

# **QUANTUM PHYSICS AND APPLICATIONS**

## **A SHORT NOTES - FIRST EDITION**

DEPARTMENT OF PHYSICS  
ATME COLLEGE OF ENGINEERING,  
Affiliated to Visveswaraya Technological Univesrsity - Belagavi  
13<sup>th</sup> km Stone, Mysuru-Bannur Road, Mysuru, PIN - 570 028.



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## **Part I**

# **MODULE 1 : Quantum Mechanics**



# Chapter 1

## Quantum Mechanics

### 1.1 Wave-Particle dualism

The phenomena like Interference, Diffraction and Polarization are attributed to the wave properties of radiation. The Quantum theory of radiation and experiments like Photoelectric effect and Compton Effect describe the particle nature of radiation. Thus radiation behaves like waves and like particles under different suitable circumstances. Hence radiation exhibits dual nature.

### 1.2 de Broglie hypothesis

In the year 1924 French physicist Louis de Broglie extended wave-particle dualism through a hypothesis stating *If radiant energy could behave like waves in some experiments and particles or photons in others and since nature loves symmetry, then one can expect the particles like protons and electrons to exhibit wave nature under suitable circumstances*. This is well known as de Broglie's hypothesis.

According to Einstein's energy mass relation

$$E = mc^2 \quad (1.1)$$

According to Planck's quantum theory of radiation

$$E = h\nu = \frac{hc}{\lambda} \quad (1.2)$$

equating equation (1.1) and (1.2) we get

$$mc^2 = \frac{hc}{\lambda} \quad (1.3)$$

$$mc = \frac{h}{\lambda} \quad (1.4)$$

$$\lambda = \frac{h}{mc} \quad (1.5)$$

For a particle of mass  $m$  moving with a velocity  $v$  by analogy the de Broglie wavelength of matter waves is given by

$$\lambda = \frac{h}{mv} \quad (1.6)$$

Here  $m$  is the mass of the moving particle and  $v$  is its velocity.

Expressing the de Broglie wave length in different forms

$$\lambda = \frac{h}{mv} = \frac{h}{p} = \frac{h}{\sqrt{2mE}} = \frac{h}{\sqrt{2meV}} = \frac{12.27}{\sqrt{V}} \text{ \AA} \quad (1.7)$$

Therefore some sort of waves can be even associated with moving material particles called *Matter waves* or *de-Broglie waves* and the wavelength associated with matter waves is called *de Broglie wavelength*.

#### Note

- For a particle, charged or uncharged, moving with kinetic energy  $E$  the de Broglie wavelength is given by  $\lambda = \frac{h}{p} = \frac{h}{\sqrt{2mE}}$ .
- For a charged particle accelerated with a potential  $V$  volt, the de Broglie wavelength is given by  $\lambda = \frac{h}{\sqrt{2mqV}}$ .
- For an electron accelerated through a potential difference of  $V$  volt, the de Broglie wavelength is given by  $\lambda = \frac{h}{\sqrt{2meV}}$ . Further substituting the values of  $h, m$  and  $e$ , the de Broglie wavelength is given by the expression  $\lambda = \frac{12.27}{\sqrt{V}} \text{ \AA}$ .

### 1.2.1 Properties of matter waves

The following are the properties associated with the matter waves

1. Matter waves are associated only with particles in motion
2. They are not electromagnetic in nature
3. Group velocity is associated with matter waves
4. As a result of superposition of large number of component waves which slightly differ in frequency, matter waves are localized.
5. The phase velocity has no physical meaning for matter waves

6. The amplitude of the matter wave at a given point is associated with the probability density of finding the particle at that point.
7. The wave length of matter waves is given by  $\lambda = \frac{h}{mv}$

### 1.3 Heisenberg's Uncertainty Principle

**Statement:** The simultaneous determination of the exact position and momentum of a moving particle is impossible.

**Explanation :** According to this principle if  $\Delta x$  is the error involved in the measurement of position and  $\Delta p_x$  is the error involved in the measurement of momentum during their simultaneous measurement, then the product of the corresponding uncertainties is given by

$$\Delta x \Delta p_x \geq \frac{h}{4\pi} \quad (1.8)$$

$$\Delta E \Delta t \geq \frac{h}{4\pi} \quad (1.9)$$

$$\Delta \theta \Delta L \geq \frac{h}{4\pi} \quad (1.10)$$

The product of the errors is of the order of Planck's constant. If one quantity is measured with high accuracy then the simultaneous measurement of the other quantity becomes less accurate.

**Physical significance :** According to Newtonian physics the simultaneous measurement of position and momentum are *exact*. But the existence of matter waves induces serious problems due to the limit to accuracy associated with the simultaneous measurement. Hence the *exactness* in Newtonian physics is replaced by *probability* in quantum mechanics.

#### 1.3.1 Application of uncertainty principle

**Broadening of Spectral lines :** Broadening of spectral lines refers to the increase in width of absorption or emission spectral lines beyond their natural, sharp frequency or wavelength. Ideally, a spectral line should appear as a very narrow line corresponding to a well-defined energy transition between two atomic or molecular states. However, in reality, various physical effects cause these lines to spread out over a finite range of frequencies or wavelengths instead of being infinitely sharp. This phenomenon is called broadening of spectral lines. According to Uncertainty principle, When an excited atom has a finite lifetime ( $\Delta t$ , the energy of the emitted photon cannot be sharply defined. This causes a spread in energy values ( $\delta E$ ), which translates to a broadening of the spectral line. When an atom absorbs a photon it rises to the excited state and it can

stay for a definite time in the excited state, called the lifetime. Lifetimes of excited levels are typically of the order of 10-8s. As the atom completes its lifetime in the excited state, on its own, it comes to the ground state by emitting a photon of energy exactly equal to the energy difference between the two levels. The energy of the emitted photon is given by

$$E = h\nu = \frac{hc}{\lambda} \quad (1.11)$$

$$\Delta E = hc \Delta \left( \frac{1}{\lambda} \right) \quad (1.12)$$

$$\Delta E = hc \left( \frac{\Delta \lambda}{\lambda^2} \right) \quad (1.13)$$

("On ignoring the negative sign")

From Uncertainty principle, we have

$$\Delta E \Delta t \geq \frac{h}{4\pi} \quad (1.14)$$

$$\Delta t = \frac{h}{4\pi \Delta E} \quad (1.15)$$

$$\Delta t = \frac{h}{4\pi hc \left( \frac{\Delta \lambda}{\lambda^2} \right)} \quad (1.16)$$

$$\Delta t = \frac{\lambda^2}{4\pi c (\Delta \lambda)} \quad (1.17)$$

Spectral lines are never perfectly sharp; they always appear broadened due to natural quantum effects and external factors like temperature, motion, and collisions.

### 1.4 Principle of Complementarity

**Statement** Bohr stated as "In a situation where the wave aspect of a system is revealed, its particle aspect is concealed; and, in a situation where the particle aspect is revealed, its wave aspect is concealed. Revealing both simultaneously is impossible; the wave and particle aspects are complementary."

**Explanation** We know that the consequence of the uncertainty principle is both the Wave and particle nature of the matter can not be measured simultaneously. In other words, we can not precisely describe the dual nature of Light.

- If an experiment is designed to measure the particle nature of the matter, during this experiment, errors of measurement of both position and the time coordinates must be zero and hence the momentum, energy and the wave nature of the matter are completely unknown.

- Similarly, if an experiment is designed for measuring the wave nature of the particle, then the errors in the measurement of the energy and the momentum will be zero, whereas the position and the time coordinates of the matter will be completely unknown.

From the above explanation, we can conclude that, when the particle nature of the matter is measured or displayed, the wave nature of the matter is necessarily suppressed and vice versa.

## 1.5 Wave Function

According to the de Broglie's hypothesis the relation between momentum and wavelength is found to be experimentally valid for both photons and particles. The quanta of matter or radiation can be represented in agreement with uncertainty principle by wave packets. Thus it suggests that concentrated bunches of waves might be used to describe localized particles and quanta of radiation. **The variations of which make up the matter wave is called wave function.** Hence we shall consider a wave function that depends on space (x, y and z) and time(t) and is denoted by  $\psi(r, t)$ . The wave function for a wave packet moving along +ve x axis is given by

$$\psi = \psi_0 e^{i(kx - \omega t)} \quad (1.18)$$

The quantity  $\psi$  is assumed to have the following three basic properties

1. It can interfere with itself so that it can account for diffraction experiments.
2. It is large in magnitude where is particle or photon is likely to be found and small else where.
3. It will be regarded as describing the behavior of single particle or photon and not statistical distribution of number of quanta.

## 1.6 Time Independent Schrödinger Wave Equation

The wave equation which has variations only with respect to position and describes the steady state is called Time Independent Schrodinger wave equation. Consider a particle of mass  $m$  moving with velocity  $v$  along +ve x-axis. The de Broglie wave length  $\lambda$  is given by

$$\lambda = \frac{h}{mv} \quad (1.19)$$

The wave equation for one dimensional propagation of waves is given by

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2} \quad (1.20)$$

The wave function is given by

$$\psi = \psi_0 e^{i(kx - \omega t)} \quad (1.21)$$

here  $\psi_0$  is the amplitude at the point of consideration,  $\omega$  is angular frequency and  $k$  is the wave number. Differentiating  $\psi$  twice with respect to  $t$ , we get

$$\frac{\partial^2 \psi}{\partial t^2} = -\omega^2 \psi_0 e^{i(kx - \omega t)} \quad (1.22)$$

$$\frac{\partial^2 \psi}{\partial t^2} = -\omega^2 \psi \quad (1.23)$$

substituting equation 1.23 in equation 1.20

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{v^2} (-\omega^2 \psi) \quad (1.24)$$

substituting for  $\omega$  and  $v$  we get

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{(f\lambda)^2} (-(2\pi f)^2 \psi) \quad (1.25)$$

$$\frac{\partial^2 \psi}{\partial x^2} = -\frac{4\pi^2}{\lambda^2} \psi \quad (1.26)$$

substituting for  $\lambda$  from equation 1.19

$$\frac{\partial^2 \psi}{\partial x^2} = -\frac{4\pi^2}{(\frac{h}{mv})^2} \psi \quad (1.27)$$

$$\frac{\partial^2 \psi}{\partial x^2} = -\frac{4\pi^2 (mv)^2}{h^2} \psi \quad (1.28)$$

$$\frac{\partial^2 \psi}{\partial x^2} = -\frac{8\pi^2 m (\frac{1}{2}mv^2)}{h^2} \psi \quad (1.29)$$

$$\frac{\partial^2 \psi}{\partial x^2} = -\frac{8\pi^2 m (E - U)}{h^2} \psi \quad (1.30)$$

here

$$\frac{1}{2}mv^2 = E - U \quad (1.31)$$

here  $E$  is the total and  $U$  is potential energy of the particle.

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{8\pi^2 m (E - U)}{h^2} \psi = 0 \quad (1.32)$$

This can be extended to three dimension

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} + \frac{8\pi^2 m (E - U)}{h^2} \psi = 0 \quad (1.33)$$

$$\nabla^2 \psi + \frac{8\pi^2 m (E - U)}{h^2} \psi = 0 \quad (1.34)$$

here the operator  $\nabla^2$  is given by

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \quad (1.35)$$

Hence the Time Independent Schrodinger equation.

## 1.7 Physical significance of Wave Function : Physical Interpretation

The wave function  $\psi$  just as itself has no direct physical meaning. It is more difficult to give a physical interpretation to the amplitude of the wave. The amplitude of the wave function  $\psi$  is certainly not like displacement in water wave or the pressure wave nor the waves in stretched string. It is a very different kind of wave. The quantity squared absolute value of the amplitude gives the probability of finding the particle per unit volume at the given location in space and is referred to as probability density. This is also referred to as *Born interpretation*. It is given by

$$P(x) = |\psi|^2 \quad (1.36)$$

Thus, in one dimension the probability of finding a particle in the width  $dx$  of length  $x$

$$P(x)dx = |\psi|^2 dx \quad (1.37)$$

Similarly, in three dimension, the probability of finding a particle in a given small volume  $dV$  of volume  $V$  is given by

$$P dV = |\psi|^2 dV \quad (1.38)$$

here  $dV = dx dy dz$  and  $P$  is the probability of finding the particle at given location per unit volume and is called *Probability Density*. Since  $\psi$  is a complex quantity  $|\psi|^2 = \psi\psi^*$  and the product is a real number.  $\psi^*$  is the complex conjugate of  $\psi$ .

## 1.8 Expectation Value

In quantum mechanics ***The expectation value is the probabilistic expected value of the result (measurement) of an experiment. It can be thought of as an average of all the possible outcomes of a measurement as weighted by their likelihood.*** Expectation value as such it is not the most probable value of a measurement. In the real sense the expectation value may have zero probability of occurring. Let us consider a particle moving along the  $x$  axis.

The result of a measurement of the position  $x$  is a continuous random variable. Consider a wave function  $\psi(x, t)$ . The  $|\psi(x, t)|^2$  value is a probability density for the position observable and  $|\psi(x, t)|^2 dx$  is the probability of finding the particle between  $x$  and  $x + dx$  at time  $t$ . Thus, if a measurement of position is repeated many times in an identical way on an identical particle in identical circumstances, many possible outcomes are possible and the expectation value of these outcomes is, according to the following equation

$$\langle x \rangle = \int_{-\infty}^{+\infty} \psi^*(x, t) \hat{x} \psi(x, t) dx \quad (1.39)$$

Let " $r$ " be the position vector in a co-ordinate system, expectation value of the components along the  $x, y$  and  $z$  co-ordinate is as shown below

$$\langle x \rangle = \int_{-\infty}^{+\infty} \psi^*(x, t) \hat{x} \psi(x, t) dx \quad (1.40)$$

$$\langle y \rangle = \int_{-\infty}^{+\infty} \psi^*(y, t) \hat{y} \psi(y, t) dy \quad (1.41)$$

$$\langle z \rangle = \int_{-\infty}^{+\infty} \psi^*(z, t) \hat{z} \psi(z, t) dz \quad (1.42)$$

**Expectation Value of momentum** The momentum  $p$  has dependence on both position and time. Let the 3 components of  $p$  are  $p_x, p_y$  and  $p_z$ . The expectation values for momentum could be written as

$$\langle p_x \rangle = \int_{-\infty}^{+\infty} \psi^*(x, t) \hat{p}_x \psi(x, t) dx \quad (1.43)$$

$$\langle p_y \rangle = \int_{-\infty}^{+\infty} \psi^*(y, t) \hat{p}_y \psi(y, t) dy \quad (1.44)$$

$$\langle p_z \rangle = \int_{-\infty}^{+\infty} \psi^*(z, t) \hat{p}_z \psi(z, t) dz \quad (1.45)$$

**Expectation Value of Energy** The momentum  $p$  has dependence on both position and momentum. It means in time dependent cases the particle may have different energies at different time. That is  $V = V(r, t)$  The expectation values for energy could be written as

$$\langle V_{(x,t)} \rangle = \int_{-\infty}^{+\infty} \psi^*(x, t) \hat{V}_{(x,t)} \psi(x, t) dx \quad (1.46)$$

$$\langle V_{(x,t)} \rangle = \int_{-\infty}^{+\infty} \psi^*(y, t) \hat{V}_{(x,t)} \psi(y, t) dy \quad (1.47)$$

$$\langle V_{(x,t)} \rangle = \int_{-\infty}^{+\infty} \psi^*(z, t) \hat{V}_{(x,t)} \psi(z, t) dz \quad (1.48)$$

The result of a measurement of the position  $x$  is a continuous random variable. Consider a wave function  $\psi(x, t)$ . The  $|\psi(x, t)|^2$  value is a probability density for the position observable and  $|\psi(x, t)|^2 dx$  is the probability of finding the particle between  $x$  and  $x + dx$  at time  $t$ . Thus, if a measurement of position is repeated many times in an identical way on an identical particle in identical circumstances, many possible outcomes are possible and the expectation value of these outcomes is, according to the following equation

$$\langle x \rangle = \int_{-\infty}^{+\infty} x |\psi(x, t)|^2 dx \quad (1.49)$$

## 1.9 Eigen values and eigen functions

The Schrodinger wave equation is a second order differential equation. Thus solving the Schrodinger wave equation to a particular system we get many expressions for wave function ( $\psi$ ). However, all wave functions are not acceptable. Only those wave functions which satisfy certain conditions are acceptable. Such wave functions are called **Eigen functions** for the system. The energy values corresponding to the Eigen functions are called **Eigen values**. The wave functions are acceptable if they satisfy the following conditions.

1.  $\psi$  must be finite everywhere (Cannot be infinite)
2.  $\psi$  must be single valued which implies that solution is unique for a given position in space.
3.  $\psi$  and its first derivatives with respect to its variables must be continuous everywhere.

## 1.10 Applications of schrödinger wave equation

### 1.10.1 Particle in one-dimensional potential well of infinite height

Consider a particle of mass  $m$  bouncing back and forth between the walls of one dimensional potential well as shown in figure 1.1. The particle is said to be under bound state. Let the motion of the particle be confined along the  $x$ -axis in between two infinitely hard walls at  $x = 0$  and  $x = a$ . Since the walls are infinitely hard, no energy is lost by the particle during the collision with walls and the total energy remains constant.

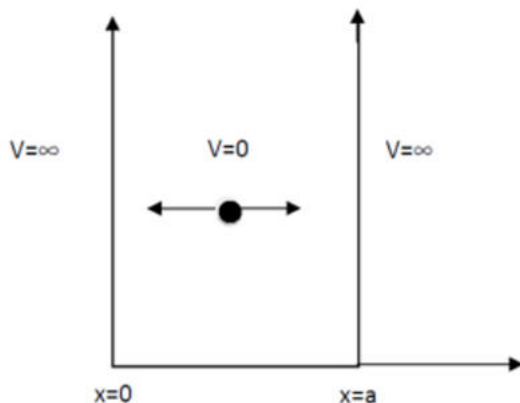


Figure 1.1: One dimensional potential well of infinite height

The description of the potential well is as follows. In between walls i.e.  $0 < x < a$ , the potential  $V = 0$ . Beyond the walls i.e.  $x \leq 0$  and  $x \geq a$ , the potential  $V = \infty$ .

#### Beyond the walls or outside the potential well

Since the particle is unable to penetrate the hard walls it exists only inside the potential well. Hence  $\psi = 0$  and the probability of finding the particle outside the potential well is also zero.

#### In between the walls or inside the potential well

The Schrodinger wave equation is given by

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{8\pi^2 m(E - U)}{h^2} \psi = 0 \quad (1.50)$$

Since the potential inside the well  $V = 0$ , hence potential energy  $U = 0$ , the Schrodinger wave equation becomes

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{8\pi^2 m(E - 0)}{h^2} \psi = 0 \quad (1.51)$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{8\pi^2 mE}{h^2} \psi = 0 \quad (1.52)$$

$$\frac{\partial^2 \psi}{\partial x^2} + k^2 \psi = 0 \quad (1.53)$$

$$k^2 = \frac{8\pi^2 mE}{h^2} \quad (1.54)$$

here  $k$  is a constant for a given value of energy  $E$ . The general solution for equation 1.53 is given by

$$\psi(x) = A \sin kx + B \cos kx \quad (1.55)$$

Here in the above equation  $A$  and  $B$  are arbitrary constants which can be evaluated by applying boundary conditions.

#### Applying Boundary Conditions

1. The first boundary condition is, at  $x = 0$ ,  $\psi(x) = 0$ . Applying this to equation 1.55, we get  $0 = A \sin 0 + B \cos 0 \Rightarrow B = 0$  hence equation 1.55 reduces to

$$\psi(x) = A \sin kx \quad (1.56)$$

2. The second boundary condition is, at  $x = a$ ,  $\psi(x) = 0$ . Applying this to equation 1.56, we get  $0 = A \sin ka$ . Since  $A \neq 0$  then  $\sin ka = 0$ . This results in  $ka = n\pi$  which further could be written as  $k = \frac{n\pi}{a}$ .  $n$  can take integer values. Hence equation 1.56 could be written as

$$\psi(x) = A \sin \left( \frac{n\pi x}{a} \right). \quad (1.57)$$

also from equation 1.54

$$k^2 = \frac{n^2 \pi^2}{a^2} = \frac{8\pi^2 mE_n}{h^2} \quad (1.58)$$

$$E_n = \frac{n^2 h^2}{8ma^2} \quad (1.59)$$

### Quantization of Energy States

Substituting for  $n = 1, 2, 3, 4, \dots$  in the above equation **Energy Eigen Values** are obtained. The lowest energy state corresponds to lowest integral value of  $n = 1$  which is also called as **Zero Point Energy** is given by  $E_1 = \frac{h^2}{8ma^2}$ . The energy values of a bound particle in one dimensional potential well are quantized (discrete) and are represented by the equation  $E_n = n^2 E_1$ .

### Normalization of wave function

The wave function for a particle in one dimensional potential well of infinite height is given by the equation 1.57  $\psi(x) = A \sin\left(\frac{n\pi x}{a}\right)$ . In this equation  $A$  is an arbitrary constant and it can take any value. The process of determination of value of the arbitrary constant is called **Normalization of wave function**. The particle has to exist somewhere inside the potential the probability of finding the particle inside the potential well is given by

$$\int_0^a |\psi(x)|^2 dx = \int_0^a P dx = 1 \quad (1.60)$$

Substituting for the wave function in the integral

$$\int_0^a A^2 \sin^2\left(\frac{n\pi x}{a}\right) dx = 1 \quad (1.61)$$

from trigonometry  $\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta)$ . Therefore the above equation could be written as

$$\int_0^a \frac{A^2}{2} \left[ 1 - \cos\left(\frac{2n\pi x}{a}\right) \right] dx = 1 \quad (1.62)$$

integrating the above equation we get

$$\frac{A^2}{2} \left[ x - \frac{a}{2n\pi} \sin\left(\frac{2n\pi x}{a}\right) \right]_0^a = 1 \quad (1.63)$$

$$\Rightarrow A = \sqrt{\frac{2}{a}} \quad (1.64)$$

Substituting this in equation 1.57 the normalized wave function or eigen function for a particle in one dimensional potential well of infinite height is given by

$$\psi(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right). \quad (1.65)$$

The wave functions and the probability densities for the first three values of  $n$  are as shown in fig 1.2

Thus for ground state ( $n = 1$ ). The probability of finding the particle at the walls is zero and at the center  $\frac{a}{2}$  is maximum. The first excited state has three nodes and the second excited state has four nodes.

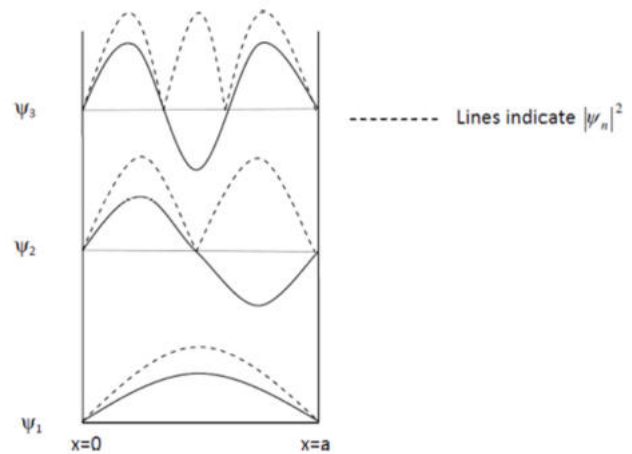


Figure 1.2: Wave function and Probability density for  $n = 1, 2, 3$

### 1.10.2 Particle in higher dimension potential well of infinite height

The solution of Particle in infinite potential well can be extended to 2D and 3D as follows. Consider the equation of energy eigen values in 3 independent variables  $x$ ,  $y$  and  $z$  are

$$E_x = \frac{n_x^2 h^2}{8ma_x^2} \quad (1.66)$$

$$E_y = \frac{n_y^2 h^2}{8ma_y^2} \quad (1.67)$$

$$E_z = \frac{n_z^2 h^2}{8ma_z^2} \quad (1.68)$$

Where  $n_x, n_y$  and  $n_z$  are any set of positive integers

**Particle in 2D potential well:** In a two dimensional potential box, the particle (electron) can move in ' $x$ ' and ' $y$ ' directions. The dimension of the well are  $0 < x < a$  and  $0 < y < a$ . Within this square the potential energy  $V_{(x,y)} = 0$ . Therefore, instead of one quantum number ' $n$ ', two quantum numbers  $n_x$  and  $n_y$  will be taken, corresponding to the two co-ordinate axis  $x$  and  $y$  respectively. The **eigen value equation** could be written as

$$E_{x,y} = \frac{h^2}{8ma^2} (n_x^2 + n_y^2) \quad (1.69)$$

Here  $n_x$  and  $n_y$  takes the value  $1, 2, 3, \dots$

Similarly the eigen function for 2D could be written as

$$\psi_{n_x, n_y}(x, y) = \left( \sqrt{\frac{2}{a}} \sin\left(\frac{n_x \pi x}{a}\right) \right) \left( \sqrt{\frac{2}{a}} \sin\left(\frac{n_y \pi y}{a}\right) \right) \quad (1.70)$$

**Particle in 3D potential well:** In a three dimensional potential box, the particle (electron) can move in 'x' 'y' and 'z' directions. The dimension of the well are  $0 < x < a$ ,  $0 < y < a$  and  $0 < z < a$ . Within this square the potential energy  $V_{(x,y,z)} = 0$ . Therefore, instead of one or two quantum number 'n', three quantum numbers  $n_x$ ,  $n_y$  and  $n_z$  will be taken, corresponding the two co-ordinate axis x and y respectively. The **eigen value equation** could be written as

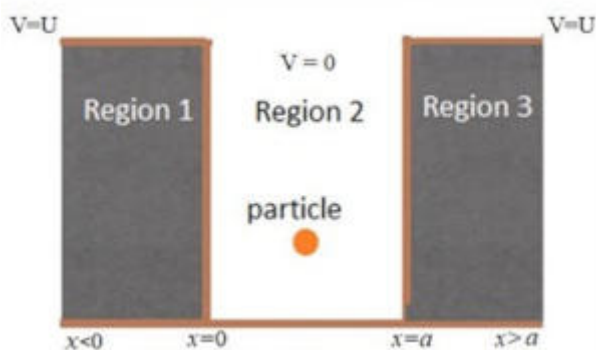
$$E_{x,y,z} = \frac{h^2}{8ma^2} (n_x^2 + n_y^2 + n_z^2) \quad (1.71)$$

Here  $n_x$ ,  $n_y$  and  $n_z$  takes the value 1,2,3,....

Similarly the eigen function for 3D could be written

### 1.11 Particle in finite potential well

In case of finite potential well we consider the height of the potential well to be finite. The potential of the particle will be zero ( $V = 0$ ) in the region 2 and the potential of particle will be  $V = U$  in the region 1 and 3.



Now, consider the region 2, the range is  $x = 0$  to  $x = a$ . The Schrödinger wave equation is

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{8\pi^2 m(E - U)}{h^2} \psi = 0 \quad (1.72)$$

Since the potential is zero in the region 2 we can have

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{8\pi^2 m(E - 0)}{h^2} \psi = 0 \quad (1.73)$$

The solution of the above equation is

$$\psi(x) = A \sin kx + B \cos kx \quad (1.74)$$

The potential is  $V$  in the region 1 and region 3, that is  $x < 0$  and  $x > a$  respectively. The Schrodinger wave equation is

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{8\pi^2 m(E - V)}{h^2} \psi = 0 \quad (1.75)$$

The solution of the above equation is

$$\psi = Ae^{\alpha x} \quad (1.76)$$

For the region 1 where  $x < 0$

$$\psi(x) = A \sin kx + B \cos kx \quad (1.77)$$

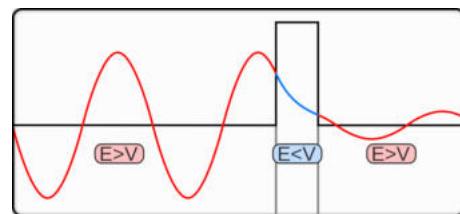
For the region 2 where  $0 < x < a$

$$\psi = Ae^{-\alpha x} \quad (1.78)$$

For the region 3 where  $x > a$

### 1.12 Quantum Tunneling

In classical mechanics, when a particle has insufficient energy, it would not be able to overcome a potential barrier. In the quantum world the particles can often behave like waves. On encountering a barrier, a quantum wave will not end abruptly. Rather its amplitude decrease exponentially. This drop in amplitude corresponds to a drop in the probability of finding a particle further into the barrier. If the barrier is thin enough, then the amplitude may be non-zero on the other side. This would imply that there is a finite probability that some of the particles will tunnel through the barrier.



In regions where the potential energy is higher than the wave's energy, the amplitude of the wave decays exponentially. If the region is narrow enough, the wave can have a non-zero amplitude on the other side.

### Summary

Black Body radiation spectrum posed challenges during the early 19th century. Many experiments were conducted and theories were put forward to understand the phenomenon. Wein and Rayleigh-Jeans were able to explain the black-body radiation spectrum in the lower and higher wavelength region respectively. Planck's radiation law, based on discrete energy packet 'Quanta' or 'Photon', is able to explain the energy distribution in the black body radiation spectrum completely. Several experiments and theories were put forward to understand the particle and wave nature of radiations. This proved dual nature of radiation. Further, based on analogy, Louis de Broglie extended the same concept to moving particles and proposed

a hypothesis which indicates the wave nature of moving particles. Diffraction experiments prove the wave nature of moving particles. Attempts were successful in knowing the different quantities related to the wave associated with particles called matter waves. Irwin Schrodinger set up a differential equation for the moving particle implementing de Broglie hypothesis. The Schrodinger wave equation is set up for a bound particle and free particles. Subsequent solutions prove that the energies of bound particles are quantized and that of free particles is continuous.

### 1.13 Model Questions

1. State and Explain de Broglie hypothesis.
2. What are matter waves and mention the properties.
3. State and Explain Heisenberg's uncertainty principle. Mention its physical significance.
4. Explain broadening of spectral lines using uncertainty principle.
5. State and Explain the Principle of Complementarity.
6. Define wave function? Mention its basic properties.
7. Derive Time independent Schrodinger wave equation for a particle moving in three dimension.
8. Discuss the physical interpretation of wave function?
9. Elucidate expectation value.
10. Describe Eigen functions and Eigen values.
11. Discuss the motion of a particle in one dimensional potential well of infinite height.
12. Discuss the energies of a free particle using time independent schrodinger equation.
13. Role of higher dimensions in Potential well.
14. Discuss the Particle inside a finite potential well

### 1.14 Numerical Problems

1. The speed of electron is measured to within an uncertainty of  $2.2 \times 10^4 \text{ ms}^{-1}$  in one dimension. What is the minimum width required by the electron to be confined in an atom?
2. Estimate the time spent by an atom in the excited state during the excitation and de-excitation processes, when a spectral line of wavelength  $546 \text{ nm}$  and width  $10^{-14} \text{ m}$  is emitted.
3. A spectral line of wavelength  $4000 \text{ \AA}$  has a width of  $8 \times 10^{-5} \text{ \AA}$ . Evaluate the minimum time spent by the electrons in the upper energy state between the excitation and de-excitation processes.

4. The inherent uncertainty in the measurement of time spent by Iridium 191 nuclei in the excited state is found to be  $1.4 \times 10^{-10} \text{ s}$ . Estimate the uncertainty that results in its energy in eV in the excited state.
5. An electron is bound in one dimensional potential well of width  $0.18 \text{ nm}$ . Find the energy value in eV of the second excited state.
6. The first excited state energy of an electron in an infinite well is  $240 \text{ eV}$ . What will be its ground state energy when the width of the potential well is doubled?
7. A quantum particle confined to one-dimensional box of width  $a$  is in its first excited state. What is the probability of finding the particle over an interval of  $\frac{a}{2}$  marked symmetrically at the center of the box.

## **Part II**

# **MODULE 2 : Electrical Properties of Metals and Semiconductors**



## Chapter 2

# Electrical Properties of Metals and Semiconductors

### 2.1 Electrical Conductivity and Resistivity

Consider a conductor carrying electric current  $I$  with area of cross section  $A$  perpendicular to the current. The current density  $J$  is defined as the ratio of current  $I$  to the area of cross section  $A$ . Hence

$$J = \frac{I}{A} \quad (2.1)$$

It is observed that the current density is proportional to the applied electric field in a conductor. And hence

$$J \propto E \quad (2.2)$$

$$J = \sigma E \quad (2.3)$$

The constant of proportionality  $\sigma$  is called **Electrical Conductivity** of the conductor. The Electrical Resistivity  $\rho = \frac{1}{\sigma}$  of the material is the reciprocal of the Electrical Conductivity of the material and is the property of the material by the virtue of which it opposes the flow of current through it.

### 2.2 Mobility of conduction electrons

It is found that the drift velocity depends on the applied field strength and is mathematically given by  $v_d = \mu E$ . Here  $\mu$  is called the mobility of the free electrons. It is defined as the drift velocity acquired by the conduction electrons per unit field strength.

$$\mu = \frac{v_d}{E} = \frac{e\tau}{m} = \frac{\sigma}{ne} = \frac{1}{\rho ne} \text{ m}^2 \text{V}^{-1} \text{s}^{-1} \quad (2.4)$$

### 2.3 Concept of Phonon

A Phonon is a quantum of lattice vibration, the collective motion of atoms constituting a crystal. The Energies and Momenta of Phonons are quantized. It is often characterized as Heat Energy. The study of phonon is an important

part of solid state physics. The phonon plays an important role in many of the physical properties of solids such as the thermal conductivity and the electrical conductivity. The conduction electrons in a metal collide against lattice ions during the motion. The interaction is considered to be of type phonon exchange. This results in non-radioactive transitions.

### 2.4 Mechanisms of Electron Scattering in Solids:

In solids, electrons do not move freely as in a vacuum. They interact with lattice atoms/ions, impurities, structural defects, other electrons, and boundaries of the material. These interactions cause electron scattering, which determines the electrical resistivity, mobility, and conductivity of the solid. The scattering mechanisms define the mean free path and the relaxation time  $\tau$  of the electrons.

#### 2.4.1 Major Scattering Mechanisms

##### 1. Electron-Phonon (Lattice) Scattering:

Caused by thermal vibrations of the lattice atoms (phonons). Resistivity increases approximately linearly with  $T$  at high temperatures.

##### 2. Electron-Impurity Scattering

Caused by foreign atoms or dopants, which disturb the periodic lattice potential. Scattering is elastic. Resistivity contribution is nearly temperature independent (residual resistivity).

##### 3. Electron-Defect / Dislocation Scattering

Caused by vacancies, interstitials, grain boundaries, and dislocations. Defects break lattice periodicity, reducing mobility.

##### 4. Electron-Electron Scattering

Due to Coulomb interactions between conduction electrons. In pure metals at low temperature, it contributes significantly.

### 5. Surface or Boundary Scattering

Caused by electron reflections at material surfaces. Significant in thin films, nanowires, and 2D materials, where dimensions are comparable to electron mean free path.

## 2.5 Mathiesen's Rule

A Metal consists of lattice ions and impurity atoms that are held together by free electrons. Free electrons wander inside the crystal. During the motion electrons undergo scattering by lattice ions and impurity atoms. The resistivity  $\rho$  of a conductor is mainly attributed to two reasons

1. Scattering of electrons with the vibrating lattice ions. The resistivity of the metal due to electron lattice ion scattering is given by

$$\rho_{ph} = \frac{m}{ne^2\tau_{ph}} \quad (2.5)$$

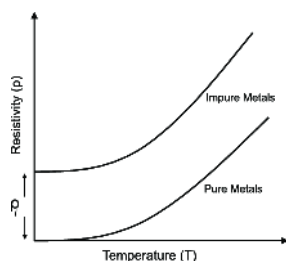
2. Scattering of electrons by the presence of impurities present in the metal. The scattering also occurs from the lattice dislocations and grain boundaries. The resistivity of the metal due to such scattering is given by

$$\rho_i = \frac{m}{ne^2\tau_i} \quad (2.6)$$

Thus net resistivity of the conductor is given by

$$\rho = \rho_{ph} + \rho_i = \frac{m}{ne^2\tau_{ph}} + \frac{m}{ne^2\tau_i} \quad (2.7)$$

The above equation is called Matthiessen's Rule, Which states that the net resistivity of conductor is equal to the sum of the resistivity due to the phonon scattering which is temperature dependent and resistivity due to the presence of impurity which is temperature independent. Graphically the variation of resistance with temperature and impurity is as follows.



## 2.6 Failures of classical free electron theory of metals

Classical free electron theory of metal is successful in explaining the certain experimentally observed facts of

electronic conduction in solids and thermal conductivity. This theory fails to explain certain other experimental observations. The following are the failures of classical free electron theory of metals.

### 2.6.1 Electronic specific heat of solids

According to the Classical Free Electron Theory Metals the electronic specific heat is given by

$$C_v = \frac{3}{2}R = 12.5 \text{ J mole}^{-1} \text{ K}^{-1} \quad (2.8)$$

The experimental value of electronic specific heat is  $C_v = 10^{-4}RT$ . It is very small and also temperature dependent. Hence Classical theory fails to explain the electronic specific heat of solids.

### 2.6.2 Dependence of $\sigma$ on temperature

According to classical free electron theory of metals the electrical conductivity  $\sigma$  is inversely proportional to square root of temperature ( $\sqrt{T}$ ). But experiments reveal that electrical conductivity ( $\sigma$ ) is inversely proportional to temperature ( $T$ ). Hence classical free electron theory fails to explain dependence of electrical conductivity ( $\sigma$ ) on the temperature ( $T$ ).

### 2.6.3 Dependence of $\sigma$ on $n$ , the number density

The theory predicts the direct dependence of electrical conductivity ( $\sigma$ ) on number of free electrons per unit volume ( $n$ ) called number density. But experiments have revealed different with  $\sigma_{Cu} > \sigma_{Al}$  even though the number densities  $n_{Cu} < n_{Al}$ . Hence it fails to explain the dependence of electrical conductivity  $\sigma$  on the number free electrons per unit volume  $n$ . The experimental observations are as in the table below.

| Metal     | $\sigma (\Omega^{-1}m^{-1})$ | $n (m^{-3})$           |
|-----------|------------------------------|------------------------|
| Copper    | $5.88 \times 10^7$           | $8.45 \times 10^{28}$  |
| Aluminium | $3.65 \times 10^7$           | $18.06 \times 10^{28}$ |

## 2.7 Quantum free electron theory of metals

The failures of classical free electron theory led to the rise of Quantum Free electron theory and was proposed by Sommerfield in the year 1928. The quantum free electron theory is based on the following assumptions.

### 2.7.1 Assumptions

1. Unlike classical free electron theory, in quantum free electron theory, energy values of free electrons are

quantized. The energy values of free electrons are discrete since their motion is confined within the boundaries of the metal.

2. Thus in a metal there exists large number of closely spaced energy levels for free electrons which form a band.
3. The distribution of free electrons in the energy levels is as per the Pauli's exclusion principle. Only a maximum of two electrons can occupy a given an energy level. This also suggests the availability of two energy states for free electrons in an energy level corresponding to spin up and spin down states.
4. The potential setup by the lattice ions is assumed to be constant throughout the metal.
5. The mutual repulsion between electrons and the attraction between electrons and lattice ions are neglected.

### 2.7.2 Fermi energy

For a metal consisting of  $N$  atom there exists  $N$  number of energy levels in each band. These energy levels are very closely spaced. The energy levels in bands fillup as per Pauli's exclusion principle. Thus free electrons in a metal start filling up the available energy levels from the lower most level of the valence band. **The highest filled energy level in a metal at absolute zero by free electrons is called Fermi Level and the corresponding energy is called Fermi Energy ( $E_F$ ).** Thus, at absolute zero and with no electric field applied, all levels below Fermi level are completely filled and above Fermi level are empty.

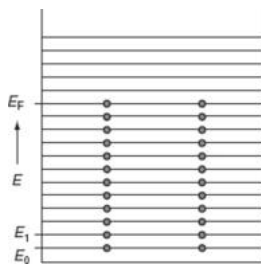


Figure 2.1: Energy Band diagram and Fermi Energy

### 2.7.3 Density of States (DoS)

According to band theory Energy bands are formed in solids and in a band the spacing between two successive energy levels decreases with increase in energy.

**The Density of States is defined as the number of energy states available per unit volume of the material in**

**the unit energy range in the valence band of the material. It is mathematically a continuous function denoted by  $g(E)$ .** The number of energy levels in the energy range  $E$  and  $E + dE$  per unit volume of the material is given by  $g(E)dE$ .

$$g(E)dE = \left( \frac{8\sqrt{2}\pi m^{3/2}}{h^3} \right) E^{-\frac{1}{2}} dE \quad (2.9)$$

The variation of  $g(E)dE$  as a function of  $E$  is given by

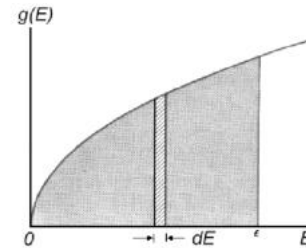


Figure 2.2: Density of states function vs Energy

### 2.7.4 Fermi-Dirac distribution and Fermi factor

The occupation of energy levels by free electrons in the valence band of a metal is according to Pauli's exclusion principle. This distribution of electrons is not random. It follows a certain universal rule of distribution called Fermi-Dirac Statistics. A metal contains large number free of electrons.

1. A metal contains large number free of electrons. The occupation of energy levels in the valence band is according to Pauli's exclusion principle.
2. This distribution of electrons follows a certain universal rule called Fermi-Dirac Distribution.
3. This statistics is applicable for the distribution of particles of spin half.
4. A distribution statistics provides the probability of occupation of an energy level with energy ( $E$ ) at a given temperature under thermal equilibrium.
5. Thus the probability of occupation of an energy level of energy ( $E$ ) at temperature ( $T$ ) under thermal equilibrium is given by Fermi Factor  $f(E)$ .

**The probability of occupation of an energy level of energy ( $E$ ) at temperature ( $T$ ) under thermal equilibrium is evaluated using an expression called Fermi Factor.**

$$f(E) = \frac{1}{e^{\left(\frac{E-E_f}{kT}\right)} + 1} \quad (2.10)$$

The number density of electrons defined as the number of free electrons per unit volume the material in the energy range  $E$  and  $E + dE$  in the valence band of the material is given by  $n(E)dE$ . Mathematically

$$N(E)dE = g(E)dEf(E) \quad (2.11)$$

Here  $f(E)$  is the Fermi factor. The number density of electrons is the product of the number of available energy states and their occupation probability. This is called Fermi-Dirac distribution.

### 2.7.5 Dependence of Fermi factor on energy and temperature (Variation of Fermi Factor With Temperature and Energy)

As described, the Fermi factor is a function of energy and temperature. This dependence could be explained for energy levels below and above Fermi level at absolute zero and higher temperatures.

#### Probability of occupation of levels with energy $E < E_F$ and at $T = 0K$

The Fermi factor or Fermi function is given by

$$f(E) = \frac{1}{e^{\left(\frac{E-E_f}{kT}\right)} + 1} \quad (2.12)$$

Here  $E - E_f$  is negative. Substituting the value for  $T = 0$

$$f(E) = \frac{1}{e^{\left(\frac{E-E_f}{k \cdot 0}\right)} + 1} = \frac{1}{e^{-\infty} + 1} = \frac{1}{0 + 1} = 1$$

There fore  $f(E) = 1$ . Hence, at  $T = 0K$ , all energy levels below the Fermi level are completely filled.

#### Probability of occupation of levels with energy $E > E_F$ and at $T = 0K$

The Fermi factor or fermi function is given by Here  $E - E_f$  is positive. Substituting the value for  $T = 0$

$$f(E) = \frac{1}{e^{\left(\frac{E-E_f}{k \cdot 0}\right)} + 1} = \frac{1}{e^{\infty} + 1} = \frac{1}{\infty + 1} = 0$$

There fore  $f(E) = 0$ . Hence, at  $T = 0K$ , all energy levels above the Fermi level are empty.

#### Probability of occupation of levels with energy $E = E_F$ and at $T > 0K$

Here  $E - E_f = 0$ . Substituting the values

$$f(E) = \frac{1}{e^{\left(\frac{0}{kT}\right)} + 1} = \frac{1}{1 + 1} = \frac{1}{2} = 0.5 \quad (2.13)$$

Thus for all temperatures above  $0 K$  the probability of occupation of Fermi level is  $1/2$ . Thus the variation of Fermi factor with temperature is as shown in the graph 2.3

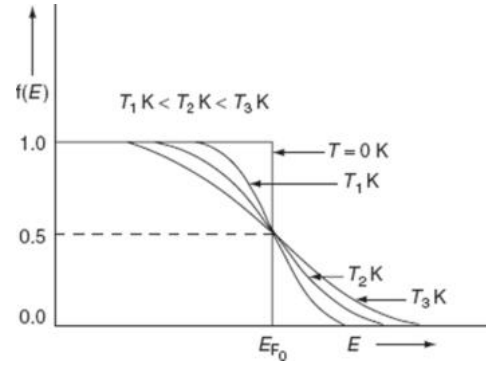


Figure 2.3: Variation of  $f(E)$  as a function of Temperature and Energy

## 2.8 Expression for Fermi energy at $0K (E_f)$

The number of allowed energy levels per unit volume is known as density of energy states. The number of electrons in these levels per unit volume at any temperature is given by

$$N(E)dE = g(E)dEf(E) \quad (2.14)$$

The number of electrons per unit volume up to fermi level is given by

$$n = \int_0^{E_f} N(E)dE \quad (2.15)$$

$$n = \int_0^{E_f} g(E)dEf(E) \quad (2.16)$$

$$n = \int_0^{E_f} \left( \frac{8\sqrt{2}\pi m^{3/2}}{h^3} \right) E^{\frac{1}{2}} dEf(E) \quad (2.17)$$

Put  $f(E) = 1$  at  $0K$

$$n = \int_0^{E_f} \left( \frac{8\sqrt{2}\pi m^{3/2}}{h^3} \right) E^{\frac{1}{2}} dE \quad (2.18)$$

$$n = \left( \frac{8\sqrt{2}\pi m^{3/2}}{h^3} \right) \int_0^{E_f} E^{\frac{1}{2}} dE \quad (2.19)$$

$$n = \left( \frac{8\sqrt{2}\pi m^{3/2}}{h^3} \right) \left[ \frac{E^{3/2}}{3/2} \right]_0^{E_f} \quad (2.20)$$

$$n = \left( \frac{16\sqrt{2}\pi m^{3/2}}{3h^3} \right) \left[ E_F^{3/2} \right] \quad (2.21)$$

This is the expression for free electron density. The above equation can be written as

$$E_F^{3/2} = \frac{3nh^3}{16\sqrt{2}\pi m^{3/2}} \quad (2.22)$$

$$E_F = \left[ \frac{3nh^3}{16\sqrt{2}\pi m^{3/2}} \right]^{2/3} \quad (2.23)$$

After simplifying this we get

$$E_F = \left( \frac{h^2}{8m} \right) \left( \frac{3n}{\pi} \right)^{2/3} \quad (2.24)$$

According to this equation

$$E_F \propto n^{2/3} \quad (2.25)$$



## Chapter 3

# Semiconductors

### 3.1 Introduction

Semiconductors are the materials which possess negative temperature coefficient of resistance. Their electrical properties are different from those of conductors and insulators. The pure form of semiconductors are known as intrinsic semiconductors. The conductivity in intrinsic semiconductors is due to both the motion of electrons in the conduction band and the motion of holes in the valence band. The electrons in the valence band of intrinsic semiconductors need more energy to move to conduction band. This energy could be reduced by doping a pure semiconductor with certain impurities. The process of adding impurity is called doping. Such semiconductors are called extrinsic semiconductors. Based on the type of dopant, extrinsic semiconductors are classified into n-type semiconductors and p-type semiconductors. The conductivity of extrinsic semiconductors is due to the motion of majority charge carriers: electrons in n-type and holes in p-type.

Doping of pentavalent impurity atoms to pure semiconductor results in the fifth electron which does not participate in bonding and is loosely bound to the impurity atoms. The energy of these electrons lies in the energy gap of semiconductor. Thus a donor level is created near the lower edge of the conduction band. Given a small amount of energy, electrons move from donor level to conduction band. Thus there will be a large number of free electrons in the conduction band. Thus electrons are the majority charge carriers in n-type semiconductors. Similarly, doping of trivalent impurity atoms to pure semiconductor results in the formation of the acceptor level near the upper edge of the valence band. Holes are the majority carriers in p-type semiconductors.

### 3.2 Significance of Fermi level

Fermi level acts as a distinguishing energy position between filled and unfilled energy states in metals and semiconductors. In case of pure semiconductors, at normal temperatures, electrons are most probably found either in conduction band or in valence band. This is because of electrons in the top most levels of valence band absorb en-

ergy and move to conduction band. But electrons cannot spend more time there and hence they return to a lower energy level in valence band. Thus electrons will be under constant excitation and de-excitation process. Thus the average energy of the electron lies at the center of the energy gap and is called Fermi energy.

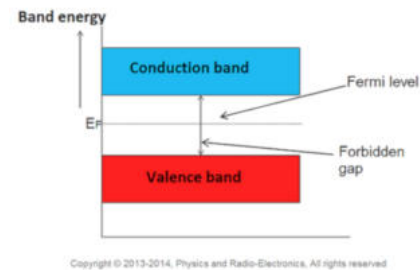


Figure 3.1: Fermi Level in Intrinsic Semiconductors



Figure 3.2: Fermi Level in Extrinsic semiconductors

In case of extrinsic semiconductors the Fermi energy lies towards conduction band in n-type and towards valence band in p-type semiconductors. Fermi level depends on temperature and changes with temperature.

### 3.3 Intrinsic semiconductor:

An intrinsic semiconductor is a pure, undoped semiconductor material with an equal number of free electrons and

holes. Electrical current is carried by both free electrons (in the conduction band) and holes (vacancies in the valence band). At absolute zero (0 K), an intrinsic semiconductor behaves as a perfect insulator because all electrons are bound in covalent bonds. As temperature increases, thermal energy excites some electrons into the conduction band, and increasing conductivity. The electrical conductivity is relatively low compared to doped (extrinsic) semiconductors and metals.

### 3.3.1 Expression for Concentration of electron in an intrinsic semiconductor:

The number of electrons in the conduction band per unit volume of the material is called electron concentration  $N_e$ . The number of holes in the valence band per unit volume of the material is called hole concentration  $N_h$ .

The number of electrons whose energy lies in the energy interval  $E$  and  $E + dE$  in the conduction band. Then

$$N(E)dE = g(E)dE f(E) \quad (3.1)$$

where  $g(E)dE$  is the density of states in the energy interval  $E$  and  $E + dE$  and  $f(E)$  is the probability that a state of energy is occupied by an electron. The electron density in the conduction band is given by integrating eq. (1) between the limits  $E_c$  and  $\infty$ .  $E_c$  is the energy corresponding to the bottom edge of the conduction band and  $\infty$  is the energy corresponding to the top edge of the conduction band. As the probability of electrons occupying upper levels of conduction band  $f(E)$  readily approaches zero for higher energies, the upper limit namely the top of the conduction band is taken as  $\infty$ . Thus

$$N_e = \int_{E_c}^{\infty} N(E)dE \quad (3.2)$$

$$N_e = \int_{E_c}^{\infty} g(E)dE f(E) \quad (3.3)$$

But, for electron concentration in an intrinsic semiconductor

$$g(E)dE = \left( \frac{8\sqrt{2}\pi m_e^{*3/2}}{h^3} \right) (E - E_c)^{1/2} dE \quad (3.4)$$

We know that

$$f(E) = \frac{1}{e^{\left(\frac{E-E_f}{kT}\right)} + 1} \quad (3.5)$$

Under practical condition,

$$e^{\left(\frac{E-E_f}{kT}\right)} + 1 \gg 1 \quad (3.6)$$

Therefore,

$$f(E) = \frac{1}{e^{\left(\frac{E-E_f}{kT}\right)}} \quad (3.7)$$

$$f(E) = e^{-\left(\frac{E-E_f}{kT}\right)} \quad (3.8)$$

$$f(E) = e^{\left(\frac{E_f-E}{kT}\right)} \quad (3.9)$$

Now using equation (3.4) and (3.9) in equation (3.3) we get,

$$N_e = \int_{E_c}^{\infty} \left( \frac{8\sqrt{2}\pi m_e^{*3/2}}{h^3} \right) (E - E_c)^{1/2} e^{\left(\frac{E_f-E}{kT}\right)} dE \quad (3.10)$$

$$N_e = \int_{E_c}^{\infty} \left( \frac{8\sqrt{2}\pi m_e^{*3/2}}{h^3} \right) (E - E_c)^{1/2} e^{\left(\frac{E_f-E_c+E_c-E}{kT}\right)} dE \quad (3.11)$$

$$N_e = \left( \frac{8\sqrt{2}\pi m_e^{*3/2}}{h^3} \right) e^{\left(\frac{E_f-E_c}{kT}\right)} \int_{E_c}^{\infty} (E - E_c)^{1/2} e^{\left(\frac{E_c-E}{kT}\right)} dE \quad (3.12)$$

$$N_e = \left( \frac{8\sqrt{2}\pi m_e^{*3/2}}{h^3} \right) e^{\left(\frac{E_f-E_c}{kT}\right)} \int_{E_c}^{\infty} (E - E_c)^{1/2} e^{-\left(\frac{E - E_c}{kT}\right)} dE \quad (3.13)$$

Let,

$$x = \frac{E - E_c}{kT} \quad (3.14)$$

$$(E - E_c) = xkT \quad (3.15)$$

$$E = E_c + xkT \quad (3.16)$$

Differentiate with respect to 'x'

$$\frac{dE}{dx} = kT \quad (3.17)$$

$$dE = kT dx \quad (3.18)$$

Now, the limit of integral changes. When  $E = E_c$ ,  $x = 0$  and  $E = \infty$ ,  $x = \infty$ . Thus, limit of integral varies from 0 to  $\infty$ . Therefore, use equation (3.14), (3.15) and (3.18) in eq (3.13), we get

$$N_e = \left( \frac{8\sqrt{2}\pi m_e^{*3/2}}{h^3} \right) e^{\left(\frac{E_f-E_c}{kT}\right)} \int_0^{\infty} (xkT)^{1/2} e^{-x} (kT) dx \quad (3.19)$$

$$N_e = \left( \frac{8\sqrt{2}\pi m_e^{*3/2}}{h^3} \right) e^{\left(\frac{E_f-E_c}{kT}\right)} (kT)^{3/2} \int_0^{\infty} (x)^{1/2} e^{-x} dx \quad (3.20)$$

We know that,

$$\int_0^{\infty} (x)^{1/2} e^{-x} dx = \frac{\sqrt{\pi}}{2} \quad (3.21)$$

Therefore,

$$N_e = \left( \frac{8\sqrt{2}\pi m_e^{*3/2}}{h^3} \right) e^{\left(\frac{E_f-E_c}{kT}\right)} (kT)^{3/2} \frac{\sqrt{\pi}}{2} \quad (3.22)$$

$$N_e = \left( \frac{4\sqrt{2}(\pi m_e^* kT)^{3/2}}{h^3} \right) e^{\left(\frac{E_f-E_c}{kT}\right)} \quad (3.23)$$

This is the expression for concentration of electrons in an intrinsic semiconductor.

Similarly, expression for hole concentration in an intrinsic semiconductor is given by,

$$N_h = \left( \frac{4\sqrt{2}(\pi m_h^* kT)^{3/2}}{h^3} \right) e^{\left(\frac{E_v-E_f}{kT}\right)} \quad (3.24)$$

### 3.4 Carrier Concentration in intrinsic semiconductors

Intrinsic semiconductor is the pure form of semiconductor. When sufficient energy is supplied to electrons in the valence band they absorb energy and move to conduction band leaving behind a hole. Thus the number of electrons in the conduction band is equal to number of holes in the valence band. The total number of electrons in the conduction band per unit volume of the semiconductor is known as electron concentration. Similarly number of holes in the valence band per unit volume of the semiconductor is known as hole concentration. The total number of charge carriers per unit volume in the semiconductor is called carrier concentration. In case of pure semiconductors electron concentration and hole concentration are same.

#### 3.4.1 Expression for Electron concentration ( $N_e$ )

The following equation 3.25 determines the electron concentration in intrinsic semiconductors.

$$N_e = \frac{4\sqrt{2}}{h^3} (\pi m_e^* kT)^{\frac{3}{2}} e^{\left(\frac{E_F - E_g}{kT}\right)} \quad (3.25)$$

Here

$h$  is Planck's constant

$m_e^*$  is the effective mass of the electron

$k$  is Boltzmann constant

$T$  is absolute temperature

$E_F$  is Fermi energy

$E_g$  is the energy gap of the semiconductor.

#### 3.4.2 Expression for Hole concentration ( $N_h$ )

The following equation 3.26 determines the electron concentration in intrinsic semiconductors.

$$N_h = \frac{4\sqrt{2}}{h^3} (\pi m_h^* kT)^{\frac{3}{2}} e^{\left(\frac{-E_F}{kT}\right)} \quad (3.26)$$

Here

$h$  is Planck's constant

$m_h^*$  is the effective mass of the electron

$k$  is Boltzmann constant

$T$  is absolute temperature

$E_F$  is Fermi energy

$E_g$  is the energy gap of the semiconductor.

#### 3.4.3 Expression relating Fermi energy and Energy gap for an intrinsic semiconductor

The expressions for electron and hole concentrations in a pure semiconductor are given by equations 3.25 and 3.26

$$N_e = \frac{4\sqrt{2}}{h^3} (\pi m_e^* kT)^{\frac{3}{2}} e^{\left(\frac{E_F - E_g}{kT}\right)} \quad (3.27)$$

$$N_h = \frac{4\sqrt{2}}{h^3} (\pi m_h^* kT)^{\frac{3}{2}} e^{\left(\frac{-E_F}{kT}\right)} \quad (3.28)$$

Both electron concentration and hole concentration depend on fermi energy  $E_F$ . In case of intrinsic semiconductor  $N_e = N_h$  and hence equating the equations 3.25 and 3.26 we get

$$(m_e^*)^{\frac{3}{2}} e^{\left(\frac{E_F - E_g}{kT}\right)} = (m_h^*)^{\frac{3}{2}} e^{\left(\frac{-E_F}{kT}\right)} \quad (3.29)$$

$$e^{\left(\frac{2E_F - E_g}{kT}\right)} = \left[\frac{m_h^*}{m_e^*}\right]^{\frac{3}{2}} \quad (3.30)$$

$$\frac{2E_F - E_g}{kT} = \frac{3}{2} \ln\left(\frac{m_h^*}{m_e^*}\right) \quad (3.31)$$

$$2E_F - E_g = \frac{3}{2} kT \ln\left(\frac{m_h^*}{m_e^*}\right) \quad (3.32)$$

$$E_F = \frac{E_g}{2} + \frac{3}{4} kT \ln\left(\frac{m_h^*}{m_e^*}\right) \quad (3.33)$$

Thus the expression for Fermi energy in intrinsic semiconductors. If we consider that  $m_h^* = m_e^*$  then equation 3.35 becomes

$$E_F = \frac{E_g}{2} \quad (3.34)$$

At a finite temperature  $T > 0K$  the Fermi level differs from its 0K value by a correction term,

$$E_F = \frac{3}{4} kT \ln\left(\frac{m_h^*}{m_e^*}\right) \quad (3.35)$$

This term is proportional to the temperature T. This Proves that the Fermi energy in an intrinsic semiconductor at finite temperature differs from its value at  $T = 0K$  by a value proportional to the temperature.

### 3.5 Fermi Level extrinsic semiconductor

In an extrinsic semiconductor, the position of the Fermi level depends on the type and concentration of impurities introduced. In an n-type semiconductor, donor atoms add extra electrons, raising the electron concentration in the conduction band. As a result, the Fermi level shifts upward, lying closer to the conduction band edge  $E_c$ . Conversely, in a p-type semiconductor, acceptor atoms create holes in the valence band, and the Fermi level moves downward, lying closer to the valence band edge  $E_v$ . The higher the doping concentration, the nearer the Fermi level approaches the respective band edge. At very high temperatures, however, intrinsic behavior dominates and the Fermi level tends back toward the mid-gap position.  $E_d$  and  $E_a$

are donor and acceptor level The Fermi Level for n-type semiconductor at T=0K

$$E_f = \frac{E_d + E_c}{2} \quad (3.36)$$

The Fermi Level for p-type semiconductor at T=0K

$$E_f = \frac{E_a + E_v}{2} \quad (3.37)$$

### 3.6 Hall Effect

**Introduction** This was observed first by E. H. Hall in the year 1879. When electric current is passed through a conductor or semiconductor placed in a magnetic field, a potential difference proportional to the current and to the magnetic field is developed across the material in a direction perpendicular to both the current and to the magnetic field. This effect is known as the Hall effect. The potential difference developed in a direction perpendicular to both current and magnetic field is called Hall voltage and the corresponding electric field is called Hall field.

**Explanation** Consider a conductor carrying current along +ve X axis. Magnetic field is applied along +ve Z axis. Thus electrons experience Lorentz force along the -ve Y axis. Thus electrons accumulate on the lower surface which results in the accumulation of +ve charges on the upper surface. This results in an electric field developed between the two surfaces. The force on the electrons due to electric field further opposes the force due to magnetic field. A stage is reached where in both the forces are equal and an equilibrium state is reached. The electric field developed across the material at equilibrium is called Hall Field.

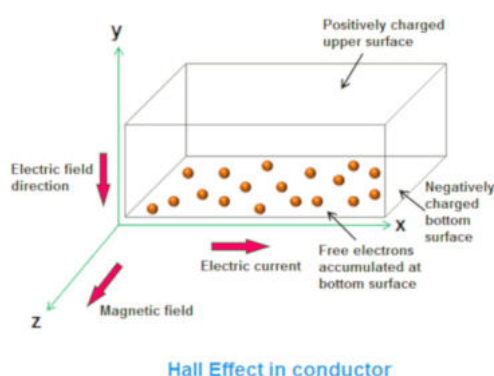


Figure 3.3: Hall effect in conductors

**Expression for Hall Co-efficient** Under equilibrium the Lorentz force on electrons is equal to force due to hall field

$$eE_H = Bev \quad (3.38)$$

Here  $e$  is electronic charge,  $E_H$  is Hall Field,  $B$  is Magnetic field,  $v$  is the velocity of electrons. The current density  $J$  is given by

$$J = nev \quad (3.39)$$

Here  $n$  is the number density of charges. Dividing equation 3.38 by 3.39 we get

$$\frac{eE_H}{J} = \frac{Bev}{nev} \quad (3.40)$$

$$\frac{eE_H}{J} = \frac{B}{n} \quad (3.41)$$

$$E_H = \frac{BJ}{ne} \quad (3.42)$$

$$E_H = R_H BJ \quad (3.43)$$

Here  $R_H$  is called Hall coefficient and is given by

$$R_H = \frac{1}{ne} \quad (3.44)$$

The Hall voltage is given by the equation

$$V_H = E_H d = R_H BJd \quad (3.45)$$

Here  $d$  is the thickness of the material along y-axis.

**Note:** In case of n-type semiconductors  $R_H$  is negative and for p-type semiconductors  $R_H$  is positive.

#### Applications

1. This effect is the basis of many practical applications and devices such as magnetic field measurements, and position and motion detectors.
2. Hall effect is also used in the determination of type of extrinsic semiconductor (P-Type and N-Type).
3. Hall effect is also used to find semiconductor properties like carrier concentration and mobility of carriers.

### 3.7 Model Questions

1. Explain the failures of Classical free electron theory.
2. Give the assumptions on which quantum free electron theory is based.
3. Describe Fermi level and Fermi energy.
4. Explain the density of states and mention density of states function.
5. Write a note of Fermi function and Fermi-Dirac distribution statistics.
6. Explain the variation of Fermi factor with energy and temperature.
7. Derive an expression for fermi energy at absolute temperature.

8. Derive an expression for electron density at absolute temperature.
9. Derive an expression for concentration of electron in an intrinsic semiconductor.
10. Derive an expression relating fermi energy and the energy gap for an intrinsic semiconductor.
11. Describe hall effect. Derive an expression for hall co-efficient and hall voltage.
11. A rectangular plane sheet of semiconductor material has dimensions 2cm along Y-direction and 1mm along Z-direction. Hall probes are attached on its surfaces parallel to X-Y plane and a magnetic field of flux density  $1\text{ weber}/m^2$  is applied along Z-direction. A current of 3 mA is flowing in it in the X-direction. Calculate the hall voltage measured by the probe, if the hall co-efficient of the material is  $3.66 \times 10^{-4}/\text{coulomb}$ . Also calculate the charge carrier concentration. Ans.  $E_H = 0.0549V$ ,  $V_H = 1.1mV$ ,  $n = 1.71 \times 10^{22}/m^3$ .

### 3.8 Numerical Problems

1. The Fermi level in silver is  $5.5eV$  at  $0K$ . Calculate the number of free electrons per unit volume and the probability of occupation for electrons with energy  $5.6eV$  in silver at the same temperature.
2. Calculate the probability of an electron occupying an energy level  $0.02eV$  above the Fermi level at  $200K$  and  $400K$ .
3. Calculate the probability of an electron occupying an energy level  $0.02eV$  below the Fermi level at  $400K$ .
4. The Fermi level for silver is  $5.5eV$ . What is the energy for which probability of occupancy at  $300K$  is  $0.01$
5. Find the temperature at which there is  $1\%$  probability that a state with an energy  $0.5eV$  above Fermi energy is occupied.
6. Find the probability that an energy level at  $0.2eV$  below Fermi level being occupied at temperature  $300K$  and  $1000K$
7. Calculate the probability that an electron occupies an energy level  $0.02\text{ eV}$  above Fermi level at  $300\text{ K}$ .
8. The conductivity and Hall co-efficient of an n-type silicon specimen are  $112/\text{ohm-m}$  and  $1.25 \times 10^{-3}m^3/C$ , respectively. Calculate the charge carrier concentration and electron mobility. Ans.  $N_e = 5 \times 10^{21}m^{-3}$ ,  $\mu_e = 0.14$ .
9. The hall co-efficient of a material is  $-3.68 \times 10^{-5}m^3/C$ . What is the type of charge carriers? Also calculate the carrier concentration. Ans.  $R_H$  is - Ve, electrons,  $1.7 \times 10^{23}/m^3$
10. The hall co-efficient of a specimen of a doped silicon is found to be  $3.66 \times 10^{-4}/\text{coulomb}$ . The resistivity of the specimen is  $9.93 \times 10^{-3}\text{ohm} - m$ . Find the mobility and density of charge carrier, Assuming single carrier concentration. Ans.  $1.7055 \times 10^{22}/m^3$  and  $0.041m^2V^{-1}s^{-1}$



## **Part III**

### **MODULE 3 : Superconductivity**



## Chapter 4

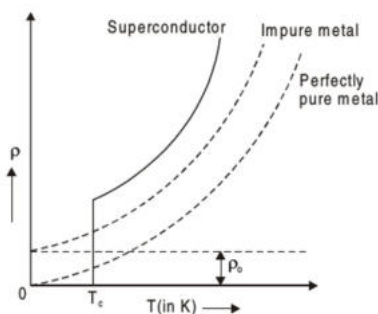
# Superconductivity

### 4.1 Introduction to Superconductivity

**Discovery :** Lord Kamerlingh Onnes discovered the phenomenon of superconductivity in the year 1911. When he was studying the temperature dependence of resistance of Mercury at very low temperature he found that resistance of Mercury decreases with temperature with the decrease in temperature up to a particular temperature  $T_c = 4.15\text{K}$ . Below this temperature the resistance of mercury abruptly drops to zero. Between  $4.15\text{K}$  and  $0\text{K}$  Mercury offered no resistance for the flow of electric current. This phenomenon is reversible and material becomes normal once again when temperature was increased above  $4.15\text{K}$ . This phenomenon is called superconductivity and material which exhibits the property is named superconductor.

**Definition :** Thus the Superconductivity is defined as *“The phenomenon in which resistance of certain metals, alloys and compounds drops to zero abruptly, below certain temperature is called superconductivity”*

**Variation of Resistivity with Temperature** The variation of the resistivity of a superconductor, pure and impure metals with temperature is as shown below.



**Critical Temperature :** The temperature, below which materials exhibit superconducting property is called critical temperature, denoted by  $T_c$ . Critical temperature  $T_c$  is different for different substances. The materials, which

exhibit superconducting property, are called superconductors.

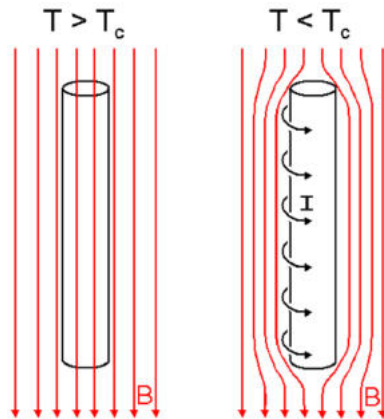
Above critical temperature material is said to be in normal state and offers resistance for the flow of electric current. Below critical temperature material is said to be in superconducting state. Thus  $T_c$  is also called as transition temperature.

### 4.2 Persistent current

In a superconducting material, current will continue to flow in a closed loop as long as the loop is held below the critical temperature  $T_c$ . Such a steady current which flows with undiminished strength is called a persistent current. The concept of persistent current is a direct consequence of both zero electrical resistance and the ability of superconductors to sustain screening currents. Consider a ring of superconducting material placed in the magnetic field. For temperature greater than the critical temperature,  $T > T_c$ , the magnetic flux will pass through the body of the loop as depicted in Fig. When the temperature of the loop is decreased below the critical temperature,  $T < T_c$ , magnetic flux is expelled from the body of the loop. Because resistance vanishes, an electric current once established in a superconducting loop continues indefinitely. If a magnetic flux is trapped in a superconducting ring, the induced current  $I_p$  satisfies the flux quantization condition:

### 4.3 Meissner's Effect

In 1933, Meissner and Ochsenfeld showed that when a superconducting material is placed in a magnetic field it allows magnetic lines of force to pass through, if its temperature is above  $T_c$ . If the temperature is reduced below the critical temperature  $T_c$  then it expels all the flux lines completely out of the specimen and exhibits perfect diamagnetism. This is known as Meissner's effect. Since superconductor exhibits perfect diamagnetism below the critical temperature  $T_c$ , magnetic flux density inside the material is zero.



The expression for magnetic flux density is given by

$$B = \mu_0(M + H) \quad (4.1)$$

Here  $B$  is Magnetic Flux Density,  $M$  is Magnetization and  $H$  is the applied magnetic field strength. For a superconductor,  $B = 0$  at  $T < T_c$ . Thus we get

$$M = -H \quad (4.2)$$

Thus Meissner's Effect signifies the negative magnetic moment associated with superconductors.

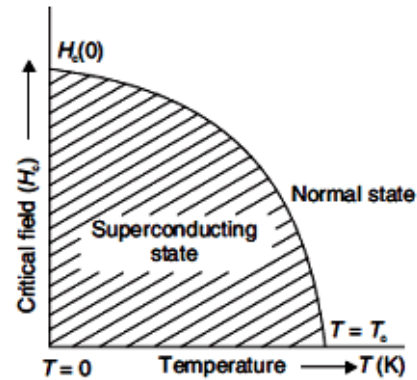
## 4.4 Critical Field and its Temperature Dependence

**Critical field** We know that when superconductor is placed in a magnetic field it expels magnetic flux lines completely out of the body and exhibits a perfect diamagnetism. But if the strength of the magnetic field is further increased, it is found that for a particular value of the magnetic field, material loses its superconducting property and becomes a normal conductor. The value of the magnetic field at which the transition occurs from the Superconducting state to Normal Conducting state is called **Critical Field or Critical Magnetic Field** and is denoted by  $H_c$ . It is found that by reducing the temperature of the material further superconducting property of the material could be restored. Thus, critical field does not destroy the superconducting property of the material completely but only reduces the critical temperature of the material.

The variation of Critical field with temperature below the critical temperature is given by

$$H_c = H_0 \left[ 1 - \frac{T^2}{T_c^2} \right] \quad (4.3)$$

Here  $H_c$  is the Critical field at any temperature  $T$  less than  $T_c$ ,  $H_0$  is the Critical field at  $T = 0K$ .



## 4.5 Critical Current:

The critical current,  $I_c$  of a superconductor represents the maximum electrical current that a superconducting material can carry without losing its superconducting state and reverting to a normal, resistive state. In the absence of an external magnetic field, we can write the critical current,  $I_c$ , that can be passed through a superconducting wire of radius  $r$  as

$$I_c = 2\pi r H_c \quad (4.4)$$

where  $H_c$  is the critical magnetic field of the superconductor. The bulk current that can flow through a superconducting wire per unit area of cross-section without destroying its superconducting properties is called critical current density,  $J_c$  given as

$$J_c = \frac{I_c}{A} = \frac{2\pi r H_c}{\pi r^2} = \frac{2H_c}{r} \quad (4.5)$$

where  $A$  is the area of cross-section of the wire.

## 4.6 Silsbee effect:

Superconductivity is also destroyed if the current exceeds a maximum value, called the critical current  $I_c$ . According to the Silsbee effect, this occurs when the magnetic field at the surface reaches the critical field  $H_c$ . If the strength of the applied magnetic field is increased, the screening (shielding) current must also increase to maintain perfect diamagnetism. However, there is a limit to the density of current which can be carried by the superconducting specimen. Thus, if the gradually increasing current in the superconductor due to increasing applied magnetic field attains this limiting value of the current density, the superconductor makes a transition from the superconducting to normal state. And, this value of the applied magnetic field is known as critical magnetic field.

## 4.7 Derivation for a cylindrical wire using ampere's law:

Consider a circular Ampèrian path of radius  $r$  centered on the wire axis. By symmetry the magnetic induction  $B$  is azimuthal (circumferential) and its magnitude depends only on Ampère's law (magnetostatic form) is

$$\oint B \cdot dl = \mu_0 I_{enc} \quad (4.6)$$

**outside the wire** ( $r \geq 0$ ) the enclosed current is the total current  $I$ , so

$$\oint B dl \cos \theta = \mu_0 I_{enc} \quad (4.7)$$

$\theta = 0, \cos 0 = 1 \Rightarrow \oint B dl \cos(0) = \oint B dl$  here,  $dl$  is the small elemental length,  $R$  is the radius of the amperian loop

$$B \oint dl = \mu_0 I_{enc} \quad (4.8)$$

But

$$\oint dl = 2\pi R \quad (4.9)$$

By substituting in eq (3.7) we get,

$$B(2\pi R) = \mu_0 I \quad (4.10)$$

This is the Magnetic field. If the current is critical current then

$$B = \frac{\mu_0 I_c}{2\pi R} \quad (4.11)$$

But

$$B = \mu_0 H \quad (4.12)$$

$\Rightarrow$

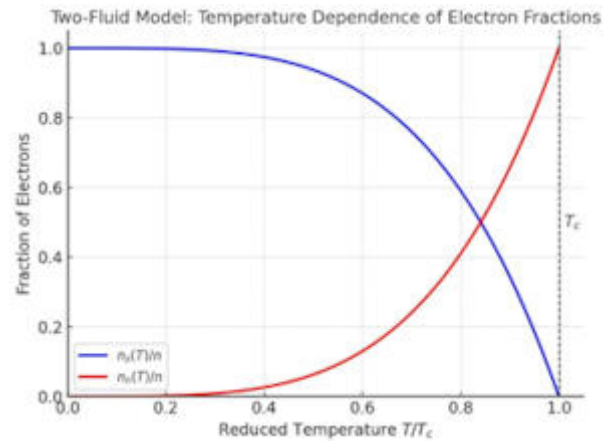
$$\mu_0 H(2\pi R) = \mu_0 I \quad (4.13)$$

This is the Magnetic field. If the current density is critical current then

$$H = \frac{I_c}{2\pi R} \quad (4.14)$$

## 4.8 Two-Fluid Model of Superconductivity:

The two-fluid model, proposed by Gorter and Casimir (1934), is one of the earliest phenomenological approaches to explain the behavior of superconductors. According to this model, below the critical temperature  $T_c$ , the conduction electrons in a superconductor are divided into two interpenetrating components: the normal fluid and the superfluid. The normal fluid consists of electrons that behave like ordinary conduction electrons and are responsible for resistive processes, while the superfluid consists of Cooper-paired electrons that move without scattering and hence carry current without dissipation.



This model explains many important superconducting properties. The zero resistivity arises because the current is carried by the superfluid component, which does not experience scattering. The finite thermal conductivity at low temperatures is explained by the normal fluid, which carries entropy and heat. Furthermore, the electrodynamic behavior (such as the London penetration depth of magnetic fields) can be described by associating magnetic field screening currents with the superfluid component.

Thus, the two-fluid model provides a qualitative and semi-quantitative understanding of superconductivity by postulating a coexistence of two distinct electron populations within the material, one responsible for dissipation and the other for perfect conductivity.

## 4.9 Types of Superconductors

Superconductors are classified into two types

1. Type I Superconductor or Soft Superconductor
2. Type II Superconductor or Hard Superconductor

### Type I Superconductors

paragraph Type I superconductors exhibit complete Meissner's Effect and possess a single value of critical field. The graph of magnetic moment Vs magnetic field is as shown in the Fig.4.1. As the field strength increases the material becomes more and more diamagnetic until  $H$  becomes equal to  $H_c$ . Above  $H_c$  the material allows the flux lines to pass through and exhibits normal conductivity. The value of  $H_c$  is very small for soft superconductors. Therefore soft superconductors cannot withstand high magnetic fields. Therefore they cannot be used for making superconducting magnets. Ex. Hg, Pb and Zn.

### Type II Superconductors

paragraph Superconducting materials, which can withstand high value of critical magnetic fields, are called Hard

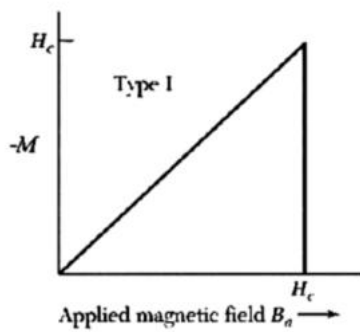


Figure 4.1: Type I Superconductor

Superconductors.

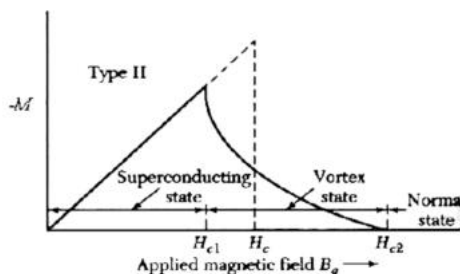
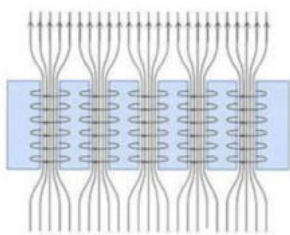


Figure 4.2: Type II Superconductor

The graph of magnetic moment Vs magnetic field is as shown in the Fig.4.2. Hard superconductors are characterized by two critical fields  $H_{c1}$  and  $H_{c2}$ . When applied magnetic field is less than  $H_{c1}$  material exhibits perfect diamagnetism. Beyond  $H_{c1}$  partial flux penetrates and the material is said to be **Vortex State**. Thus flux penetration occurs through small-channelized regions called filaments. As the strength of the field increases further, more and more flux fills the body and thereby decreasing the diamagnetic property of the material. At  $H_{c2}$  flux fills the body completely and material loses its diamagnetic property as well as superconducting property completely.



The value of  $H_{c2}$  is hundreds of times greater than  $H_c$  of soft superconductors. Therefore they are used for making powerful superconducting magnets. Examples:  $NbTi$ ,  $Nb_3Sn$

#### 4.9.1 Explanation for upper critical field

The upper critical field ( $H_{c2}$ ) in superconductors is the maximum magnetic field that destroys superconductivity, occurring when external fields overcome the binding energy of Cooper pairs, driven by two main effects: orbital pair-breaking (Lorentz force on moving pairs) and Pauli paramagnetism (Zeeman splitting of electron spins). It's a key material property, especially for Type-II superconductors, defining the field limit for high-field applications, and is strongly dependent on temperature, decreasing to zero at the critical temperature ( $T_c$ ).

### 4.10 Formation of Cooper Pairs – Mediation of Phonons:

In conventional superconductors, the key microscopic mechanism is the formation of Cooper pairs, bound states of two electrons with opposite momenta and spins. Despite their mutual Coulomb repulsion, electrons effectively attract each other via lattice vibrations called phonons. As one electron passes through the lattice, it distorts the positively charged ion lattice, creating a local region of increased positive charge that attracts a second electron. This effective electron–electron attraction leads to pairing. The binding energy of a Cooper pair is related to the energy gap  $\Delta$  of the superconductor:

$$E_{pair} = 2\Delta \quad (4.15)$$

At zero temperature, the gap  $\Delta(0)$  is approximately related to the critical temperature  $T_c$  as:

$$\Delta(0) \approx 1.76k_B T_c \quad (4.16)$$

where  $k_B$  is the Boltzmann constant. Cooper pairs condense into a coherent quantum state, enabling superconductivity.

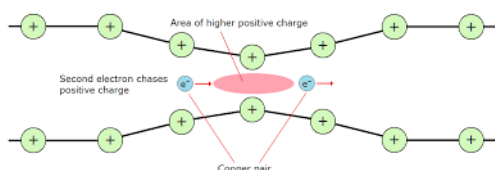
### 4.11 BCS Theory of Superconductivity

Bardeen, Cooper and Schrieffer explained the phenomenon of superconductivity in the year 1957. The essence of the BCS theory is as follows.

Consider an electron approaching a positive ion core and suffers attractive coulomb interaction. Due to this attraction ion core is set in motion and thus distorts that lattice. Let a second electron come in the way of distorted lattice

and interaction between the two occurs which lowers the energy of the second electron. The two electrons therefore interact indirectly through the lattice distortion or the phonon field which lowers the energy of the electrons. The above interaction is interpreted as electron - Lattice - electron interaction through phonon field.

It was shown by Cooper that, this attractive force becomes maximum if two electrons have opposite spins and momentum. The attractive force may exceed coulombs repulsive force between the two electrons below the critical temperature, which results in the formation of bound pair of electrons called cooper pairs.



Below the critical temperature the dense cloud of Cooper pairs form a collective state and the motion all Cooper pairs is correlated resulting in zero resistance of the material.

#### 4.11.1 Phase coherence:

The phase coherence explains why the superconducting current flows without dissipation: a disturbance to a single pair cannot destroy the collective order, since the entire condensate resists scattering in a correlated way. Moreover, the quantum nature of this phase leads to observable macroscopic effects such as the Josephson effect, where current can tunnel between two superconductors depending on the phase difference of their order parameters. Thus, the BCS phase coherent state is the cornerstone of superconductivity, transforming a collection of independent electrons into a single macroscopic quantum entity governed by a unified phase.

#### 4.11.2 Limitations of BCS Theory

Although BCS theory successfully describes conventional low-temperature superconductors, it has several limitations:

1. High- $T_c$  superconductors: BCS cannot fully explain the pairing mechanism in cuprates and iron-based superconductors.
2. Strong-coupling effects: Materials with very strong electron-phonon interactions may require Eliashberg theory, as BCS assumes weak coupling.
3. Non-phonon pairing: BCS relies on phonon-mediated attraction; in systems with other pairing mechanisms (e.g., spin fluctuations), BCS fails.

## 4.12 Examples of Systems with Low and High Electron-Phonon Coupling

1. Low electron-phonon coupling (weak-coupling): Aluminum (Al), Tin (Sn), and Lead (Pb) are conventional superconductors where the BCS approximation is accurate.
2. High electron-phonon coupling (strong-coupling): Niobium (Nb), Mercury (Hg), and lead under high pressure exhibit stronger phonon-mediated pairing, requiring corrections to the BCS predictions.

## 4.13 High Temperature Superconductivity

Superconducting materials which exhibit superconductivity at relatively higher temperatures are called high temperature superconductors. Thus high temperature superconductors possess higher value of critical temperature compared to conventional superconductors. Most of the high temperature superconductors are found to fall into the category of ceramics. In 1986 George Bednorz and Alex Muller discovered a compound containing Lanthanum, Barium, Copper and Oxygen having  $T_c = 30\text{K}$  was developed. In 1987 scientists developed a compound which is an oxide of the form  $YBa_2Cu_3O_7$  which is referred to as 1-2-3 compound with  $T_c > 90\text{K}$  was discovered.

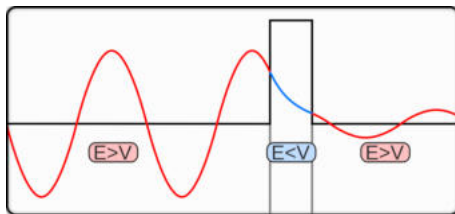
All high temperature superconductors are oxides of copper and bear Perovskite crystal structure characterized by large number of copper-oxygen layers. It was found that addition of extra copper-oxygen layer pushes the critical temperature  $T_c$  to higher values. The super currents are strong in the copper-oxygen layer and weak in the direction perpendicular to the planes. Following is the list of High Temperature Superconductors.

## 4.14 Quantum Tunneling

In classical mechanics, when a particle has insufficient energy, it would not be able to overcome a potential barrier. In the quantum world the particles can often behave like waves. On encountering a barrier, a quantum wave will not end abruptly. Rather its amplitude decreases exponentially. This drop in amplitude corresponds to a drop in the probability of finding a particle further into the barrier. If the barrier is thin enough, then the amplitude may be non-zero on the other side. This would imply that there is a finite probability that some of the particles will tunnel through the barrier.

In regions where the potential energy is higher than the wave's energy, the amplitude of the wave decays exponen-

| Superconductor          | Year | $T_c$ (K) |
|-------------------------|------|-----------|
| $K_yWO_3$               | 1967 | 6.0       |
| $LiTi_{2+y}O_4$         | 1973 | 1.2       |
| $BaPb_{1-y}Bi_yO_3$     | 1975 | 13        |
| $La_{2-y}Ba_yCuO_4$     | 1986 | 30        |
| $YBa_2Cu_3O_{7-y}$      | 1987 | 90        |
| $Ba_{1-y}K_yBiO_3$      | 1988 | 20        |
| $BiSrCaCu_2O_{6+y}$     | 1988 | 105       |
| $TlBa_2Ca_2Cu_3O_{9+y}$ | 1989 | 110       |
| $HgBa_2CaCu_2O_{6+y}$   | 1993 | 120       |
| $GdFeAsO_{1-y}$         | 2008 | 53.5      |



tially. If the region is narrow enough, the wave can have a non-zero amplitude on the other side.

## 4.15 AC and DC Josephson Junctions

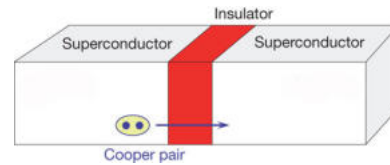
### 4.15.1 Josephson Junction

In 1962, Brian Josephson predicted that Cooper pairs could tunnel through a very thin insulating layer separating two superconductors. The superconductor-insulator-superconductor layer constitutes the Josephson junction.

### 4.15.2 DC Josephson Effects

Consider a Josephson junction consisting of two superconducting metal films separated by a thin oxide barrier of 10 to 20 Å thick. The Cooper pairs tunnel from one side of the junction to the other side easily. This is due to the phase difference introduced by the insulator between the wave function of Cooper pairs on both sides of the junction. Due to the phase difference a super current flows through the junction even if the applied voltage is zero. This is known as the **DC Josephson effect**. Josephson showed that the super current through the junction is given by

$$I_S = I_C \sin \phi_0 \quad (4.17)$$



Here  $\phi_0$  is the phase difference between the wave functions describing Cooper pairs on both sides of the barrier, and  $I_C$  is the critical current at zero voltage condition.  $I_C$  depends on the thickness and width of the insulating layer and the temperature.

### 4.15.3 AC Josephson Effect

If we apply a dc voltage across the Josephson junction, it introduces an additional phase on Cooper pairs during tunneling. As a result a strikingly new phenomenon will be observed. The dc voltage generates an alternating current  $I$  given by

$$I_S = I_C \sin(\phi_0 + \Delta\phi) \quad (4.18)$$

Because of the dc voltage  $V$  applied across the barrier, the energies of Cooper pairs on both sides of the barrier differ in energy by  $2eV$ .

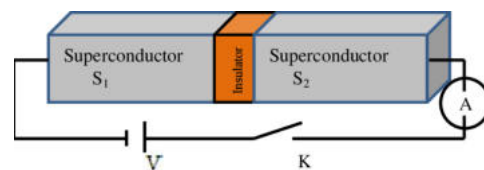


Figure 4.3: DC and AC Josephson Effects

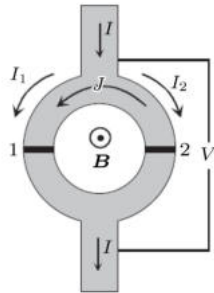
## 4.16 DC and RF SQUIDS

### 4.16.1 SQUID

A superconducting quantum interference device (SQUID) is a device used to measure extremely weak magnetic flux. Thus, it is basically a sensitive magnetometer made of a superconducting ring. The flux penetrating a superconducting loop is quantized in steps of  $\phi_0 = \frac{h}{2e}$ . The heart of a SQUID is a superconducting ring, which contains one or more Josephson junctions. There are two main types of SQUID: DC SQUID and RF (or AC) SQUID.

### 4.16.2 DC Squid

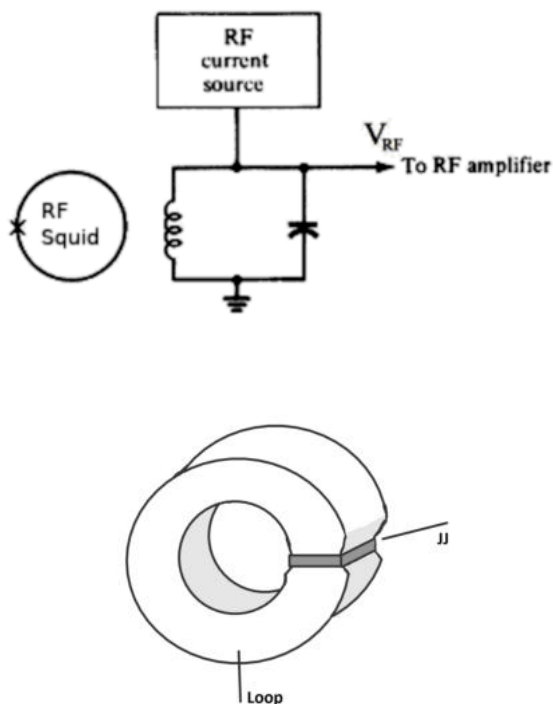
The DC SQUID was invented in 1964 by Robert Jaklevic, John Lambe, Arnold Silver, and James Mercereau. It has two Josephson junctions in parallel in a superconducting loop. It is based on the DC Josephson effect. It relies on the interference of currents from each junction.



The dc SQUID is biased with a dc current equal to about twice the superconducting current  $I_c$ . Dc voltage across the junctions is created. Change in the flux penetrating the loop enhances the current through one JJ and reduces the current through the other. This leads to JJs working asymmetrically, one JJ is driven normally and one is superconducting. This asymmetry provides a feedback current that nulls the flux penetrating the SQUID loop. Total flux within the loop is multiples of  $\phi_0$ . The feedback current is a direct measure of changes in flux applied to the SQUID.

#### 4.16.3 RF (AC) Squid

The RF SQUID was invented in 1965 by Robert Jaklevic, John J. Lambe, Arnold Silver, and James Edward Zimmerman at Ford. It is based on the AC Josephson effect and uses only one Josephson junction. It is less sensitive compared to DC SQUID but is cheaper and easier to manufacture in smaller quantities.



In RF SQUID the flux is coupled into a loop containing a single JJ through an input coil and an RF coil. RF coil is part of a high-Q resonant circuit to read out current changes due to induced flux in the SQUID loop. The tuned circuit is driven by a constant RF oscillator which is weakly coupled to the loop. Measuring the change in the input coil current is done by counting the number of periods the coil produces in the detected RF output, because the detected RF output is a periodic function.

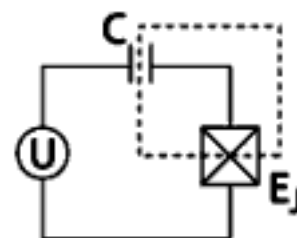
**Note :** The DC SQUIDS offer higher sensitivity, but RF SQUIDS have lower sensitivity. RF SQUIDS are commonly used form of the sensor, because of their ease and low price of manufacturing in small batches.

### 4.17 Applications of Superconductivity in Quantum Computing

Squids find application of magnetometers to measure very small fields like human brain magnetic fields. But the applications of SQUIDS in Quantum computing are as follows.

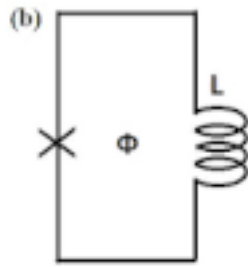
#### 4.17.1 Charge Qubit

In quantum computing, a charge qubit is also known as Cooper-pair box. it is a qubit whose basis states are charge states. The states represent the presence or absence of excess Cooper pairs in the island (dotted region in the figure). In superconducting quantum computing, a charge qubit is formed by a tiny superconducting island coupled by Josephson Junction to a superconducting reservoir.



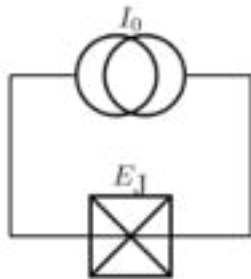
#### 4.17.2 Flux Qubit

Flux qubits (also known as persistent current qubits) are micrometer sized loops of superconducting metal that is interrupted by a number of Josephson junctions. These devices function as quantum bits. The Josephson junctions are designed so that a persistent current will flow continuously when an external magnetic flux is applied. Only an integer number of flux quanta are allowed to penetrate the superconducting ring.



### 4.17.3 Phase Qubit

A phase qubit is a current-biased Josephson junction, operated in the zero voltage state with a non-zero current bias. This employs a single Josephson junction and the two levels are defined by quantum oscillations of the phase difference between the electrodes of the junction. DC Squid is a type of phase qubit.



### 4.18 Model Questions

1. Define Phonon. Describe Mathiessen's rule.
2. Explain the failures of Classical Free Electron Theory of Metals.
3. Mention the Assumptions of Classical Free electron theory of metals.
4. Explain the concept of Fermi Level, Fermi Energy, Density of States.
5. Define Fermi Factor. Discuss the variation of Fermi Factor with Energy and Temperature and represent graphically.
6. Discuss the discovery of Superconductivity and hence discuss the variation of resistivity with temperature in superconductor with critical temperature as reference.
7. State and explain Meissner's Effect.
8. Define Critical field and hence explain its variation with temperature below critical temperature.

9. Distinguish between/Explain Type-1 and Type-2 superconductors.
10. Describe the evolution of high temperature of superconductivity.
11. Elucidate the BCS Theory of Superconductivity.
12. Explain the phenomenon of quantum tunneling.
13. Define a Josephson Junction and hence explain the DC and AC Josephson effects.
14. Define Squid and describe DC and RF Squids.
15. Brief the applications of superconductivity in quantum computing.

### 4.19 Numerical Problems

1. Lead has superconducting transition temperature of  $7.26K$ . If the initial field at  $0K$  is  $50 \times 10^3 Am^{-1}$  Calculate the critical field at  $6K$ .
2. A superconducting tin has a critical temperature of  $3.7K$  at zero magnetic field and a critical field of  $0.0306$  tesla at  $0K$ . Find the critical field at  $2K$ .
3. The superconducting transition temperature of Lead is  $7.26K$ . Calculate the initial field at  $0K$  given the critical field at  $5K$  as  $33.644 \times 10^3 Am^{-1}$
4. Calculate the ratio of critical fields for a superconductor at  $7K$  and  $5K$  give the critical temperature  $8K$ .
5. The critical field for niobium is  $1 \times 10^5 Am^{-1}$  at  $8K$  and  $2 \times 10^5 Am^{-1}$  at  $0K$ . Calculate the transition temperature of the element.

## **Part IV**

# **MODULE 4 : PHOTONICS**



# Chapter 5

# LASER

## 5.1 Introduction

**LASER** is an acronym for Light Amplification by Stimulated Emission of Radiation. The first LASER was built by Theodore H Maiman in the year 1960. Thus it finds various applications starting from industries to communication.

## 5.2 Characteristics of a LASER beam

The LASER beam has the following four characteristics.

1. LASER beam is highly monochromatic.
2. LASER beam is highly coherent.
3. LASER beam is highly directional.
4. LASER is a high intensity beam of light.

## 5.3 Interaction of radiation with matter

The interaction between radiation and matter occurs through the following three processes.

1. Induced absorption
2. Spontaneous emission
3. Stimulated emission

### 5.3.1 Induced Absorption

When a photon of right energy is incident on the atom then the photon is absorbed. This process is induced by the photon and hence it is called Induced Absorption.

Consider an atom in a lower energy state  $E_1$ , it will excite to higher energy state  $E_2$  by absorbing the incident photon of energy  $E = h\nu = E_2 - E_1$ . Here  $E_1$  energy of the lower energy state,  $E_2$  is the energy of the higher energy state,  $h$  is the Planck's constant  $\nu$  is the frequency of photon.

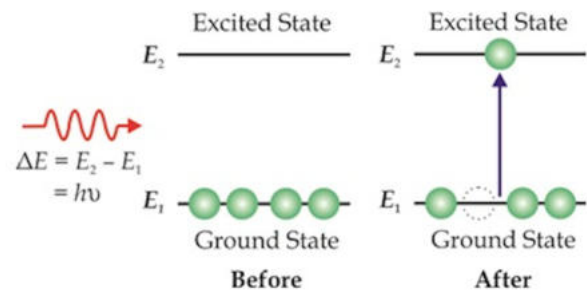


Figure 5.1: Induced absorption

### 5.3.2 Spontaneous Emission

Spontaneous emission is the process of emission of photon, when an atom transits from higher energy level to lower energy level without the influence of any external energy.

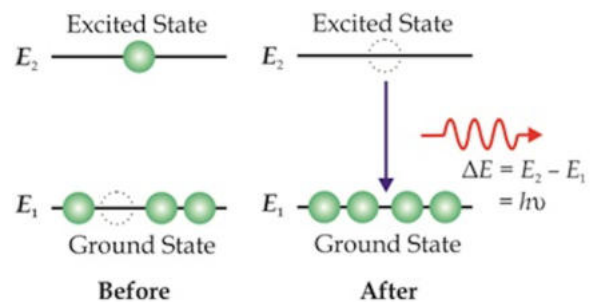


Figure 5.2: Spontaneous emission

An electron in the higher energy state of an atom makes a transition to lower energy state without the action of any external agency. The energy of the photon emitted is given by  $E = h\nu = E_2 - E_1$ . In this process the emitted photons need not travel in the same direction. Thus the light beam is not directional.

### 5.3.3 Stimulated Emission

When a photon of suitable energy interacts with an atom in the higher energy state then the atom is stimulated (Forced) to make transition from higher energy state to a lower energy state with the emission of a photon. Both the incident photon and the emitted photons are coherent and travel in the same direction. Thus the process is called stimulated emission.

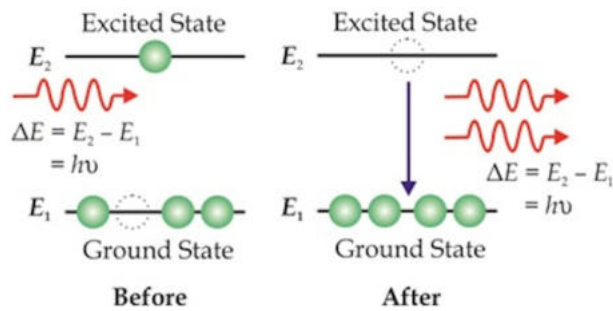


Figure 5.3: Stimulated emission

When a photon of energy  $h\nu = E_2 - E_1$  interacts with an atom in the higher energy state the stimulated emission takes place with the emission of two photons of same energy that are highly directional and coherent. Thus stimulated emission could be used to generate a highly coherent directional beam of light.

## 5.4 Einstein's A and B co-efficients and expression for energy density

Consider a system containing  $N$  atoms and is under thermal equilibrium. Let  $E_1$  and  $E_2$  be the lower and higher energy levels that contain  $N_1$  and  $N_2$  number of atoms respectively. Let the incident energy density of the radiation be  $E_\nu$ . Hence the system absorbs and emits the energy through the following processes. The energy of the photons absorbed and emitted by the atoms is  $E = h\nu = (E_2 - E_1)$

### Rate of induced absorption

The rate of induced absorption is defined as the number of induced absorption per second per unit volume in unit time. Rate of absorption depends on

1. Number of atoms in the lower energy state  $N_1$ .
2. The incident energy density  $E_\nu$ .

Hence

1. Rate of Induced absorption  $\propto N_1 E_\nu$

2. Rate of Induced absorption  $= B_{12} N_1 E_\nu$

Here  $B_{12}$  is proportionality constant called Einsteins coefficient of Induced absorption.

### Rate of spontaneous emission:

The number of spontaneous emission per unit volume in unit time is called rate of spontaneous emission. Rate of spontaneous emission depends on

Since spontaneous emission is a voluntary process it is independent of energy density  $E_\nu$ . The rate of spontaneous emission depends only on the number of atoms in the higher energy state  $N_2$ . Thus

1. Rate of spontaneous emission  $\propto N_2$
2. Rate of Spontaneous emission  $= A_{21} N_2$

Here  $A_{21}$  is the proportionality constant called Einstein's co-efficient of spontaneous emission.

### Rate of stimulated emission

The number of stimulated emission per unit volume in unit time is called rate of stimulated emission. Rate of stimulated emission depends upon,

1. Number of atoms in the higher energy state ( $N_2$ )
2. The energy density ( $E_\nu$ ).

Hence

1. The Rate of stimulated emission  $\propto N_2 E_\nu$
2. Rate of stimulated emission  $= B_{21} N_2 E_\nu$

Here the proportionality constant called  $B_{21}$  is Einstein's coefficient of stimulated emission.

Under Thermal Equilibrium the total Energy of the System remains unchanged. Hence Rate of Absorption is equal to rate of emission.

$\therefore$  Rate of Induced Absorption = [Rate of Spontaneous emission + Rate of Stimulated Emission]

$\therefore$

$$B_{12} N_1 E_\nu = A_{21} N_2 + B_{21} N_2 E_\nu \quad (5.1)$$

$$(B_{12} N_1 - B_{21} N_2) E_\nu = A_{21} N_2$$

$$E_\nu = \frac{A_{21} N_2}{B_{12} N_1 - B_{21} N_2} \quad (5.2)$$

$$E_\nu = \frac{A_{21}}{B_{12} \frac{N_1}{N_2} - B_{21}} \quad (5.3)$$

$$E_\nu = \frac{A_{21}}{B_{21}} \left[ \frac{1}{\frac{B_{12}}{B_{21}} \frac{N_1}{N_2} - 1} \right] \quad (5.4)$$

According to Boltzmann relation the we have

$$\frac{N_2}{N_1} = e^{\frac{-h\nu}{kT}} \quad (5.5)$$

or we can re-write as,

$$\frac{N_1}{N_2} = e^{\frac{h\nu}{kT}} \quad (5.6)$$

Here  $h$  is the Planck's constant,  $c$  is the speed of light in vacuum,  $\lambda$  is the wavelength of the photon,  $k$  is the Boltzmann constant and  $T$  is the absolute temperature. Substituting for  $\frac{N_1}{N_2}$  in equation 5.4

$$E_\nu = \frac{A_{21}}{B_{21}} \left[ \frac{1}{\frac{B_{12}}{B_{21}} \left( e^{\frac{h\nu}{kT}} \right) - 1} \right] \quad (5.7)$$

According to Planck's radiation law, the equation for energy density in the frequency domain is given by

$$E_\nu = \frac{8\pi h\nu^3}{c^3} \left[ \frac{1}{e^{\frac{h\nu}{kT}} - 1} \right] \quad (5.8)$$

on comparing equations 5.7 and 5.8 we can get

$$\frac{A_{21}}{B_{21}} = \frac{8\pi h\nu^3}{c^3} \quad (5.9)$$

and

$$\frac{B_{12}}{B_{21}} = 1 \quad (5.10)$$

or  $B_{12} = B_{21}$

This means that Probability of Induced absorption is equal to Probability of Stimulated emission. Hence  $A_{21}$  &  $B_{21}$  can be replaced by  $A$  &  $B$ . Thus equation 5.7 could be written as

$$E_\nu = \frac{A}{B} \left[ \frac{1}{e^{\frac{h\nu}{kT}} - 1} \right] \quad (5.11)$$

Hence the expression for energy density in terms of Einstein's co-efficient  $A$  and  $B$ .

## 5.5 Requisites of a LASER system

The three requisites of a LASER system are,

1. Excitation source for pumping action
2. Active medium that supports meta-stable states
3. LASER cavity

### 5.5.1 Energy Source or Pumping Mechanism

In order to achieve population inversion more and more atoms are to be moved to higher energy state and is called pumping. This is achieved by supplying suitable energy using an energy source. If optical energy is used then the pumping is called optical pumping and if electrical energy is used then the pumping is called electrical pumping.

### 5.5.2 Active medium

Population inversion occurs at certain stage in the active medium due to the absorption of energy. The active medium supports meta-stable states. After this stage the active medium is capable of emitting LASER light.

### 5.5.3 Resonant cavity (or) LASER cavity

The LASER Cavity consists of an active medium bound between two highly parallel mirrors. The reflection of photons from the mirrors results in multiple traverse of photons through the active medium inducing more and more stimulated emissions. Thus amplification of light is achieved. This also helps to tap certain permissible part of LASER energy from the active medium. The cavity resonates and the output will be maximum when the distance  $L$  between the mirrors is equal to an integral multiple of  $\frac{\lambda}{2}$ . Here  $\lambda$  is the wavelength of incident suitable radiation. the length of the LASER cavity is expressed as

$$L = \frac{n\lambda}{2} \quad (5.12)$$

## 5.6 Types of LASER

### 1. Gas LASER

A gas laser is a laser that uses a gas or a mixture of gases as its active medium to produce a coherent light beam.

### 2. Liquid LASER

A liquid laser is a type of laser that uses an organic dye dissolved in a liquid solvent as its active medium. These are also known as dye lasers and are notable for their ability to produce a wide, tunable range of wavelengths, from the ultraviolet to the infrared spectrum.

### 3. Solid state LASER

A solid-state laser uses a solid material, such as a crystal or glass, as the active medium, rather than a liquid or gas. The solid material is doped with specific ions that absorb energy and produce a coherent light beam through stimulated emission.

### 4. Semiconductor LASER (Diode LASER)

## 5.7 Semiconductor LASER or Diode LASER

### Introduction

Semiconductor diode LASER is an LED with heavily doped P and N sections. First semiconductor LASER was fabricated in 1962 using  $Ga - As$  by *Hall* with his co-workers. It is a low cost and high efficiency LASER.

### Construction

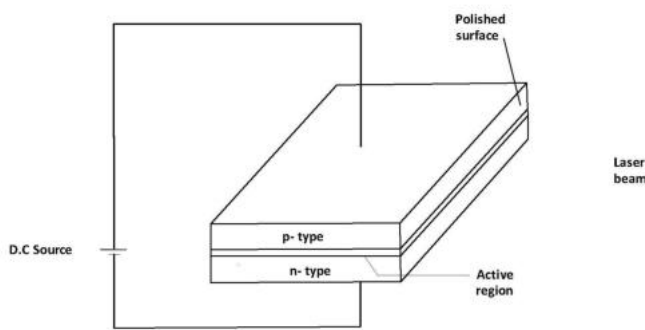
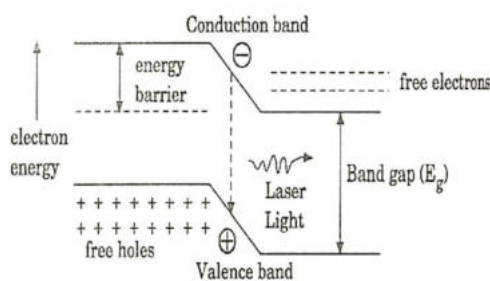


Figure 5.4: Semiconductor Diode LASER

### Construction

The  $Ga - As$  LASER diode belongs to direct band gap semiconductors. The  $n$ -section is derived by doping the substrate with *Tellurium* and  $p$ -section is derived by doping the substrate with *Zinc*. The diode used is in the form of a cube with dimension 0.4 mm. The depletion region is of thickness 0.1 micrometer and lies horizontal as shown in the figure 5.4. The current is passed through the ohmic contacts provided to the top and bottom faces. The front and back faces are polished and made highly parallel to each other to have a LASER cavity. The other two faces are roughened.



### Working

The Diode is forward biased using an external source. Therefore electrons and holes flow across junction. Hence the current flows through the diode. When a hole meets an electron it recombines with electron emitting a photon. This could be considered as the transition of electron from conduction band to valance band. When the current is low spontaneous emission is predominant. If the current is sufficiently high population inversion is achieved. The photons liberated initially due to spontaneous emissions induce further stimulated emissions. The LASER cavity helps in the amplification of light. Finally this results in an avalanche of photons and hence the LASER action is achieved. If the  $GaAs$  semiconductor is used then the wavelength of the LASER emitted is 840nm.

### Advantages

1. It has excellent efficiency
2. The output can be modulated
3. It produces both continuous wave output or pulsed output.
4. It is highly economical

### Applications

1. It is used in optical fiber communication.
2. It is used in commercial CD recording and reading.
3. It is used in Barcode Reader, Laser printing and Laser Cooling.

## 5.8 Single Photon Source

A single Photon source is a device or a system which emit one photon at a time.

### 5.8.1 Single Photon Sources are listed below

1. Quantum dot single Photon source
2. Color center single Photon source
3. Atomic and ionic single Photon source
4. Molecule based single Photon source
5. Attenuators

### 5.8.2 Attenuators

The following types of attenuators are used to obtain single photons:

1. Neutral Density (ND) Filters
2. Variable Optical Attenuators (VOA)
3. Polarization-based Attenuators

## 5.9 Use of Attenuators for single photon sources

It is possible to produce single photons by attenuating the light source using suitable attenuators. Actually, attenuators do not produce single photons, but they reduce the power of the beam and thereby reduce the average number of photons in the beam to exactly one. In this case, the output behaves as a pseudo-single photon source.

When a high-loss attenuator is placed in the path of a laser beam, it either absorbs or scatters or polarizes the light and removes a very large number of photons from the beam. The term  $\mu$  (called mean photon number per pulse) decides the photon number in the transmitted beam. By setting  $\mu = 0.1$ , it is possible to have only one photon in the transmitted beam.

### 5.9.1 Optical Modulators

An optical modulator is a device used to control or modify the property of light, such as its amplitude, phase, polarization, or frequency. Optical modulators are essential components in a wide range of applications, including telecommunications, signal processing, and laser systems.

1. **Electro-optic modulators** : They utilize the electro-optic effect, where the refractive index of a material changes in response to an electric field.

**Electro-optic modulators** There are two types in this namely:

- (a) Pockel's effect
- (b) Kerr effect

### 5.10 Pockel's effect (Linear electro-optic effect)

When an electric field is applied to some transparent crystals, the refractive index (RI) of the crystal changes. As the applied voltage increases, the refractive index increases linearly. The incident light splits into two parts due to the change in RI, and hence birefringence occurs. This phenomenon is known as the Pockel's effect.

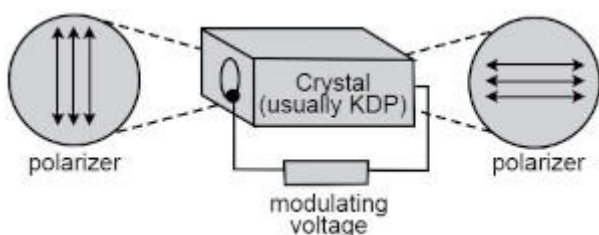


Figure 5.5: Pockel effect

Pockel's effect occurs only in non-centrosymmetric materials such as:

1. Lithium niobate ( $LiNbO_3$ )
2. Lithium tantalate ( $LiTaO_3$ )
3. Potassium di-deuterium phosphate (KDP)
4.  $\beta$  Barium borate (BBO)

**It also occurs in compound semiconductors such as:**

1. Gallium arsenide (GaAs)
2. Indium phosphide (InP)

Using Pockel's effect, the phase, polarization state, and amplitude of the light passing through the crystals can be modulated by applying an electric field.

#### Pockel's Cell

A Pockel's cell is a device that works on the principle of the Pockel's effect. It consists of a crystal placed between two electrodes. When light passes through the crystal and a very high voltage is applied across it, the phase of the light changes due to the change in refractive index. By using suitable polarizers in the cell, one can also modulate the amplitude of the light.

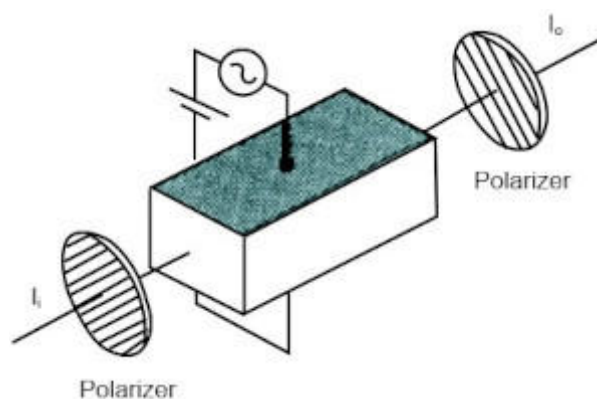


Figure 5.6: Pockel cell

### 5.11 Kerr effect (linear electro-optic effect)

When electric field is applied to transparent crystals, the refractive index (RI) of the crystal changes. The change in the RI is directly proportional to square of the applied voltage or square of the applied field strength. This is known as Kerr effect.

The RI is different for light polarized parallel to and perpendicular to applied field. This change makes the crystal birefringent (even though it was not birefringent earlier!). Hence the crystal can be used as 'electrically controllable wave plate'. If  $\Delta n$  is the change in RI of the beams,  $\lambda$  is the wavelength of the incident light and  $E$  is the applied field strength then

$$\Delta n = \lambda K E^2 \quad (5.13)$$

Where  $K$  is known as Kerr constant.

The Kerr cell is placed between two crossed polarizers. With no voltage applied, the polarizers block all light from passing through them. When voltage is applied, the Kerr effect causes the polarization of the light to rotate.

This rotation allows some light to pass through the second polarizer, effectively "opening" the shutter. By modulating the voltage, one can precisely control the amount of light transmitted.

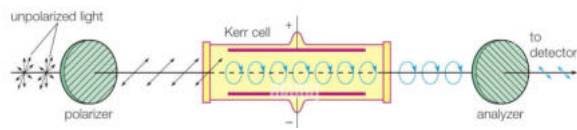


Figure 5.7: Kerr cell

## 5.12 Photodetectors

A photo-detector is an opto-electronic device that converts light into an electrical signal.

### 5.12.1 Types of Photo-detectors

1. Photo-diodes
2. Photo-transistors
3. Photo-multiplier tubes
4. Single Photon Avalanche Diode

#### 5.12.2 Photo-multiplier tubes

A photomultiplier tube (PMT) converts weak light into an electrical signal through the principles of the photoelectric effect and secondary emission.

##### Principle:

The operation of a PMT is based on two fundamental physics principles:

**Photo-electric effect:** When a suitable frequency of electromagnetic radiation strikes a specific photo-emissive material, they transfer their energy to electrons, causing the electrons to be released.

**Secondary electron emission:** A single, accelerated electron striking an electrode can knock multiple secondary electrons free from the electrode surface.

##### Construction:

A photomultiplier tube is an evacuated glass tube containing several key components:

**Photocathode:** A photoemissive material, often a cesium-antimony alloy, that is mounted inside the tube at the end where light enters. When light strikes the photo-cathode, it emits electrons via the photoelectric effect.

**Focusing electrodes:** These guide and accelerate the photo-electrons from the photo-cathode toward the first electrode.

**Electrodes:** A series of multiple electrodes (typically 8 to

19) arranged in a cascade. Each electrode is held at a progressively higher positive potential than the previous one.

**Anode:** The final electrode in the tube, which is at the highest positive potential. It collects the amplified cloud of electrons.

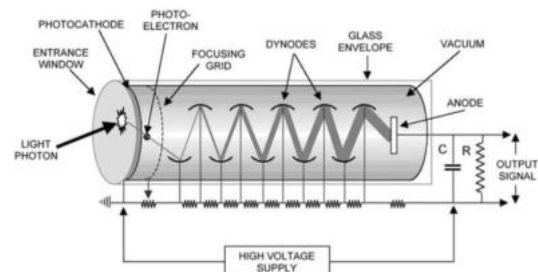


Figure 5.8: Photo-multiplier tubes

##### Working:

The working of a PMT involves a three-step process:

**Photo-emission:** When a photon enters the tube and strikes the photo-cathode, its energy causes an electron to be ejected. This initial electron is called a photo-electron.

**Electron multiplication:** The photo-electron is accelerated by a high-voltage electric field toward the first electrode, which is set at a higher potential. When the electron strikes the first electrode, it releases several secondary electrons. These new electrons are then accelerated toward the second electrode, which is at an even higher potential. This process is repeated down the chain of electrode, creating a cascade or avalanche of electrons. The gain, or multiplication factor, of the tube is determined by the number of electrode stages and the voltage difference between them.

**Signal collection:** The large, multiplied cloud of electrons finally reaches the anode. This surge of electrons produces a measurable output current, which is then sent to an external circuit for signal processing. The output current is directly proportional to the initial number of photons that hit the photo-cathode.

#### 5.12.3 Single Photon Avalanche Diode (SPAD):

##### Principle:

The biasing a p-n semiconductor junction with a reverse voltage that exceeds its characteristic breakdown voltage, placing it in a metastable state where an extremely high electric field is present. When a single photon is absorbed within the SPAD's active region, it generates a single electron-hole pair. This carrier is then accelerated by the intense electric field, triggering an avalanche of impact ionization events. This chain reaction, or avalanche multiplication, results in a massive, self-sustaining avalanche

of charge carriers, which is registered as a sharp, macroscopic current pulse. This *pulse*<sup>s</sup> leading edge indicates the precise arrival time of the photon, converting a single quantum of light into a strong electrical signal.

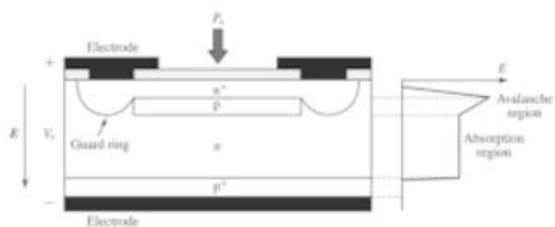
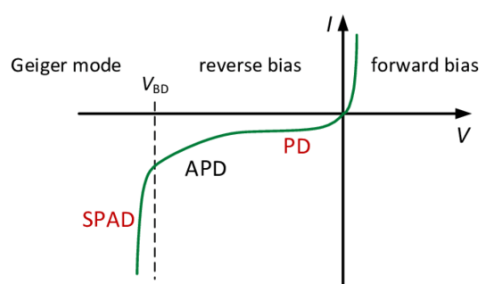


Figure 5.9: Single Photon Avalanche Diode (SPAD)



#### Construction:

SPADs are typically fabricated using standard semiconductor processes, with silicon being the most common material for detecting visible light. A crucial feature is the guard ring, an implant surrounding the active junction that prevents premature breakdown at the device edges by confining the high electric field to the intended avalanche region. To minimize noise, high-performance SPADs are often built on high-quality epitaxial layers. Advanced designs may utilize back-illumination to increase quantum efficiency by allowing light to enter from the thinned back of the wafer. For applications requiring high-resolution imaging, SPADs are frequently arranged into arrays, sometimes integrated monolithically with readout electronics using 3D stacking techniques to improve the pixel fill factor.

#### Working:

The working cycle of a SPAD is a continuous sequence of arming, detection, quenching, and resetting. In the quiescent state, the SPAD is biased at a voltage above its breakdown level but remains non-conductive until a charge carrier initiates an avalanche. Upon detection, the single photon triggers the avalanche current, which is then swiftly terminated by a quenching circuit. Quenching can be passive, using a high-value series resistor, or active, using a fast electronic circuit to quickly lower the bias voltage. After quenching, the device is reset by restoring the bias

voltage, making it ready to detect the next photon. This robust and repeatable process effectively converts single light quanta into standardized electrical pulses, making SPADs ideal for a wide range of applications that demand single-photon sensitivity and high timing resolution. **I-V characteristics of Single Photon Avalanche Diode,**

#### 5.12.4 Superconducting nanowire single photon detectors (SNSPD's):

Superconducting nanowire single photon detectors (SNSPD's) are the advanced devices for single photon detection. They work on the principle that when a photon is incident it breaks cooper pairs and reduces the local critical current. By measuring, the changes in the current can detect the photon.

#### Construction:

The SNSPD consists of a thin and narrow superconducting wire. The wire is arranged in a curved fashion (called Meander geometry) to achieve high detection efficiency. The wire is kept at a temperature below the critical temperature and connected to DC source. The biasing current ( $I_B$ ) in the wire is maintained just below its critical current ( $I_C$ ). The whole arrangement is connected to a special type of electronic circuit called as readout.

#### Working:

When a photon is incident on the wire, it breaks the cooper pairs and local critical current decreases (fig i). If the critical current decreases below the bias current then the superconductivity is spoiled and a local non-superconducting region or 'Hotspot' is created. The resistance of the hotspot is larger than the resistance of the readout (about 1 Kilo-ohm). As a result bias current flows through the readout. (In fact the readout acts as a shunt for the current). This produces a measurable voltage pulse (fig ii) such that one photon pulse. By measuring this voltage, we can detect the incident photon. With most of the bias current flowing through the readout, the non superconducting region rapidly cools and returns to the superconducting state, ready to detect another photon.

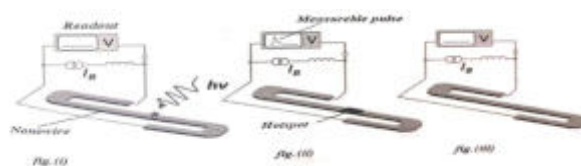


Figure 5.10: Photo-multiplier tubes

**Advantages:**

SNSPD's offers many important advantages over SPADs. Some of them are 1. Unparalleled detection efficiency and timing precision. 2. Negligible intrinsic dark counts. 3. Ultrafast detection rates. 4. Broadband operation.

**Uses:**

They are becoming more popular in the applications of bio photonics, quantum optics and quantum computing.

**5.13 Model Questions**

1. What is LASER? Enumerate the Characteristics of a LASER Beam.
2. Discuss the three possible ways through which radiation and matter interaction can take place.
3. Explain the terms, (i) Induced absorption, (ii) Spontaneous emission, (iii) Stimulated emission, (iv) Population inversion, (v) Meta-stable state & (vi) Resonant cavity.
4. Explain the rates of absorption and emission and hence derive an expression for energy density using Einstein's A and B coefficients.
5. Explain requisites of LASER system.
6. What is Semiconductor LASER? Describe with energy band diagram the construction & working of Semiconductor diode LASER along with applications.
7. Describe the Single Photon Source and Explain the use of Attenuators for the single photon sources.
8. Explain Optical Modulators.
9. Describe Pockel's effect, and Explain the construction & working of Pockel's effect along with pockel's cell.
10. Explain the Kerr effect.
11. Describe the construction & working of Photo-multiplier tube.
12. Sketch the Construction & working of Single Photon Avalanche Diode(SPAD).
3. A pulsed LASER has an average power output 1.5 mW per pulse and pulse duration is 20 ns. The number of photons emitted per pulse is estimated to be  $1.047 \times 10^8$ . Find the wavelength of the emitted LASER.
4. A pulsed LASER with power 1 mW lasts for 10 ns. If the number of photons emitted per pulse is  $5 \times 10^7$ . Calculate the wavelength of LASER.
5. A Ruby LASER emits a pulse of 20 ns duration with average power per pulse being 100 kW. If the number of photons in each pulse is  $6.981 \times 10^{15}$ , calculate the wavelength of photons.
6. In a LASER system when the energy difference between two energy levels is  $2 \times 10^{-19}$  J, the average power output of LASER beam is found to be 4 mW. Calculate number of photons emitted per second.

**5.14 Numerical Problems**

1. The average power output of a LASER beam of wavelength  $6500 \text{ \AA}$  is 10 mW. Find the number of photons emitted per second by the LASER source.
2. The average power of a LASER beam of wavelength  $6328 \text{ \AA}$  is 5 mW. Find the number of photons emitted per second by the LASER source.

## Chapter 6

# Optical Fibers

### 6.1 Introduction

Optical fibers are the wires and strands made of transparent dielectrics which guide light over longer distances using the phenomenon of **Total Internal Reflection**. Many optical fibers are bundled together and are given a protective layer of covering using an insulating material. This bundle is called Optical Fiber Cable or Fiber Bundle (Bundle Fiber).

**Construction:** The sectional view of a typical optical fiber is as shown in the figure. It has three regions named Core, Cladding and Sheath.

1. The innermost light guiding region is called Core.
2. The layer covering core and helps in total internal reflection of light is called Cladding or Clad.
3. The outermost protective layer is called Sheath (Coating). The sheath protects the fiber from mechanical stress and chemical reactions.

The optical fiber is designed to support total internal reflection and hence the RI of core  $n_1$  is made greater than the RI of cladding  $n_2$ . A typical fiber will be of the order of few microns.

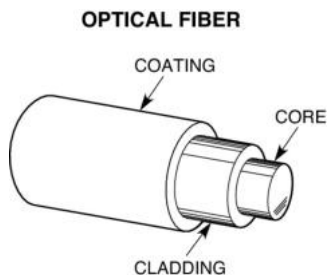


Figure 6.1: Optical fiber construction

### 6.2 Total Internal Reflection

Consider a ray of light moving from a denser medium to rarer medium. As a result the incident ray of light bends away from the normal. Hence the angle of refraction is greater than the angle of incidence. As the angle of incidence increases the angle of refraction also increases. For a particular angle of incidence  $\theta_c$  the refracted ray grazes the interface separating the two media. The corresponding angle of incidence  $\theta_c$  is called Critical Angle. If the angle of incidence is greater than the critical angle then all the light is turned back into the same medium and is called Total Internal Reflection.

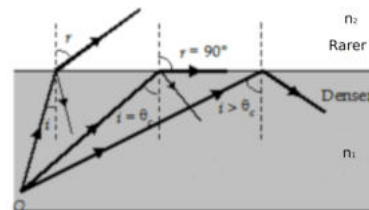


Figure 6.2: Total Internal Reflection

According to Snell's law

$$n_1 \sin \theta_1 = n_2 \sin \theta_2$$

$$\text{when } \theta_1 = \theta_c \text{ then, } \theta_2 = 90^\circ$$

$$n_1 \sin \theta_c = n_2 \sin 90^\circ$$

$$\sin \theta_c = \frac{n_2}{n_1}$$

$$\theta_c = \sin^{-1} \left( \frac{n_2}{n_1} \right) \quad (6.1)$$

### 6.3 Angle of acceptance and Numerical aperture

Acceptance angle ( $\theta_0$ ) is the maximum angle of incidence with which the ray is sent into the fiber core which allows

the incident light to be guided by the core. It is also called as waveguide acceptance angle or acceptance cone half angle.

In optics, the numerical aperture (NA) of an optical fiber is a dimensionless number that characterizes the range of angles over which the fiber can accept light. Numerical aperture represents the light gathering capability of optical fiber and it is given by  $NA = \sin\theta_0$ .

### 6.3.1 Condition for propagation

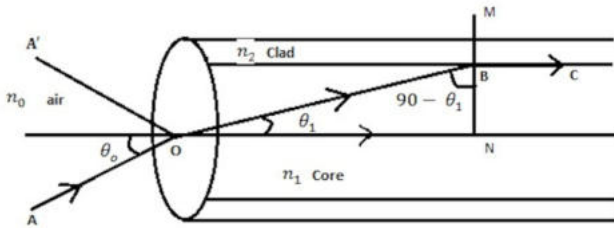


Figure 6.3: Ray propagation in the fiber

Consider an optical fiber with core made of refractive index  $n_1$  & cladding made of material refractive index  $n_2$ . Let  $n_0$  be the refractive index of the surrounding medium. Let a ray of light  $AO$  entering into core at an angle of incidence  $\theta_0$  w.r.t fiber axis. Then it is refracted along  $OB$  at an angle  $\theta_1$  & meet core-cladding interface at critical angle of incidence ( $\theta_c = 90 - \theta_1$ ). Then the refracted ray grazes along  $BC$ . On applying Snell's law at  $O$ , we get

$$\begin{aligned} n_0 \sin\theta_0 &= n_1 \sin\theta_1 \\ \therefore \sin\theta_0 &= \frac{n_1}{n_0} \sin\theta_1 \end{aligned} \quad (6.2)$$

On applying Snell's law at point  $B$ , we get

$$\begin{aligned} n_1 \sin(90^\circ - \theta_1) &= n_2 \sin 90^\circ \\ n_1 \cos\theta_1 &= n_2 \\ \therefore \cos\theta_1 &= \frac{n_2}{n_1} \end{aligned} \quad (6.3)$$

**From trigonometric identity**

$$\begin{aligned} \sin^2\theta_1 + \cos^2\theta_1 &= 1 \\ \sin\theta_1 &= \sqrt{1 - \cos^2\theta_1} \end{aligned}$$

using equation 1.25

$$\begin{aligned} \sin\theta_1 &= \sqrt{1 - \left(\frac{n_2}{n_1}\right)^2} \\ \sin\theta_1 &= \sqrt{\frac{n_1^2 - n_2^2}{n_1^2}} \end{aligned}$$

$$\sin\theta_1 = \frac{1}{n_1} \sqrt{n_1^2 - n_2^2} \quad (6.4)$$

use equation (1.26) in equation (1.24) we have,

$$\begin{aligned} \sin\theta_0 &= \frac{n_1}{n_0} \frac{1}{n_1} \sqrt{n_1^2 - n_2^2} \\ \sin\theta_0 &= \frac{1}{n_0} \sqrt{n_1^2 - n_2^2} \end{aligned} \quad (6.5)$$

Numerical aperture  $N.A = \sin\theta_0$

$$N.A = \frac{1}{n_0} \sqrt{n_1^2 - n_2^2} \quad (6.6)$$

If the fiber is in air  $n_0 = 1$  then,

$$N.A = \sin\theta_0 = \sqrt{n_1^2 - n_2^2} \quad (6.7)$$

Light is transmitted through the fiber only when

$$\theta_i \leq \theta_0 \quad (6.8)$$

$$\sin\theta_i \leq \sin\theta_0 \quad (6.9)$$

$$\sin\theta_i \leq \sqrt{n_1^2 - n_2^2} \quad (6.10)$$

$$\sin\theta_i \leq N.A \quad (6.11)$$

This is the condition for propagation. Light will be transmitted through the optical fiber with multiple total internal reflections when the above condition is satisfied.

### 6.3.2 Fractional RI Change

Fractional index change ( $\Delta$ ) is defined as the ratio of difference in refractive indices of core & cladding to the refractive index of the core.

$$\Delta = \frac{n_1 - n_2}{n_1} \quad (6.12)$$

$$n_1 \Delta = n_1 - n_2 \quad (6.13)$$

### 6.3.3 Relation between NA and $\Delta$

consider the equation

$$\begin{aligned} N.A &= \sin\theta_0 = \sqrt{n_1^2 - n_2^2} \\ N.A &= \sqrt{(n_1 + n_2)(n_1 - n_2)} \end{aligned} \quad (6.14)$$

For small difference of  $n_1$  &  $n_2$ , we can have,

$$n_1 \approx n_2 \quad (6.15)$$

$$\begin{aligned} \therefore n_1 + n_2 &\approx 2n_1 \\ \therefore N.A &= \sqrt{(2n_1)(n_1 \Delta)} \\ N.A &= \sqrt{2n_1^2 \Delta} \\ N.A &= n_1 \sqrt{2\Delta} \end{aligned} \quad (6.16)$$

Thus the Numerical aperture can be increased by increasing the fractional index change.

## 6.4 Modes of propagation

Though optical fiber should support any numbers of rays for propagation practically. But it is found that the optical fiber allows only a certain restricted number of rays for propagation. The maximum number of rays or paths supported by the fiber for the propagation of light is called *Modes of propagation*.

**V-number (Normalised Frequency of the fiber)** An Optical fiber may be characterized by one more parameter called V-number. This determines the Number of modes supported by an optical fiber for the propagation.

$$V = \frac{\pi d}{\lambda} \sqrt{n_1^2 - n_2^2} \quad (6.17)$$

$$V = \frac{\pi d}{\lambda} N.A \quad (6.18)$$

here  $d$  is the diameter of the core,  $\lambda$  is wavelength,  $n_1$  is the refractive index of the core and  $n_2$  is the refractive index of the cladding. N.A is numerical Aperture. If the fiber is surrounded by a medium of refractive index  $n_0$ , then

$$V = \frac{\pi d}{\lambda} \frac{\sqrt{n_1^2 - n_2^2}}{n_0} \quad (6.19)$$

If  $V \gg 1$ , the number of modes supported by fiber can be determined using the formula

$$N \cong \frac{V^2}{2}$$

$$N \cong \frac{\pi^2 d^2}{2 \lambda^2} \left( \frac{n_1^2 - n_2^2}{n_0^2} \right) \quad (6.20)$$

## 6.5 Attenuation

The optical energy (signal) passing through the optical fiber gets reduced progressively. This is due to attenuation. It is also called the fiber loss or significant loss. The attenuation is measured in terms of attenuation co-efficient. The attenuation co-efficient  $\alpha$  is defined as the ratio of optical power output to the optical power input for a fiber of length  $L$  and for a given wavelength of propagating light. It is expressed in  $dB/km$ . Attenuation co-efficient is given by

$$\alpha = \frac{-10}{L} \log_{10} \left( \frac{P_{out}}{P_{in}} \right) \quad dB/km \quad (6.21)$$

Here  $L$  is the length of the cable in  $km$ ,  $P_{in}$  is Power of optical signal at launching end (input power) &  $P_{out}$  is Power of optical signal at receiving end (output power)

The attenuation in fibers gives is due to the following three losses

1. Absorption losses
2. Scattering loss (due to Rayleigh Scattering)
3. Geometric Effects (Radiation losses)

### 6.5.1 Absorption loss

In this type of loss, the loss of signal power occurs due to absorption of photons associated with the signal. Photons are absorbed either by impurities in the glass fiber or by pure glass material itself. Absorption loss is wavelength dependent. Thus absorption loss is classified in to two types.

**Extrinsic absorption :** Extrinsic loss in an optical fiber is due to the absorption of light by the impurities such as hydroxide ions and transition metal ions such as iron, chromium, cobalt and copper.

**Intrinsic absorption** Intrinsic loss in fiber is due to the absorption of light by the material of the fiber glass itself. The intrinsic losses are insignificant.

### 6.5.2 Scattering loss

Light traveling through the core can get scattered by impurities or small regions with sudden change in refractive index. Rayleigh scattering varies as  $\alpha = \frac{1}{\lambda^4}$  and leads to significant power loss at smaller wavelengths. The scattering results in loss of photons. Rayleigh scattering is responsible for maximum losses in optical fibers.

### 6.5.3 Geometric effects

These may occur due to manufacturing defects like irregularities in fibre dimensions during drawing process or during coating, cabling or insulation processes. The microscopic bends are the bends with radii greater than fiber diameter. The microscopic bends couple light between the various guided modes of the fiber and some of them then leak through the fiber.

## 6.6 Mach-Zehnder interferometer (MZI)

The Mach-Zehnder interferometer (MZI) is a type of beam division interferometer used in electro optic modulator, quantum cryptography, quantum computing, quantum entanglement etc. It is highly configurable instrument compare to other interferometers.

### 6.6.1 Construction and working

MZI consists of a monochromatic source S (usually laser source), two half silvered mirrors called beam splitters BS-1 & BS-2, two plane mirrors M1 & M2 and two detectors D1 and D2 arranged as shown in fig. The light emitted by

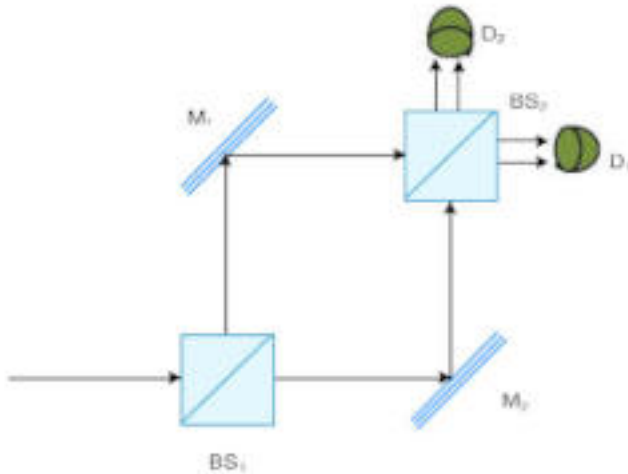


Figure 6.4: Mach-Zehnder interferometer

the source is incident on BS-1 where it splits into two. The reflected beam R travels along X-axis and the transmitted beam T travels along Y-axis. If the reflected beam R and the transmitted beam T are then reflected by M1 & M2 and finally arrive at BS-2 and the path length remains same, they then interfere constructively.

If may be observed that beam R at D1 and T at D2 with a phase difference of  $180^\circ$  (as shown in fig). Hence, they interfere destructively at D2. Beam proceeding towards D1 will have same phase and interfere constructively. Hence only D1 receives the light or now some sample is placed in the path of R beam or T, the path of S is blocked then both D1 and D2 receives the light.

MZI can be used to estimate the phase shift of photon going through one interferometer by estimating the probabilities. In that case beams splitters are matrix operators. This estimation is very crucial in the understanding of quantum super positions, quantum computing, QKD etc.

## 6.7 Model Questions

1. Define the terms: (i) angle of acceptance, (ii) numerical aperture, (iii) fractional index change (iv) modes of propagation & (v) refractive index profile.
2. Obtain an expression for numerical aperture and arrive at the condition for propagation.
3. What is attenuation? Explain the factors contributing to the fiber loss.

4. What is attenuation coefficient? Mention the expression for the attenuation coefficient.
5. What are the advantages of optical communications over other conventional types of communication?
6. Discuss the advantages and disadvantages of an optical communication.
7. Describe the construction & working of Mach-Zehnder interferometer

## 6.8 Numerical Problems

1. Calculate the numerical aperture and angle of acceptance for an optical fiber having refractive indices 1.563 and 1.498 for core and cladding respectively.
2. The refractive indices of the core and cladding of a step index optical fiber are 1.45 and 1.4 respectively and its core diameter is  $45\mu\text{m}$ . Calculate its fractional refractive index change and numerical aperture.
3. Calculate numerical aperture, acceptance angle and critical angle of a fiber having a core RI 1.50 and cladding RI 1.45.
4. An optical fiber has a numerical aperture of 0.32. The refractive index of cladding is 1.48. Calculate the refractive index of the core, the acceptance angle of the fiber and the fractional index change.
5. An optical signal propagating in a fiber retains 85% of input power after traveling a distance of 500 m in the fiber. Calculate the attenuation coefficient.
6. An optical fiber has core RI 1.5 and RI of cladding is 3% less than the core index. Calculate the numerical aperture, angle of acceptance critical angle.
7. The numerical aperture of an optical fiber is 0.2 when surround by air. Determine the RI of its core, given the RI of the cladding is 1.59. Also find the acceptance angle when the fiber is in water of RI 1.33.
8. The angle of acceptance of an optical fiber is  $30^\circ$  when kept in air. Find the angle of acceptance when it is in medium of refractive index 1.33.
9. Calculate NA, V-number and number of modes in an optical fiber of core diameter  $50\mu\text{m}$ , core and cladding refractive indices 1.41 and 1.4 respectively at wavelength 820 nm.
10. For a step index optical fiber RI of core is 1.45 and RI of cladding is 1.40 and its core diameter is  $45\mu\text{m}$ . Calculate its relative refractive index difference, V-number at wavelength 1000 nm and the number of modes.

11. Calculate the number of modes of an optical fiber will transmit using the following data  $n_{core} = 1.50$ ,  $n_{clad} = 1.48$ , core radius =  $50\mu m$ ,  $\lambda = 1\mu m$ .
12. An optical fiber of 600 m long has input power of 120 mW which emerges out with power of 90 mW. Find attenuation in fiber.
13. The attenuation of light in an optical fiber is 3.6 dB/km. What fraction of its initial intensity is remains after i) 1 km and ii) 3 km ?
14. The attenuation of light in an optical fiber is 2.2 dB/km. What fraction of its initial intensity is remains after i) 2 km and ii) 6 km ?