}(y_1) = \frac{d}{dx} \left[\frac{\sin\theta}{1 + \cos\theta} \right]$$

$$\Rightarrow \frac{d}{dx}(y_1) = \frac{d}{d\theta} \left[\frac{\sin\theta}{1 + \cos\theta} \right] \frac{d\theta}{dx}$$

$$= \frac{(1 + \cos\theta) \cos\theta - \sin\theta (-\sin\theta)}{(1 + \cos\theta)^2} \frac{d\theta}{dx}$$

$$= \frac{\cos\theta + (\cos^2\theta + \sin^2\theta)}{(1 + \cos\theta)^2} \frac{1}{dx/d\theta}$$

$$= \frac{1 + \cos\theta}{(1 + \cos\theta)^2} \times \frac{1}{a(1 + \cos\theta)}$$

$$\Rightarrow y_2 = \frac{1}{a(1+\cos\theta)^2}$$

WKT,

$$\Rightarrow \rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$\Rightarrow \rho = \frac{\left[1 + \frac{\sin^2\theta}{(1+\cos\theta)^2}\right]^{3/2}}{\frac{1}{a + [\cos\theta + 1]^2}}$$

$$\Rightarrow \rho = a(1+\cos\theta)^2 \left[\frac{[(1+\cos\theta)^2 + \sin^2\theta]^{3/2}}{[(1+\cos\theta)^2]^{3/2}} \right]$$

$$\Rightarrow \rho = \frac{a(1+\cos\theta)^2}{(1+\cos\theta)^3} [1 + 2\cos\theta + \cos^2\theta + \sin^2\theta]^{3/2}$$

$$\Rightarrow \rho = \frac{a}{1+\cos\theta} [1 + 2\cos\theta + 1]^{3/2}$$

$$\Rightarrow \rho = \frac{a}{1+\cos\theta} [2 + 2\cos\theta]^{3/2}$$

$$\Rightarrow \rho = \frac{2^{3/2} a}{(1+\cos\theta)} (1+\cos\theta)^{3/2}$$

9. Find the radius of curvature of curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ at the point $\left[\frac{a}{4}, \frac{a}{4}\right]$

$$\Rightarrow \sqrt{x} + \sqrt{y} = \sqrt{a} \quad \text{--- (1)} \quad \text{P}\left[\frac{a}{4}, \frac{a}{4}\right]$$

diff w.r to x

$$\textcircled{1} \Rightarrow \frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{1}{\sqrt{y}} \frac{dy}{dx} = -\frac{1}{\sqrt{x}}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}} = y_1 \quad \text{--- (2)}$$

$$\therefore (y_1)_p = \frac{-\sqrt{a/x}}{\sqrt{a/x}} = -1$$

diff w.r to x

$$\Rightarrow \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{\sqrt{x} \cdot \frac{1}{2\sqrt{y}} \cdot y_1 - \sqrt{y} \cdot \frac{1}{2\sqrt{x}}}{(\sqrt{x})^2}$$

$$\Rightarrow y_2 = \left[-\frac{1}{2} \cdot \frac{y_1 \sqrt{x}/\sqrt{y} - \sqrt{y}/\sqrt{x}}{x} \right]$$

$$\Rightarrow (y_2)_p = \left[\frac{(-1) \sqrt{a/4} - \sqrt{a/4}}{\sqrt{a/4} - \sqrt{a/4}} \right] \cdot \frac{-1/2}{a/4}$$

$$\Rightarrow (y_2)_p = -\frac{1}{2} \left[\frac{-1-1}{a/4} \right]$$

$$\Rightarrow y_2 = -\frac{1}{2} \left[\frac{-2}{a/4} \right]$$

$$\Rightarrow y_2 = \frac{4}{a} \neq 0$$

$\therefore$ WKT,

$$\Rightarrow \rho = \frac{(1 + y_1^2)^{3/2}}{y_2}$$

$$\Rightarrow \rho = \frac{(1 + (-1)^2)^{3/2}}{4/a}$$

$$\Rightarrow \rho = \frac{2\sqrt{2}}{24/a}$$

$$\Rightarrow \rho = \frac{\sqrt{2}}{2} a$$

$$\Rightarrow \rho = \frac{\sqrt{2} a}{\sqrt{2}\sqrt{2}}$$

$$\Rightarrow \rho = \frac{a}{\sqrt{2}}$$

10. Find the radius of curvature for the curve

$$r^n = a^n \cos n\theta$$

$\Rightarrow$ Given:-

$$\Rightarrow r^n = a^n \cos n\theta \quad \text{--- (1)}$$

diff w.r to ' θ '

$$\textcircled{1} \Rightarrow n r^{n-1} \frac{dr}{d\theta} = a^n [-\sin n\theta] [n]$$

$$\Rightarrow \frac{r^n}{r} \frac{dr}{d\theta} = -a^n \sin n\theta$$

$$\Rightarrow \frac{1}{r} \cdot \frac{dr}{d\theta} = -\frac{a^n \sin n\theta}{r^n}$$

$$\Rightarrow \frac{1}{r} \cdot \frac{dr}{d\theta} = -\frac{a^n \sin n\theta}{a^n \cos n\theta}$$

$$\Rightarrow \cot \phi = -\tan n\theta \quad \text{--- (2)}$$

$\therefore$ WKT

$$\Rightarrow \frac{1}{\rho^2} = \frac{1}{r^2} [1 + \cot^2 \phi]$$

$$\Rightarrow \frac{1}{\rho^2} = \frac{1}{r^2} [1 + \tan^2 n\theta]$$

$$\Rightarrow \frac{1}{\rho^2} = \frac{1}{r^2} [\sec^2 n\theta]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2 \cos^2 n\theta}$$

$$\Rightarrow p^2 = r^2 \cos^2 n\theta$$

$$\Rightarrow p = r \cos n\theta$$

$$\Rightarrow p = r \frac{r^n}{a^n}$$

$$\Rightarrow a^n p = r^{n+1} \text{ --- (2)}$$

diff w.r to 'p'

$$\textcircled{2} \Rightarrow a^n \cdot (1) = (n+1) r^n \cdot \frac{dr}{dp}$$

$$\Rightarrow r^n \frac{dr}{dp} = \frac{a^n}{n+1}$$

$$\Rightarrow r^{n-1} \cdot r \frac{dr}{dp} = \frac{a^n}{n+1}$$

$$\Rightarrow r \frac{dr}{dp} = \frac{a^n}{n+1} \cdot \frac{1}{r^{n-1}}$$

$$\Rightarrow \int = \left[\frac{a^n}{n+1} \right] \frac{1}{r^{n-1}}$$

$$\Rightarrow \int \propto \frac{1}{r^{n-1}}$$

$$11. \quad r^n = a^n \sin n\theta$$

$$\Rightarrow r^n = a^n \sin n\theta \text{ --- (i)}$$

diff w.r to 'θ'

$$\Rightarrow n r^{n-1} \frac{dr}{d\theta} = a^n (\cos n\theta) (n)$$

$$\Rightarrow \frac{r^n}{r} \frac{dr}{d\theta} = a^n \cos n\theta$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{a^n \cos n\theta}{r^n}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{a^n \cos n\theta}{a^n \sin n\theta}$$

$$\Rightarrow \cot \phi = \cot n\theta \quad \text{--- ②}$$

WKT ;

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} [1 + \cot^2 \phi]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} [1 + \cot^2 n\theta]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} (\operatorname{cosec}^2 n\theta)$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2 \sin^2 n\theta}$$

$$\Rightarrow p^2 = r^2 \sin^2 n\theta$$

$$\Rightarrow p = r \sin n\theta$$

$$\Rightarrow p = r \frac{r^n}{a^n}$$

$$\Rightarrow a^n p = r^{n+1} \quad \text{--- ③}$$

diff w.r to 'p'

$$\textcircled{3} \Rightarrow a^n \cdot 1 = (n+1) r^n \cdot \frac{dr}{dp}$$

$$\Rightarrow r^n \frac{dr}{dp} = \frac{a^n}{n+1}$$

$$\Rightarrow r \frac{dr}{dp} = \frac{a^n}{n+1} \cdot \frac{1}{r^{n-1}}$$

$$\Rightarrow p = \left(\frac{a^n}{n+1} \right) \frac{1}{r^{n-1}}$$

$$\Rightarrow p \propto \frac{1}{r^{n-1}}$$

12. Find $\frac{2a}{r} = 1 + \cos \theta$ and hence show that $p^2 \propto r^3$ (or) $\frac{p^2}{r^3}$ be a constant.

$$\Rightarrow \frac{2a}{r} = 1 + \cos \theta$$

$$\Rightarrow r(1 + \cos \theta) = 2a \text{ --- ①}$$

diff w. r to ' θ '

$$\Rightarrow (1 + \cos \theta) \frac{dr}{d\theta} + r(0 - \sin \theta) = 0$$

$$\Rightarrow (1 + \cos \theta) \frac{dr}{d\theta} = r \sin \theta$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{\sin \theta}{1 + \cos \theta}$$

$$\Rightarrow \cot \phi = \frac{2 \sin \theta/2 \cos \theta/2}{2 \cos^2 \theta/2}$$

$$\Rightarrow \cot \phi = \tan(\theta/2)$$

WKT;

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^3} (1 + \cot^2 \phi)$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^3} (1 + \tan^2(\theta/2))$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^3} (\sec^2 \theta/2)$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^3 \cos^2 \theta/2}$$

$$\Rightarrow p^2 = r^3 \cos^2 \theta/2$$

$$\Rightarrow p^2 = r^3 \left[\frac{1 + \cos \theta}{2} \right]$$

$$\Rightarrow p^2 = \frac{r^3}{2} [1 + \cos \theta]$$

$$\Rightarrow p^2 = \frac{r^2}{2} \left[\frac{2a}{r} \right]$$

$$\Rightarrow p^2 = ar \text{ ——— ②}$$

diff w.r to 'p'

$$\textcircled{2} \Rightarrow 2p = a \frac{dr}{dp}$$

$$\Rightarrow \frac{dr}{dp} = \frac{2}{a} p$$

$$\Rightarrow r \frac{dr}{dp} = \frac{2}{a} pr$$

$$\Rightarrow \int = \frac{2}{a} pr$$

$$\Rightarrow p^2 = \frac{4}{a^2} p^2 r^2$$

$$\Rightarrow p^2 = \frac{4}{a^2} (ar)(r^2)$$

$$\Rightarrow p^2 = \frac{4}{a} r^3$$

$$\Rightarrow p^2 = Kr^3$$

$$\Rightarrow p^2 \propto r^3$$

(or)

$$\Rightarrow \frac{p^2}{r^3} = K = \text{constant.}$$

$$13. \frac{2a}{r} = (1 - \cos \theta)$$

$$\Rightarrow 2a = r(1 - \cos \theta) \text{ ——— ①}$$

diff w.r to 'θ'

$$\Rightarrow 0 = (1 - \cos \theta) \frac{dr}{d\theta} - (-r \sin \theta)$$

$$\Rightarrow -r \sin \theta = (1 - \cos \theta) \frac{dr}{d\theta}$$

$$\Rightarrow \frac{-\sin \theta}{1 - \cos \theta} = r \frac{dr}{d\theta}$$

$$\Rightarrow \frac{-2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}}{2 \sin^2 \frac{\theta}{2}} = \cot \phi$$

$$\Rightarrow -\cot \frac{\theta}{2} = \cot \phi$$

$$\Rightarrow \cot \phi = \cot \left[-\frac{\theta}{2} \right]$$

WKT;

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} [1 + \cot^2 \phi]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} [1 + \cot^2 \left(-\frac{\theta}{2} \right)]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} [\operatorname{cosec}^2 \frac{\theta}{2}]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2 \sin^2 \frac{\theta}{2}}$$

$$\Rightarrow p^2 = r^2 \sin^2 \frac{\theta}{2}$$

$$\Rightarrow p^2 = r^2 \sin^2 \frac{\theta}{2}$$

$$\Rightarrow p^2 = r^2 \left[\frac{1 - \cos \theta}{2} \right]$$

$$\Rightarrow p^2 = \frac{r^2}{2} [1 - \cos \theta]$$

$$\Rightarrow p^2 = \frac{r^2}{2} \left(\frac{2a}{r} \right)$$

$$\Rightarrow p^2 = ar \quad \text{--- (2)}$$

$$\Rightarrow 2p = r \frac{dr}{dp}$$

$$\Rightarrow \frac{dr}{dp} = \frac{2}{a} p$$

$$\Rightarrow r \frac{dr}{dp} = \frac{2}{a} pr$$

$$\Rightarrow p = \frac{2}{a} pr$$

$$\Rightarrow p^2 = \frac{4}{a^2} p^2 r^2$$

$$\Rightarrow p^2 = \frac{4}{a^2} (dr)(r^2)$$

$$\Rightarrow p^2 = \frac{4}{a^2} r^3$$

$$\Rightarrow p^2 = Kr^3$$

$$\therefore K = 4/a$$

$$\Rightarrow p^2 \propto r^3$$

(or)

$$\frac{p^2}{r^3} = K = \text{constant}.$$

$$14. r = a(1 - \cos \theta)$$

$$\Rightarrow r = a(1 - \cos \theta) \text{ --- ①}$$

diff w.r to 'θ'

$$\Rightarrow \frac{dr}{d\theta} = a \sin \theta$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{a \sin \theta}{r}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{a \sin \theta}{a(1 - \cos \theta)}$$

$$\Rightarrow \cot \phi = \frac{r \sin \theta/2 \cos \theta/2}{r \sin^2 \theta/2}$$

$$\Rightarrow \cot \phi = \cot \frac{\theta}{2}$$

WKT,

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} [1 + \cot^2 \phi]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} \left[1 + \cot^2 \frac{\theta}{2}\right]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} \operatorname{cosec}^2 \frac{\theta}{2}$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2 \sin^2 \theta/2}$$

$$\Rightarrow p^2 = r^2 \sin^2 \theta/2$$

$$\Rightarrow p^2 = r^2 \left[\frac{1 - \cos \theta/2}{2} \right]$$

$$\Rightarrow p^2 = \frac{r^2}{2} [1 - \cos \theta/2]$$

$$\Rightarrow p^2 = \frac{r^2}{2} \left[\frac{r}{a} \right]$$

$$\Rightarrow p^2 = \frac{r^3}{2a}$$

$$\Rightarrow 2ap^2 = r^3 \quad \text{--- (2)}$$

diff (2) w.r to 'p'

$$(2) \Rightarrow 4ap = 3r^2 \cdot \frac{dr}{dp}$$

$$\Rightarrow r^2 \cdot \frac{dr}{dp} = \frac{4a}{3} p$$

$$\Rightarrow r \cdot \frac{dr}{dp} = \frac{4a}{3} \cdot \frac{p}{r}$$

$$\Rightarrow p = \frac{4a}{3} \cdot \frac{p}{r}$$

$$\Rightarrow p^2 = \frac{16a^2}{9} \cdot \frac{p^2}{r^2}$$

$$\Rightarrow p^2 = \frac{16a^2}{9} \cdot \frac{1}{r^2} \left[\frac{r^3}{2a} \right]$$

$$\Rightarrow p^2 = \frac{8a}{9} r$$

$$\Rightarrow p^2 \propto r$$

15. Find the radius of curvature of the curve

$$\theta = \frac{\sqrt{r^2 - a^2}}{a} - \cos^{-1} \left[\frac{a}{r} \right]$$

$$\Rightarrow \theta = \frac{\sqrt{r^2 - a^2}}{a} - \cos^{-1} \left[\frac{a}{r} \right] \quad \text{--- (1)}$$

diff w. r to 'r'

$$\Rightarrow \frac{d\theta}{dr} = \frac{1}{a} \frac{r}{2\sqrt{r^2 - a^2}} (2r) - \frac{(-1)}{\sqrt{1 - \left[\frac{a}{r} \right]^2}} \cdot \frac{d}{dr} \left[\frac{a}{r} \right]$$

$$\Rightarrow \frac{d\theta}{dr} = \frac{r}{a\sqrt{r^2 - a^2}} + \frac{1}{\sqrt{1 - \frac{a^2}{r^2}}} \left[\frac{-a}{r^2} \right]$$

$$\Rightarrow \frac{d\theta}{dr} = \frac{r}{a\sqrt{r^2 - a^2}} - \frac{a}{\sqrt{r^2 - a^2}} \left[\frac{1}{r^2} \right]$$

$$\Rightarrow \frac{d\theta}{dr} = \frac{r}{a\sqrt{r^2 - a^2}} - \frac{a}{r\sqrt{r^2 - a^2}}$$

$$\Rightarrow \frac{d\theta}{dr} = \frac{1}{\sqrt{r^2 - a^2}} \left[\frac{r}{a} - \frac{a}{r} \right]$$

$$\Rightarrow \frac{d\theta}{dr} = \frac{1}{\sqrt{r^2 - a^2}} \left[\frac{r^2 - a^2}{ar} \right]$$

$$\Rightarrow \frac{d\theta}{dr} = \frac{1}{\sqrt{r^2 - a^2}} \cdot \frac{\sqrt{r^2 - a^2} \cdot \sqrt{r^2 - a^2}}{ar}$$

$$\Rightarrow \frac{d\theta}{dr} = \frac{\sqrt{r^2 - a^2}}{ar}$$

$$\Rightarrow \frac{dr}{d\theta} = \frac{ar}{\sqrt{r^2 - a^2}}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{a}{\sqrt{r^2 - a^2}}$$

$$\Rightarrow \cot \phi = \frac{a}{\sqrt{r^2 - a^2}}$$

$$\Rightarrow \cot^2 \phi = \frac{a^2}{r^2 - a^2}$$

$$\Rightarrow 1 + \cot^2 \phi = 1 + \frac{a^2}{r^2 - a^2}$$

$$\Rightarrow 1 + \cot^2 \phi = \frac{r^2 - a^2 + a^2}{r^2 - a^2} = \frac{r^2}{r^2 - a^2}$$

WKT,

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} [1 + \cot^2 \phi]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} \times \frac{r^2}{r^2 - a^2}$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2 - a^2}$$

$$\Rightarrow p^2 = r^2 - a^2$$

$$\Rightarrow p = \sqrt{r^2 - a^2} \quad \text{--- (2)}$$

diff wr to 'p'

$$(2) \Rightarrow 1 = \frac{1}{2\sqrt{r^2 - a^2}} \cdot 2r \cdot \frac{dr}{dp} \Rightarrow r \frac{dr}{dp} = \sqrt{r^2 - a^2}$$

$$\Rightarrow \int = \sqrt{r^2 - a^2}$$