

Module 2: Fluid Kinematics

Lagrangian and Eulerian Description of fluid flow, Continuity, Momentum, Bernoulli, Energy Equation and Momentum equation, problems solving using any programming language

KINEMATICS OF FLOW

INTRODUCTION

Kinematics is defined as that branch of science which deals with motion of particles without considering the forces causing the motion. The velocity at any point in a flow field at any time is studied in this branch of fluid mechanics. Once the velocity is known, then the pressure distribution and hence forces acting on the fluid can be determined. In this chapter, the methods of determining velocity and acceleration are discussed.

METHODS OF DESCRIBING FLUID MOTION

The fluid motion is described by two methods. They are —(i) Lagrangian Method, and (ii) Eulerian Method. In the Lagrangian method, a **single fluid particle** is followed during its motion and its velocity, acceleration, density, etc., are described. In case of Eulerian method, the velocity, acceleration, pressure, density etc., are described **at a point** in flow field. The Eulerian method is commonly used in fluid mechanics.

TYPES OF FLUID FLOW

The fluid flow is classified as :

- (i) Steady and unsteady flows ;
- (ii) Uniform and non-uniform flows ;
- (iii) Laminar and turbulent flows ;
- (iv) Compressible and incompressible flows ;
- (v) Rotational and irrotational flows ; and
- (vi) One, two and three-dimensional flows.

Steady and Unsteady Flows. Steady flow is defined as that type of flow in which the fluid characteristics like velocity, pressure, density, etc., at a point do not change with time. Thus for steady flow, mathematically, we have

$$\left(\frac{\partial V}{\partial t}\right)_{x_0, y_0, z_0} = 0, \left(\frac{\partial p}{\partial t}\right)_{x_0, y_0, z_0} = 0, \left(\frac{\partial \rho}{\partial t}\right)_{x_0, y_0, z_0} = 0$$

where (x_0, y_0, z_0) is a fixed point in fluid field.

Unsteady flow is that type of flow, in which the velocity, pressure or density at a point changes with respect to time. Thus, mathematically, for unsteady flow

$$\left(\frac{\partial V}{\partial t}\right)_{x_0, y_0, z_0} \neq 0, \left(\frac{\partial p}{\partial t}\right)_{x_0, y_0, z_0} \neq 0 \text{ etc.}$$

Uniform and Non-uniform Flows. Uniform flow is defined as that type of flow in which the velocity at any given time does not change with respect to space (*i.e.*, length of direction of the flow). Mathematically, for uniform flow

$$\left(\frac{\partial V}{\partial s}\right)_{t = \text{constant}} = 0$$

where ∂V = Change of velocity

∂s = Length of flow in the direction S .

Non-uniform flow is that type of flow in which the velocity at any given time changes with respect to space. Thus, mathematically, for non-uniform flow

$$\left(\frac{\partial V}{\partial s}\right)_{t = \text{constant}} \neq 0.$$

Laminar and Turbulent Flows. Laminar flow is defined as that type of flow in which the fluid particles move along well-defined paths or stream line and all the stream-lines are straight and parallel. Thus the particles move in laminas or layers gliding smoothly over the adjacent layer. This type of flow is also called stream-line flow or viscous flow.

Turbulent flow is that type of flow in which the fluid particles move in a *zig-zag* way. Due to the movement of fluid particles in a *zig-zag* way, the eddies formation takes place which are responsible

for high energy loss. For a pipe flow, the type of flow is determined by a non-dimensional number $\frac{VD}{\nu}$

called the Reynold number,

where D = Diameter of pipe

V = Mean velocity of flow in pipe

and ν = Kinematic viscosity of fluid.

If the Reynold number is less than 2000, the flow is called laminar. If the Reynold number is more than 4000, it is called turbulent flow. If the Reynold number lies between 2000 and 4000, the flow may be laminar or turbulent.

Compressible and Incompressible Flows. Compressible flow is that type of flow in which the density of the fluid changes from point to point or in other words the density (ρ) is not constant for the fluid. Thus, mathematically, for compressible flow

$$\rho \neq \text{Constant}$$

Incompressible flow is that type of flow in which the density is constant for the fluid flow. Liquids are generally incompressible while gases are compressible. Mathematically, for incompressible flow

$$\rho = \text{Constant.}$$

Rotational and Irrotational Flows. Rotational flow is that type of flow in which the fluid particles while flowing along stream-lines, also rotate about their own axis. And if the fluid particles while flowing along stream-lines, do not rotate about their own axis then that type of flow is called irrotational flow.

One-, Two- and Three-Dimensional Flows. **One-dimensional flow** is that type of flow in which the flow parameter such as velocity is a function of time and one space co-ordinate only, say x . For a steady one-dimensional flow, the velocity is a function of one-space-co-ordinate only. The variation of velocities in other two mutually perpendicular directions is assumed negligible. Hence mathematically, for one-dimensional flow

$$u = f(x), v = 0 \text{ and } w = 0$$

where u , v and w are velocity components in x , y and z directions respectively.

Two-dimensional flow is that type of flow in which the velocity is a function of time and two rectangular space co-ordinates say x and y . For a steady two-dimensional flow the velocity is a function of two space co-ordinates only. The variation of velocity in the third direction is negligible. Thus, mathematically for two-dimensional flow

$$u = f_1(x, y), v = f_2(x, y) \text{ and } w = 0.$$

Three-dimensional flow is that type of flow in which the velocity is a function of time and three mutually perpendicular directions. But for a steady three-dimensional flow the fluid parameters are functions of three space co-ordinates (x , y and z) only. Thus, mathematically, for three-dimensional flow

$$u = f_1(x, y, z), v = f_2(x, y, z) \text{ and } w = f_3(x, y, z).$$

RATE OF FLOW OR DISCHARGE (Q)

It is defined as the quantity of a fluid flowing per second through a section of a pipe or a channel. For an incompressible fluid (or liquid) the rate of flow or discharge is expressed as the volume of fluid flowing across the section per second. For compressible fluids, the rate of flow is usually expressed as the weight of fluid flowing across the section. Thus

(i) For liquids the units of Q are m^3/s or litres/s

(ii) For gases the units of Q is kgf/s or Newton/s

Consider a liquid flowing through a pipe in which

A = Cross-sectional area of pipe

V = Average velocity of fluid across the section

Then discharge

$$Q = A \times V.$$

...(5.1)

CONTINUITY EQUATION

The equation based on the principle of conservation of mass is called continuity equation. Thus for a fluid flowing through the pipe at all the cross-section, the quantity of fluid per second is constant. Consider two cross-sections of a pipe as shown in Fig. 5.1.

Let V_1 = Average velocity at cross-section 1-1

ρ_1 = Density at section 1-1

A_1 = Area of pipe at section 1-1

and V_2, ρ_2, A_2 are corresponding values at section, 2-2.

Then rate of flow at section 1-1 = $\rho_1 A_1 V_1$

Rate of flow at section 2-2 = $\rho_2 A_2 V_2$

According to law of conservation of mass

Rate of flow at section 1-1 = Rate of flow at section 2-2

or
$$\rho_1 A_1 V_1 = \rho_2 A_2 V_2 \quad \dots(5.2)$$

Equation (5.2) is applicable to the compressible as well as incompressible fluids and is called **Continuity Equation**. If the fluid is incompressible, then $\rho_1 = \rho_2$ and continuity equation (5.2) reduces to

$$A_1 V_1 = A_2 V_2 \quad \dots(5.3)$$

Problem 5.1 The diameters of a pipe at the sections 1 and 2 are 10 cm and 15 cm respectively. Find the discharge through the pipe if the velocity of water flowing through the pipe at section 1 is 5 m/s. Determine also the velocity at section 2.

Solution. Given :

At section 1,

$$D_1 = 10 \text{ cm} = 0.1 \text{ m}$$

$$A_1 = \frac{\pi}{4} (D_1)^2 = \frac{\pi}{4} (.1)^2 = 0.007854 \text{ m}^2$$

$$V_1 = 5 \text{ m/s.}$$

At section 2,

$$D_2 = 15 \text{ cm} = 0.15 \text{ m}$$

$$A_2 = \frac{\pi}{4} (.15)^2 = 0.01767 \text{ m}^2$$

(i) Discharge through pipe is given by equation (5.1)

or

$$Q = A_1 \times V_1$$

$$= 0.007854 \times 5 = \mathbf{0.03927 \text{ m}^3/\text{s. Ans.}}$$

Using equation (5.3), we have $A_1 V_1 = A_2 V_2$

$$(ii) \therefore V_2 = \frac{A_1 V_1}{A_2} = \frac{0.007854}{0.01767} \times 5.0 = \mathbf{2.22 \text{ m/s. Ans.}}$$

Problem 5.2 A 30 cm diameter pipe, conveying water, branches into two pipes of diameters 20 cm and 15 cm respectively. If the average velocity in the 30 cm diameter pipe is 2.5 m/s, find the discharge in this pipe. Also determine the velocity in 15 cm pipe if the average velocity in 20 cm diameter pipe is 2 m/s.

Solution. Given :

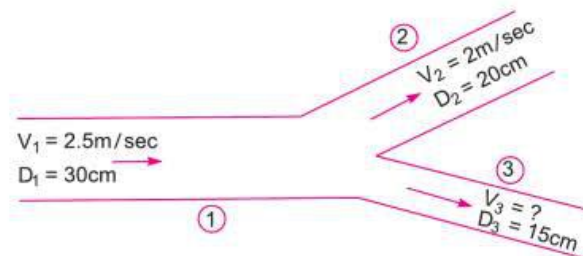


Fig. 5.3

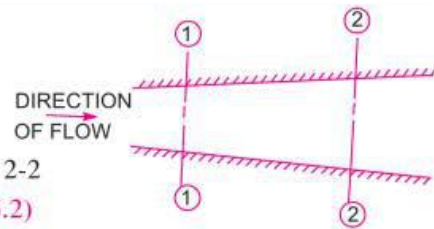


Fig. 5.1 Fluid flowing through a pipe.

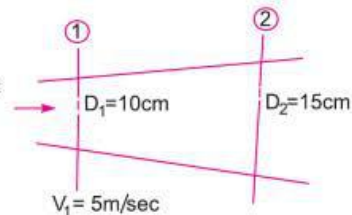


Fig. 5.2

$$\begin{aligned}
 D_1 &= 30 \text{ cm} = 0.30 \text{ m} \\
 \therefore A_1 &= \frac{\pi}{4} D_1^2 = \frac{\pi}{4} \times .3^2 = 0.07068 \text{ m}^2 \\
 V_1 &= 2.5 \text{ m/s} \\
 D_2 &= 20 \text{ cm} = 0.20 \text{ m} \\
 \therefore A_2 &= \frac{\pi}{4} (.2)^2 = \frac{\pi}{4} \times .4 = 0.0314 \text{ m}^2, \\
 V_2 &= 2 \text{ m/s} \\
 D_3 &= 15 \text{ cm} = 0.15 \text{ m} \\
 \therefore A_3 &= \frac{\pi}{4} (.15)^2 = \frac{\pi}{4} \times 0.225 = 0.01767 \text{ m}^2
 \end{aligned}$$

Find (i) Discharge in pipe 1 or Q_1

(ii) Velocity in pipe of dia. 15 cm or V_3

Let Q_1 , Q_2 and Q_3 are discharges in pipe 1, 2 and 3 respectively.

Then according to continuity equation

$$Q_1 = Q_2 + Q_3 \quad \dots(1)$$

(i) The discharge Q_1 in pipe 1 is given by

$$Q_1 = A_1 V_1 = 0.07068 \times 2.5 \text{ m}^3/\text{s} = \mathbf{0.1767 \text{ m}^3/\text{s. Ans.}}$$

(ii) Value of V_3

$$Q_2 = A_2 V_2 = 0.0314 \times 2.0 = 0.0628 \text{ m}^3/\text{s}$$

Substituting the values of Q_1 and Q_2 in equation (1)

$$0.1767 = 0.0628 + Q_3$$

$$\therefore Q_3 = 0.1767 - 0.0628 = 0.1139 \text{ m}^3/\text{s}$$

$$\text{But } Q_3 = A_3 \times V_3 = 0.01767 \times V_3 \quad \text{or} \quad 0.1139 = 0.01767 \times V_3$$

$$\therefore V_3 = \frac{0.1139}{0.01767} = \mathbf{6.44 \text{ m/s. Ans.}}$$

Problem 5.3 Water flows through a pipe AB 1.2 m diameter at 3 m/s and then passes through a pipe BC 1.5 m diameter. At C, the pipe branches. Branch CD is 0.8 m in diameter and carries one-third of the flow in AB. The flow velocity in branch CE is 2.5 m/s. Find the volume rate of flow in AB, the velocity in BC, the velocity in CD and the diameter of CE.

Solution. Given :

Diameter of pipe AB,	$D_{AB} = 1.2 \text{ m}$
Velocity of flow through AB,	$V_{AB} = 3.0 \text{ m/s}$
Dia. of pipe BC,	$D_{BC} = 1.5 \text{ m}$
Dia. of branched pipe CD,	$D_{CD} = 0.8 \text{ m}$
Velocity of flow in pipe CE,	$V_{CE} = 2.5 \text{ m/s}$
Let the flow rate in pipe	$AB = Q \text{ m}^3/\text{s}$
Velocity of flow in pipe	$BC = V_{BC} \text{ m/s}$
Velocity of flow in pipe	$CD = V_{CD} \text{ m/s}$

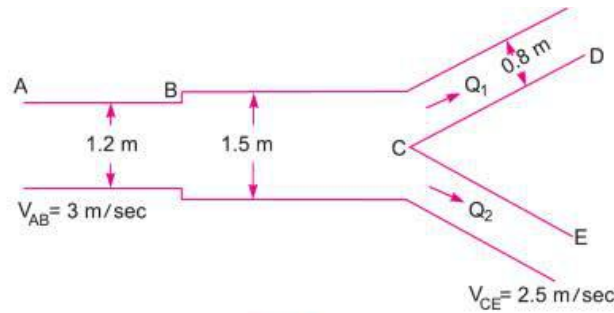


Fig. 5.4

Diameter of pipe $CE = D_{CE}$

Then flow rate through $CD = Q/3$

and flow rate through $CE = Q - Q/3 = \frac{2Q}{3}$

(i) Now volume flow rate through $AB = Q = V_{AB} \times \text{Area of } AB$

$$= 3.0 \times \frac{\pi}{4} (D_{AB})^2 = 3.0 \times \frac{\pi}{4} (1.2)^2 = 3.393 \text{ m}^3/\text{s. Ans.}$$

(ii) Applying continuity equation to pipe AB and pipe BC ,

$$V_{AB} \times \text{Area of pipe } AB = V_{BC} \times \text{Area of pipe } BC$$

$$\text{or } 3.0 \times \frac{\pi}{4} (D_{AB})^2 = V_{BC} \times \frac{\pi}{4} (D_{BC})^2$$

$$\text{or } 3.0 \times (1.2)^2 = V_{BC} \times (1.5)^2$$

Divide by $\frac{\pi}{4}$

$$\text{or } V_{BC} = \frac{3 \times 1.2^2}{1.5^2} = 1.92 \text{ m/s. Ans.}$$

(iii) The flow rate through pipe

$$CD = Q_1 = \frac{Q}{3} = \frac{3.393}{3} = 1.131 \text{ m}^3/\text{s}$$

$$\therefore Q_1 = V_{CD} \times \text{Area of pipe } CD \times \frac{\pi}{4} (D_{CD})^2$$

$$\text{or } 1.131 = V_{CD} \times \frac{\pi}{4} \times 0.8^2 = 0.5026 V_{CD}$$

$$\therefore V_{CD} = \frac{1.131}{0.5026} = 2.25 \text{ m/s. Ans.}$$

(iv) Flow rate through CE ,

$$Q_2 = Q - Q_1 = 3.393 - 1.131 = 2.262 \text{ m}^3/\text{s}$$

$$\therefore Q_2 = V_{CE} \times \text{Area of pipe } CE = V_{CE} \frac{\pi}{4} (D_{CE})^2$$

$$\text{or } 2.263 = 2.5 \times \frac{\pi}{4} \times (D_{CE})^2$$

$$\text{or } D_{CE} = \sqrt{\frac{2.263 \times 4}{2.5 \times \pi}} = \sqrt{1.152} = 1.0735 \text{ m}$$

$\therefore$ Diameter of pipe $CE = 1.0735 \text{ m. Ans.}$

Problem 5.4 A 25 cm diameter pipe carries oil of sp. gr. 0.9 at a velocity of 3 m/s. At another section the diameter is 20 cm. Find the velocity at this section and also mass rate of flow of oil.

Solution. Given :

at section 1,

$$D_1 = 25 \text{ cm} = 0.25 \text{ m}$$

$$A_1 = \frac{\pi}{4} D_1^2 = \frac{\pi}{4} \times 0.25^2 = 0.049 \text{ m}^2$$

$$V_1 = 3 \text{ m/s}$$

at section 2,

$$D_2 = 20 \text{ cm} = 0.2 \text{ m}$$

$$A_2 = \frac{\pi}{4} (0.2)^2 = 0.0314 \text{ m}^2$$

$$V_2 = ?$$

Mass rate of flow of oil = ?

Applying continuity equation at sections 1 and 2,

$$A_1 V_1 = A_2 V_2$$

$$\text{or } 0.049 \times 3.0 = 0.0314 \times V_2$$

$$\therefore V_2 = \frac{0.049 \times 3.0}{0.0314} = 4.68 \text{ m/s. Ans.}$$

$$\text{Mass rate of flow of oil} = \text{Mass density} \times Q = \rho \times A_1 \times V_1$$

$$\text{Sp. gr. of oil} = \frac{\text{Density of oil}}{\text{Density of water}}$$

$$\therefore \text{Density of oil} = \text{Sp. gr. of oil} \times \text{Density of water}$$

$$= 0.9 \times 1000 \text{ kg/m}^3 = \frac{900 \text{ kg}}{\text{m}^3}$$

$$\therefore \text{Mass rate of flow} = 900 \times 0.049 \times 3.0 \text{ kg/s} = 132.23 \text{ kg/s. Ans.}$$

Problem 5.5 A jet of water from a 25 mm diameter nozzle is directed vertically upwards. Assuming that the jet remains circular and neglecting any loss of energy, that will be the diameter at a point 4.5 m above the nozzle, if the velocity with which the jet leaves the nozzle is 12 m/s.

Solution. Given :

$$\text{Dia. of nozzle, } D_1 = 25 \text{ mm} = 0.025 \text{ m}$$

$$\text{Velocity of jet at nozzle, } V_1 = 12 \text{ m/s}$$

$$\text{Height of point A, } h = 4.5 \text{ m}$$

$$\text{Let the velocity of the jet at a height 4.5 m} = V_2$$

Consider the vertical motion of the jet from the outlet of the nozzle to the point A (neglecting any loss of energy).

$$\text{Initial velocity, } u = V_1 = 12 \text{ m/s}$$

$$\text{Final velocity, } V = V_2$$

$$\text{Value of } g = -9.81 \text{ m/s}^2 \text{ and } h = 4.5 \text{ m}$$

$$\text{Using, } V^2 - u^2 = 2gh, \text{ we get}$$

$$V_2^2 - 12^2 = 2 \times (-9.81) \times 4.5$$

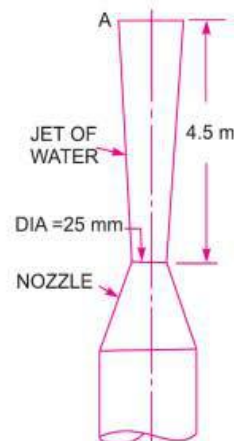


Fig. 5.5

$$\therefore V_2 = \sqrt{12^2 - 2 \times 9.81 \times 4.5} = \sqrt{144 - 88.29} = 7.46 \text{ m/s}$$

Now applying continuity equation to the outlet of nozzle and at point A, we get

$$A_1 V_1 = A_2 V_2$$

$$\text{or } A_2 = \frac{A_1 V_1}{V_2} = \frac{\frac{\pi}{4} D_1^2 \times V_1}{V_2} = \frac{\pi \times (0.025)^2 \times 12}{4 \times 7.46} = 0.0007896$$

Let D_2 = Diameter of jet at point A.

$$\text{Then } A_2 = \frac{\pi}{4} D_2^2 \text{ or } 0.0007896 = \frac{\pi}{4} \times D_2^2$$

$$\therefore D_2 = \sqrt{\frac{0.0007896 \times 4}{\pi}} = 0.0317 \text{ m} = \mathbf{31.7 \text{ mm. Ans.}}$$

CONTINUITY EQUATION IN THREE-DIMENSIONS

Consider a fluid element of lengths dx , dy and dz in the direction of x , y and z . Let u , v and w are the inlet velocity components in x , y and z directions respectively. Mass of fluid entering the face $ABCD$ per second

$$\begin{aligned} &= \rho \times \text{Velocity in } x\text{-direction} \times \text{Area of } ABCD \\ &= \rho \times u \times (dy \times dz) \end{aligned}$$

$$\text{Then mass of fluid leaving the face } EFGH \text{ per second} = \rho u \, dydz + \frac{\partial}{\partial x} (\rho u \, dydz) \, dx$$

$\therefore$ Gain of mass in x -direction

$$= \text{Mass through } ABCD - \text{Mass through } EFGH \text{ per second}$$

$$= \rho u \, dydz - \rho u \, dydz - \frac{\partial}{\partial x} (\rho u \, dydz) \, dx$$

$$= - \frac{\partial}{\partial x} (\rho u \, dydz) \, dx$$

$$= - \frac{\partial}{\partial x} (\rho u) \, dx \, dydz$$

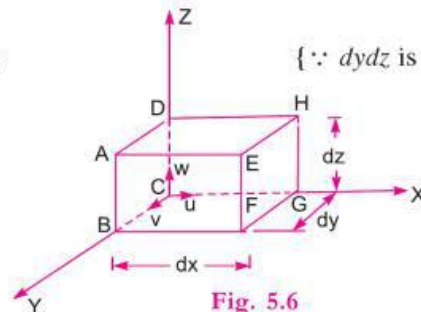
{ $\because dydz$ is constant }

Similarly, the net gain of mass in y -direction

$$= - \frac{\partial}{\partial y} (\rho v) \, dx \, dydz$$

and in z -direction

$$= - \frac{\partial}{\partial z} (\rho w) \, dx \, dydz$$



$$\therefore \text{Net gain of masses} = - \left[\frac{\partial}{\partial x} (\rho u) + \frac{\partial}{\partial y} (\rho v) + \frac{\partial}{\partial z} (\rho w) \right] dx \, dydz$$

Since the mass is neither created nor destroyed in the fluid element, the net increase of mass per unit time in the fluid element must be equal to the rate of increase of mass of fluid in the element. But mass

of fluid in the element is $\rho \cdot dx \cdot dy \cdot dz$ and its rate of increase with time is $\frac{\partial}{\partial t} (\rho \cdot dx \cdot dy \cdot dz)$ or

$$\frac{\partial \rho}{\partial t} \cdot dx \cdot dy \cdot dz.$$

Equating the two expressions,

$$\text{or} \quad - \left[\frac{\partial}{\partial x} (\rho u) + \frac{\partial}{\partial y} (\rho v) + \frac{\partial}{\partial z} (\rho w) \right] dx dy dz = \frac{\partial \rho}{\partial t} \cdot dx dy dz$$

$$\text{or} \quad \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho u) + \frac{\partial}{\partial y} (\rho v) + \frac{\partial}{\partial z} (\rho w) = 0 \quad [\text{Cancelling } dx \cdot dy \cdot dz \text{ from both sides}] \quad \dots(5.3A)$$

Equation (5.3A) is the continuity equation in cartesian co-ordinates in its most general form. This equation is applicable to :

- (i) Steady and unsteady flow,
- (ii) Uniform and non-uniform flow, and
- (iii) Compressible and incompressible fluids.

For steady flow, $\frac{\partial \rho}{\partial t} = 0$ and hence equation (5.3A) becomes as

$$\frac{\partial}{\partial x} (\rho u) + \frac{\partial}{\partial y} (\rho v) + \frac{\partial}{\partial z} (\rho w) = 0 \quad \dots(5.3B)$$

If the fluid is incompressible, then ρ is constant and the above equation becomes as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad \dots(5.4)$$

Equation (5.4) is the continuity equation in three-dimensions. For a two-dimensional flow, the component $w = 0$ and hence continuity equation becomes as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \quad \dots(5.5)$$

5.6.1 Continuity Equation in Cylindrical Polar Co-ordinates. The continuity equation in cylindrical polar co-ordinates (*i.e.*, r, θ, z co-ordinates) is derived by the procedure given below.

Consider a two-dimensional incompressible flow field. The two-dimensional polar co-ordinates are r and θ . Consider a fluid element $ABCD$ between the radii r and $r + dr$ as shown in Fig. 5.7. The angle subtended by the element at the centre is $d\theta$. The components of the velocity V are u_r in the radial direction and u_θ in the tangential direction. The sides of the element are having the lengths as

Side $AB = r d\theta$, $BC = dr$, $DC = (r + dr) d\theta$, $AD = dr$.

The thickness of the element perpendicular to the plane of the paper is assumed to be unity.

Consider the flow in radial direction

Mass of fluid entering the face AB per unit time

$$= \rho \times \text{Velocity in } r\text{-direction} \times \text{Area}$$

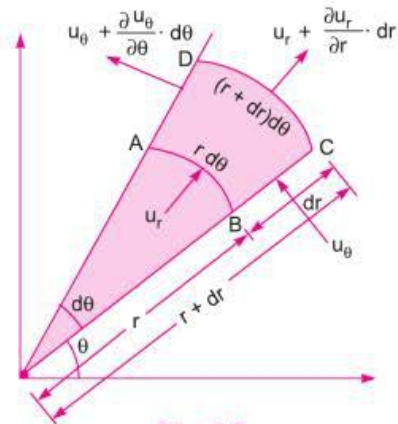


Fig. 5.7

$$\begin{aligned}
 &= \rho \times u_r \times (AB \times 1) \quad (\because \text{Area} = AB \times \text{Thickness} = rd\theta \times 1) \\
 &= \rho \times u_r \times (rd\theta \times 1) = \rho \cdot u_r \cdot rd\theta
 \end{aligned}$$

Mass of fluid leaving the face CD per unit time

$$\begin{aligned}
 &= \rho \times \text{Velocity} \times \text{Area} \\
 &= \rho \times \left(u_r + \frac{\partial u_r}{\partial r} \cdot dr \right) \times (CD \times 1) \quad (\because \text{Area} = CD \times 1) \\
 &= \rho \times \left(u_r + \frac{\partial u_r}{\partial r} \cdot dr \right) \times (r + dr)d\theta \quad [\because CD = (r + dr)d\theta] \\
 &= \rho \times \left[u_r \times r + u_r \cdot dr + r \frac{\partial u_r}{\partial r} \cdot dr + \frac{\partial u_r}{\partial r} (dr)^2 \right] d\theta \\
 &= \rho \left[u_r \times r + u_r \times dr + r \frac{\partial u_r}{\partial r} \cdot dr \right] d\theta
 \end{aligned}$$

[The term containing $(dr)^2$ is very small and has been neglected]

$\therefore$ Gain of mass in r -direction per unit time

$$\begin{aligned}
 &= (\text{Mass through } AB - \text{Mass through } CD) \text{ per unit time} \\
 &= \rho \cdot u_r \cdot rd\theta - \rho \left[u_r \cdot r + u_r \cdot dr + r \frac{\partial u_r}{\partial r} \cdot dr \right] d\theta \\
 &= \rho \cdot u_r \cdot rd\theta - \rho \cdot u_r \cdot r \cdot d\theta - \rho \left[u_r \cdot dr + r \frac{\partial u_r}{\partial r} \cdot dr \right] d\theta \\
 &= -\rho \left[u_r \cdot dr + r \frac{\partial u_r}{\partial r} \cdot dr \right] \cdot d\theta \\
 &= -\rho \left[\frac{u_r}{r} + \frac{\partial u_r}{\partial r} \right] r \cdot dr \cdot d\theta \quad \begin{array}{l} \text{[This is written in this form because} \\ \text{(} r \cdot d\theta \cdot dr \cdot 1 \text{) is equal to volume of} \\ \text{element]} \end{array}
 \end{aligned}$$

Now consider the flow in θ -direction

Gain in mass in θ -direction per unit time

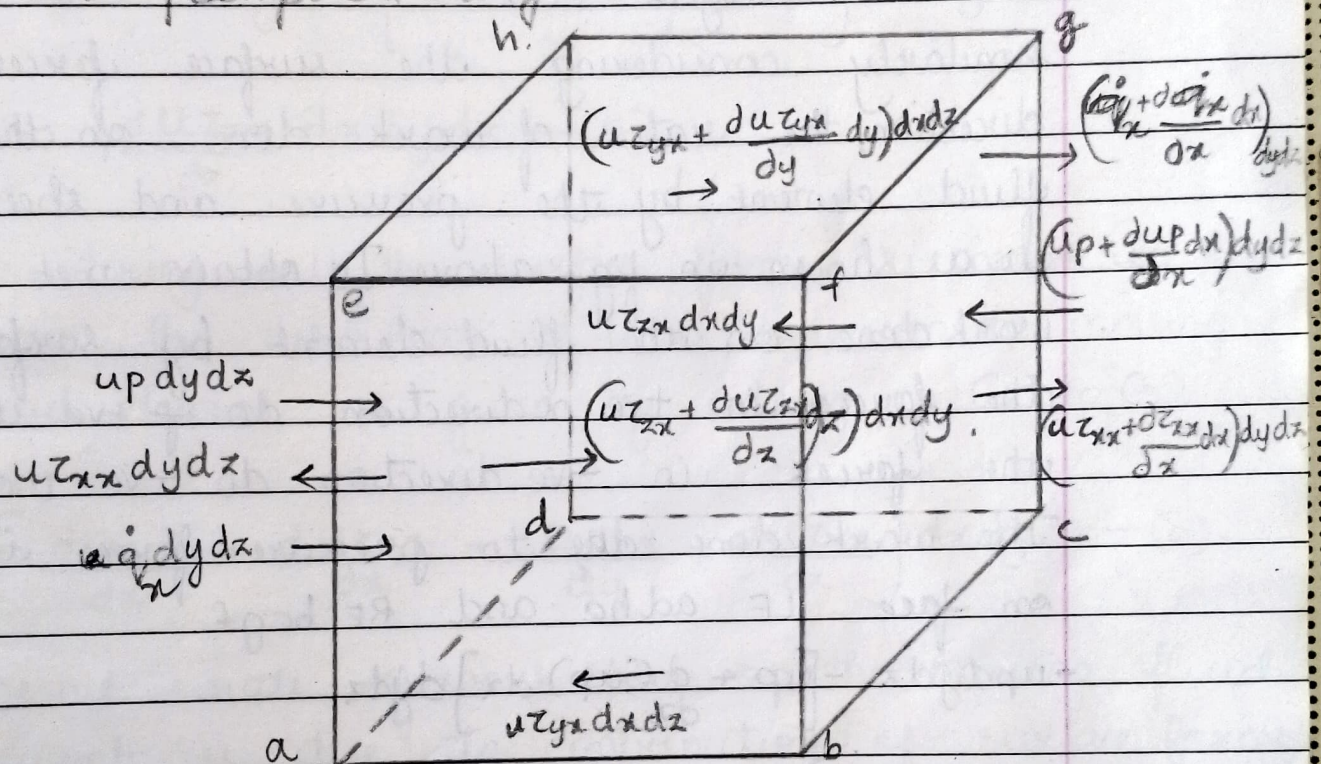
$$\begin{aligned}
 &= (\text{Mass through } BC - \text{Mass through } AD) \text{ per unit time} \\
 &= [\rho \times \text{Velocity through } BC \times \text{Area} - \rho \times \text{Velocity through } AD \times \text{Area}] \\
 &= \left[\rho \cdot u_\theta \cdot dr \times 1 - \rho \left(u_\theta + \frac{\partial u_\theta}{\partial \theta} \cdot d\theta \right) \times dr \times 1 \right] \\
 &= -\rho \left(\frac{\partial u_\theta}{\partial \theta} \cdot d\theta \right) dr \times 1 \quad (\because \text{Area} = dr \times 1) \\
 &= -\rho \frac{\partial u_\theta}{\partial \theta} \cdot \frac{r \cdot d\theta \cdot dr}{r} \quad \text{[Multiplying and dividing by } r\text{]}
 \end{aligned}$$

$\therefore$ Total gain in fluid mass per unit time

$$= -\rho \left[\frac{u_r}{r} + \frac{\partial u_r}{\partial r} \right] \cdot r \cdot dr \cdot d\theta - \rho \frac{\partial u_\theta}{\partial \theta} \cdot \frac{rd\theta \cdot dr}{r} \quad \dots(5.5A)$$

ENERGY EQUATION

Physical principle - energy is conserved.
 $u \rightarrow$ velocity component along x direction.



Consider an infinite small fluid element moving along with the flow. Applying physical principle i.e. energy is conserved which is nothing but 1st law of thermodynamics.

By applying the principle to a model of moving fluid element as sketched above, the first law of thermodynamics states that

Rate of change of ^(A) energy inside fluid = Net flow of ^(B) heat into the element + Rate of work done on element due to surface and body force. ^(C)

To evaluate term c,

The rate of work done on the fluid element due to body and surface forces exerted on the moving fluid element is equal to product of force and component of velocity in the direction of force.

Hence the rate of work done by the body force (F) acting on the fluid element moving with velocity V is

$$V = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$$

$$\rho f \cdot V dx dy dz \quad \text{--- (1)}$$

Similarly considering the surface forces in x -direction, the rate of work done on the moving fluid element by the pressure and shear force is as shown in fig. above. To obtain net rate of work done on the fluid element by surface forces, the forces in $+ve$ x -direction do $+ve$ work while the forces in $-ve$ direction do $-ve$ work.

The work done due to pressure forces in the x -direction on face LF adhe and RF bcgf

$$= up dy dz - \left[p + \frac{\partial (p)}{\partial x} dx \right] dy dz$$

$$= up dy dz - up dy dz - \frac{\partial (p)}{\partial x} dx dy dz$$

$$= - \frac{\partial (p)}{\partial x} dx dy dz \quad \text{--- (2)}$$

Similarly the net rate of work done by the normal stress τ_{xx} in x -direction on LF and RF of fluid element is given as to be

$$- U \tau_{xx} dy dz + \left[U \tau_{xx} + \frac{\partial (U \tau_{xx})}{\partial x} dx \right] dy dz$$

$$= \frac{\partial (U \tau_{xx})}{\partial x} dx dy dz \quad \text{--- (3)}$$

Similarly, the net rate of work done by shear stress (τ_{yx} and τ_{xx}) on the top and bottom and front & back faces respectively is given as Rate of work done on top and bottom faces in x-direction

$$-U\tau_{yx}dx dz + \left[U\tau_{yx} + \frac{\partial U\tau_{yx}}{\partial y} dy \right] dx dz$$

$$= \frac{\partial U\tau_{yx}}{\partial y} dx dy dz \quad \text{--- (4)}$$

Rate of work done on front and back faces in x-direction

$$-U\tau_{xx}dx dy + \left[U\tau_{xx} + \frac{\partial U\tau_{xx}}{\partial z} dz \right] dx dy$$

$$= \frac{\partial U\tau_{xx}}{\partial z} dx dy dz \quad \text{--- (5)}$$

Considering all the surface forces as shown in fig above, the net rate of work done on the moving fluid element is given by adding eq (2) to (5).

$$\left[\frac{\partial (Cup)}{\partial x} + \frac{\partial U\tau_{xx}}{\partial x} + \frac{\partial U\tau_{yx}}{\partial y} + \frac{\partial U\tau_{zx}}{\partial z} \right] dx dy dz \quad \text{--- (6)}$$

The net rate of work done on the moving fluid element is due to contribution of surface forces acting along x, y and z direction and as well as the body force contribution.

$$C = \left[- \left(\frac{\partial up}{\partial x} + \frac{\partial vp}{\partial y} + \frac{\partial wp}{\partial z} \right) + \frac{\partial U\tau_{xx}}{\partial x} + \frac{\partial U\tau_{yx}}{\partial y} + \frac{\partial U\tau_{zx}}{\partial z} + \frac{\partial V\tau_{xy}}{\partial y} + \frac{\partial V\tau_{xy}}{\partial x} + \frac{\partial V\tau_{zy}}{\partial z} + \frac{\partial W\tau_{xx}}{\partial x} + \frac{\partial W\tau_{xz}}{\partial x} + \frac{\partial W\tau_{yz}}{\partial y} \right] dx dy dz + \rho f V dx dy dz \quad \text{--- (7)}$$

To get B is the net flux of heat into the element which is due to

1. Volumetric heating such as ~~sub~~ absorption/emission of radiation.

2. Heat transfer due to existing temp. gradient across the surface by the mode of thermal conduction.

Let $\dot{q}$ be the rate of volumetric heat addition per unit mass. Therefore volumetric heating of the element is equal to $\int \dot{q} dx dy dz$ — (8)

From the fig. above the heat transferred into the moving fluid element by thermal conduction across the LF adhe is $\dot{q}_x dy dz$.

Similarly the heat transferred out of element across the RF DCGF is $(\dot{q}_x + \frac{d\dot{q}_x}{dx} dx) dy dz$

Hence the net heat transferred in x-direction into the fluid element is given as

$$\dot{q}_x dy dz - \left[\left(\dot{q}_x + \frac{d\dot{q}_x}{dx} dx \right) dy dz \right]$$

$$= - \left(\frac{d\dot{q}_x}{dx} \right) dx dy dz \text{ — (9)}$$

Similarly the heat transfer in y and z direction can be obtained. Thus the net heating of the fluid element by thermal conduction is equal to

$$= - \left(\frac{d\dot{q}_x}{dx} + \frac{d\dot{q}_y}{dy} + \frac{d\dot{q}_z}{dz} \right) dx dy dz \text{ — (10)}$$

$$\text{Therefore, } B = \left[\dot{q} - \left[\left(\frac{d\dot{q}_x}{dx} \right) + \left(\frac{d\dot{q}_y}{dy} \right) + \left(\frac{d\dot{q}_z}{dz} \right) \right] \right] dx dy dz \text{ — (11)}$$

To evaluate term A we are considering a gaseous medium in motion. Hence the fluid element has ~~two~~ 2 contributions to its energy.

1. The internal energy (e) due to random molecular motion per unit mass.

2. The kinetic energy due to translational motion of the fluid element. The kinetic energy per unit mass is $\frac{V^2}{2}$.

The total energy is sum of the above contribution which is equal to $e + v^2/2$.

Since we are following a moving fluid element, the time rate of change of total energy per unit mass is given by substantial derivative. The mass of the given fluid element is $\rho dx dy dz$.

$$\therefore A = \rho \frac{D}{Dt} (e + v^2/2) dx dy dz \quad (12)$$

From (1), (11) and (12), we have obtained the values of C, B and A and hence we have

$$\begin{aligned} \rho \frac{D}{Dt} (e + v^2/2) dx dy dz &= \left[\rho \dot{q} - \left[\frac{\partial \dot{q}_x}{\partial x} + \frac{\partial \dot{q}_y}{\partial y} + \frac{\partial \dot{q}_z}{\partial z} \right] \right] dx dy dz \\ &+ \left[- \left(\frac{\partial u_p}{\partial x} + \frac{\partial v_p}{\partial y} + \frac{\partial w_p}{\partial z} \right) + \frac{\partial u \tau_{xx}}{\partial x} + \frac{\partial u \tau_{yx}}{\partial y} + \frac{\partial u \tau_{zx}}{\partial z} + \right. \\ &\quad \left. \frac{\partial v \tau_{xy}}{\partial y} + \frac{\partial v \tau_{xx}}{\partial x} + \frac{\partial v \tau_{zy}}{\partial z} + \frac{\partial w \tau_{xz}}{\partial z} + \frac{\partial w \tau_{xy}}{\partial y} + \frac{\partial w \tau_{zx}}{\partial x} \right] dx dy dz + \rho f v dx dy dz \quad (13) \end{aligned}$$

NOTE :-

From Fourier's law of heat conduction the heat flux due to thermal conduction is proportional to local temp gradient which is given as

$$\dot{q}_x = -k \frac{\partial T}{\partial x}, \quad \dot{q}_y = -k \frac{\partial T}{\partial y}, \quad \dot{q}_z = -k \frac{\partial T}{\partial z} \quad (14)$$

Substituting eq(14) into (13) we have

$$\begin{aligned} \rho \frac{D}{Dt} (e + v^2/2) &= \left[\rho \dot{q} + \left[k \frac{\partial^2 T}{\partial x^2} + k \frac{\partial^2 T}{\partial y^2} + k \frac{\partial^2 T}{\partial z^2} \right] \right] - \left[- \left(\frac{\partial u_p}{\partial x} + \frac{\partial v_p}{\partial y} + \frac{\partial w_p}{\partial z} \right) \right. \\ &\quad \left. + \frac{\partial u \tau_{xx}}{\partial x} + \frac{\partial u \tau_{yx}}{\partial y} + \frac{\partial u \tau_{zx}}{\partial z} + \frac{\partial v \tau_{xy}}{\partial y} + \frac{\partial v \tau_{xx}}{\partial x} + \frac{\partial v \tau_{zy}}{\partial z} + \frac{\partial w \tau_{xz}}{\partial z} + \frac{\partial w \tau_{xy}}{\partial y} + \frac{\partial w \tau_{zx}}{\partial x} \right] + \rho f v \end{aligned}$$

$$\begin{aligned}
 \text{But mass of fluid element} &= \rho \times \text{Volume of fluid element} \\
 &= \rho \times [rd\theta \times dr \times 1] \\
 &= \rho \times rd\theta \cdot dr
 \end{aligned}$$

Rate of increase of fluid mass in the element with time

$$= \frac{\partial}{\partial t} [\rho \cdot rd\theta \cdot dr] = \frac{\partial \rho}{\partial t} \cdot rd\theta \cdot dr \quad \dots(5.5B)$$

($\because rd\theta \cdot dr \cdot 1$ is the volume of element and is a constant quantity)

Since the mass is neither created nor destroyed in the fluid element, hence net gain of mass per unit time in the fluid element must be equal to the rate of increase of mass of fluid in the element.

Hence equating the two expressions given by equations (5.5 A) and (5.5 B), we get

$$-\rho \left[\frac{u_r}{r} + \frac{\partial u_r}{\partial r} \right] r \cdot dr \cdot d\theta - \rho \frac{\partial u_\theta}{\partial \theta} \frac{rd\theta \cdot dr}{r} = \frac{\partial \rho}{\partial t} rd\theta \cdot dr$$

$$\text{or} \quad -\rho \left[\frac{u_r}{r} + \frac{\partial u_r}{\partial r} \right] - \rho \frac{\partial u_\theta}{\partial \theta} \cdot \frac{1}{r} = \frac{\partial \rho}{\partial t} \quad [\text{Cancelling } r dr \cdot d\theta \text{ from both sides}]$$

$$\text{or} \quad \frac{\partial \rho}{\partial t} + \rho \left[\frac{u_r}{r} + \frac{\partial u_r}{\partial r} \right] + \rho \frac{\partial u_\theta}{\partial \theta} \cdot \frac{1}{r} = 0 \quad \dots(5.5C)$$

Equation (5.5 C) is the continuity equation in polar co-ordinates for two-dimensional flow.

For steady flow $\frac{\partial \rho}{\partial t} = 0$ and hence equation (5.5 C) reduces to

$$\rho \left[\frac{u_r}{r} + \frac{\partial u_r}{\partial r} \right] + \rho \frac{\partial u_\theta}{\partial \theta} \cdot \frac{1}{r} = 0$$

$$\text{or} \quad \frac{u_r}{r} + \frac{\partial u_r}{\partial r} + \frac{\partial u_\theta}{\partial \theta} \cdot \frac{1}{r} = 0$$

$$\text{or} \quad u_r + r \frac{\partial u_r}{\partial r} + \frac{\partial u_\theta}{\partial \theta} = 0$$

$$\text{or} \quad \frac{\partial}{\partial r} (ru_r) + \frac{\partial}{\partial \theta} (u_\theta) = 0 \quad \left[\because \frac{\partial}{\partial r} (r \cdot u_r) = r \cdot \frac{\partial u_r}{\partial r} + u_r \right] \quad \dots(5.5D)$$

Equation (5.5 D) represents the continuity equation in polar co-ordinates for two-dimensional steady incompressible flow.

VELOCITY AND ACCELERATION

Let V is the resultant velocity at any point in a fluid flow. Let u , v and w are its component in x , y and z directions. The velocity components are functions of space-co-ordinates and time. Mathematically, the velocity components are given as

$$u = f_1(x, y, z, t)$$

$$v = f_2(x, y, z, t)$$

$$w = f_3(x, y, z, t)$$

and Resultant velocity, $V = ui + vj + wk = \sqrt{u^2 + v^2 + w^2}$

Let a_x , a_y and a_z are the **total acceleration** in x , y and z directions respectively. Then by the chain rule of differentiation, we have

$$a_x = \frac{du}{dt} = \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt} + \frac{\partial u}{\partial z} \frac{dz}{dt} + \frac{\partial u}{\partial t}$$

But

$$\frac{dx}{dt} = u, \frac{dy}{dt} = v \text{ and } \frac{dz}{dt} = w$$

$\therefore$

$$a_x = \frac{du}{dt} = u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}$$

Similarly,

$$a_y = \frac{dv}{dt} = u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} + \frac{\partial v}{\partial t}$$

$$a_z = \frac{dw}{dt} = u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} + \frac{\partial w}{\partial t}$$

...(5.6)

For steady flow, $\frac{\partial V}{\partial t} = 0$, where V is resultant velocity

or
$$\frac{\partial u}{\partial t} = 0, \frac{\partial v}{\partial t} = 0 \text{ and } \frac{\partial w}{\partial t} = 0$$

Hence acceleration in x , y and z directions becomes

$$\left. \begin{aligned} a_x &= \frac{du}{dt} = u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \\ a_y &= \frac{dv}{dt} = u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \\ a_z &= \frac{dw}{dt} = u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \end{aligned} \right] \quad \dots(5.7)$$

Acceleration vector
$$\left. \begin{aligned} A &= a_x i + a_y j + a_z k \\ &= \sqrt{a_x^2 + a_y^2 + a_z^2} \end{aligned} \right] \quad \dots(5.8)$$

5.7.1 Local Acceleration and Convective Acceleration. **Local acceleration** is defined as the rate of increase of velocity with respect to time at a given point in a flow field. In the equation given by (5.6), the expression $\frac{\partial u}{\partial t}$, $\frac{\partial v}{\partial t}$ or $\frac{\partial w}{\partial t}$ is known as local acceleration.

Convective acceleration is defined as the rate of change of velocity due to the change of position of fluid particles in a fluid flow. The expressions other than $\frac{\partial u}{\partial t}$, $\frac{\partial v}{\partial t}$ and $\frac{\partial w}{\partial t}$ in equation (5.6) are known as convective acceleration.

Problem 5.6 The velocity vector in a fluid flow is given

$$V = 4x^3 i - 10x^2 y j + 2t k.$$

Find the velocity and acceleration of a fluid particle at (2, 1, 3) at time $t = 1$.

Solution. The velocity components u , v and w are $u = 4x^3$, $v = -10x^2 y$, $w = 2t$

For the point (2, 1, 3), we have $x = 2$, $y = 1$ and $z = 3$ at time $t = 1$.

Hence velocity components at (2, 1, 3) are

$$u = 4 \times (2)^3 = 32 \text{ units}$$

$$v = -10(2)^2(1) = -40 \text{ units}$$

$$w = 2 \times 1 = 2 \text{ units}$$

$\therefore$ Velocity vector V at (2, 1, 3) = $32i - 40j + 2k$

or Resultant velocity = $\sqrt{u^2 + v^2 + w^2}$

$$= \sqrt{32^2 + (-40)^2 + 2^2} = \sqrt{1024 + 1600 + 4} = 51.26 \text{ units. Ans.}$$

Acceleration is given by equation (5.6)

$$a_x = u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}$$

$$a_y = u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} + \frac{\partial v}{\partial t}$$

$$a_z = u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} + \frac{\partial w}{\partial t}$$

Now from velocity components, we have

$$\frac{\partial u}{\partial x} = 12x^2, \frac{\partial u}{\partial y} = 0, \frac{\partial u}{\partial z} = 0 \text{ and } \frac{\partial u}{\partial t} = 0$$

$$\frac{\partial v}{\partial x} = -20xy, \frac{\partial v}{\partial y} = -10x^2, \frac{\partial v}{\partial z} = 0 \text{ and } \frac{\partial v}{\partial t} = 0$$

$$\frac{\partial w}{\partial x} = 0, \frac{\partial w}{\partial y} = 0, \frac{\partial w}{\partial z} = 0 \text{ and } \frac{\partial w}{\partial t} = 2.1$$

Substituting the values, the acceleration components at (2, 1, 3) at time $t = 1$ are

$$a_x = 4x^3 (12x^2) + (-10x^2y) (0) + 2t \times (0) + 0$$

$$= 48x^5 = 48 \times (2)^5 = 48 \times 32 = 1536 \text{ units}$$

$$a_y = 4x^3 (-20xy) + (-10x^2y) (-10x^2) + 2t (0) + 0$$

$$= -80x^4y + 100x^4y$$

$$= -80 (2)^4 (1) + 100 (2)^4 \times 1 = -1280 + 1600 = 320 \text{ units.}$$

$$a_z = 4x^3 (0) + (-10x^2y) (0) + (2t) (0) + 2.1 = 2.0 \text{ units}$$

$\therefore$ Acceleration is

$$A = a_x i + a_y j + a_z k = \mathbf{1536i + 320j + 2k. \text{ Ans.}}$$

or Resultant

$$A = \sqrt{(1536)^2 + (320)^2 + (2)^2} \text{ units}$$

$$= \sqrt{2359296 + 102400 + 4} = \mathbf{1568.9 \text{ units. Ans.}}$$

EQUATIONS OF MOTION

According to Newton's second law of motion, the net force F_x acting on a fluid element in the direction of x is equal to mass m of the fluid element multiplied by the acceleration a_x in the x -direction. Thus mathematically,

$$F_x = m \cdot a_x \quad \dots(6.1)$$

In the fluid flow, the following forces are present :

- (i) F_g , gravity force.
- (ii) F_p , the pressure force.
- (iii) F_v , force due to viscosity.
- (iv) F_t , force due to turbulence.
- (v) F_c , force due to compressibility.

Thus in equation (6.1), the net force

$$F_x = (F_g)_x + (F_p)_x + (F_v)_x + (F_t)_x + (F_c)_x.$$

- (i) If the force due to compressibility, F_c is negligible, the resulting net force

$$F_x = (F_g)_x + (F_p)_x + (F_v)_x + (F_t)_x$$

and equation of motions are called **Reynold's equations of motion.**

(ii) For flow, where (F_t) is negligible, the resulting equations of motion are known as **Navier-Stokes Equation.**

(iii) If the flow is assumed to be ideal, viscous force (F_v) is zero and equation of motions are known as **Euler's equation of motion.**

EULER'S EQUATION OF MOTION

This is equation of motion in which the forces due to gravity and pressure are taken into consideration. This is derived by considering the motion of a fluid element along a stream-line as :

Consider a stream-line in which flow is taking place in s -direction as shown in Fig. 6.1. Consider a cylindrical element of cross-section dA and length ds . The forces acting on the cylindrical element are:

1. Pressure force $p dA$ in the direction of flow.
2. Pressure force $\left(p + \frac{\partial p}{\partial s} ds\right) dA$ opposite to the direction of flow.
3. Weight of element $\rho g dA ds$.

Let θ is the angle between the direction of flow and the line of action of the weight of element.

The resultant force on the fluid element in the direction of s must be equal to the mass of fluid element $\times$ acceleration in the direction s .

$$\begin{aligned} \therefore \quad p dA - \left(p + \frac{\partial p}{\partial s} ds\right) dA - \rho g dA ds \cos \theta \\ = \rho dA ds \times a_s \end{aligned} \quad \dots(6.2)$$

where a_s is the acceleration in the direction of s .

Now $a_s = \frac{dv}{dt}$, where v is a function of s and t .

$$= \frac{\partial v}{\partial s} \frac{ds}{dt} + \frac{\partial v}{\partial t} = \frac{v \partial v}{\partial s} + \frac{\partial v}{\partial t} \left\{ \because \frac{ds}{dt} = v \right\}$$

If the flow is steady, $\frac{\partial v}{\partial t} = 0$

$$\therefore \quad a_s = \frac{v \partial v}{\partial s}$$

Substituting the value of a_s in equation (6.2) and simplifying the equation, we get

$$- \frac{\partial p}{\partial s} ds dA - \rho g dA ds \cos \theta = \rho dA ds \times \frac{v \partial v}{\partial s}$$

Dividing by $\rho ds dA$, $-\frac{\partial p}{\rho \partial s} - g \cos \theta = \frac{v \partial v}{\partial s}$

or $\frac{\partial p}{\rho \partial s} + g \cos \theta + v \frac{\partial v}{\partial s} = 0$

But from Fig. 6.1 (b), we have $\cos \theta = \frac{dz}{ds}$

$$\therefore \quad \frac{1}{\rho} \frac{dp}{ds} + g \frac{dz}{ds} + \frac{v dv}{ds} = 0 \quad \text{or} \quad \frac{dp}{\rho} + g dz + v dv = 0$$

or $\frac{dp}{\rho} + g dz + v dv = 0 \quad \dots(6.3)$

Equation (6.3) is known as Euler's equation of motion.

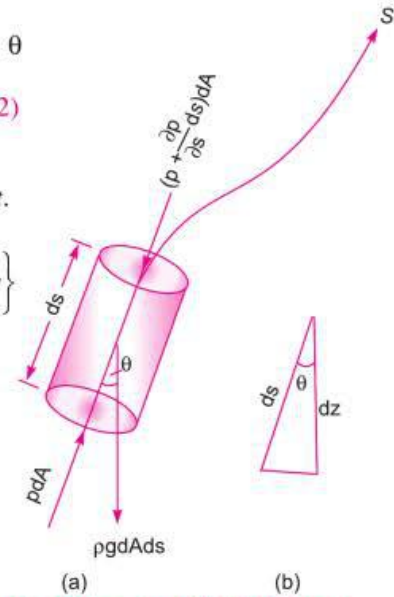


Fig. 6.1 Forces on a fluid element.

BERNOULLI'S EQUATION FROM EULER'S EQUATION

Bernoulli's equation is obtained by integrating the Euler's equation of motion (6.3) as

$$\int \frac{dp}{\rho} + \int g dz + \int v dv = \text{constant}$$

If flow is incompressible, ρ is constant and

$$\therefore \frac{p}{\rho} + gz + \frac{v^2}{2} = \text{constant}$$

or
$$\frac{p}{\rho g} + z + \frac{v^2}{2g} = \text{constant}$$

or
$$\frac{p}{\rho g} + \frac{v^2}{2g} + z = \text{constant} \quad \dots(6.4)$$

Equation (6.4) is a Bernoulli's equation in which

$$\frac{p}{\rho g} = \text{pressure energy per unit weight of fluid or pressure head.}$$

$$\frac{v^2}{2g} = \text{kinetic energy per unit weight or kinetic head.}$$

$$z = \text{potential energy per unit weight or potential head.}$$

ASSUMPTIONS

The following are the assumptions made in the derivation of Bernoulli's equation :

- (i) The fluid is ideal, i.e., viscosity is zero
- (ii) The flow is steady
- (iii) The flow is incompressible
- (iv) The flow is irrotational.

Problem 6.1 Water is flowing through a pipe of 5 cm diameter under a pressure of 29.43 N/cm² (gauge) and with mean velocity of 2.0 m/s. Find the total head or total energy per unit weight of the water at a cross-section, which is 5 m above the datum line.

Solution. Given :

Diameter of pipe = 5 cm = 0.5 m

Pressure, $p = 29.43 \text{ N/cm}^2 = 29.43 \times 10^4 \text{ N/m}^2$

Velocity, $v = 2.0 \text{ m/s}$

Datum head, $z = 5 \text{ m}$

Total head = pressure head + kinetic head + datum head

Pressure head $= \frac{p}{\rho g} = \frac{29.43 \times 10^4}{1000 \times 9.81} = 30 \text{ m} \quad \left\{ \rho \text{ for water} = 1000 \frac{\text{kg}}{\text{m}^3} \right\}$

Kinetic head $= \frac{v^2}{2g} = \frac{2 \times 2}{2 \times 9.81} = 0.204 \text{ m}$

$\therefore$ Total head $= \frac{p}{\rho g} + \frac{v^2}{2g} + z = 30 + 0.204 + 5 = \mathbf{35.204 \text{ m. Ans.}}$

Problem 6.2 A pipe, through which water is flowing, is having diameters, 20 cm and 10 cm at the cross-sections 1 and 2 respectively. The velocity of water at section 1 is given 4.0 m/s. Find the velocity head at sections 1 and 2 and also rate of discharge.

Solution. Given :

$$D_1 = 20 \text{ cm} = 0.2 \text{ m}$$

$\therefore$ Area,

$$A_1 = \frac{\pi}{4} D_1^2 = \frac{\pi}{4} (.2)^2 = 0.0314 \text{ m}^2$$

$$V_1 = 4.0 \text{ m/s}$$

$$D_2 = 0.1 \text{ m}$$

$\therefore$

$$A_2 = \frac{\pi}{4} (.1)^2 = .00785 \text{ m}^2$$

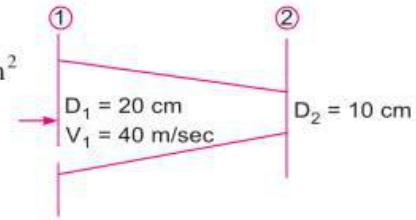


Fig. 6.2

(i) Velocity head at section 1

$$= \frac{V_1^2}{2g} = \frac{4.0 \times 4.0}{2 \times 9.81} = \mathbf{0.815 \text{ m. Ans.}}$$

(ii) Velocity head at section 2 = $V_2^2/2g$

To find V_2 , apply continuity equation at 1 and 2

$$\therefore A_1 V_1 = A_2 V_2 \quad \text{or} \quad V_2 = \frac{A_1 V_1}{A_2} = \frac{.0314}{.00785} \times 4.0 = 16.0 \text{ m/s}$$

$$\therefore \text{Velocity head at section 2} = \frac{V_2^2}{2g} = \frac{16.0 \times 16.0}{2 \times 9.81} = \mathbf{83.047 \text{ m. Ans.}}$$

(iii) Rate of discharge

$$= A_1 V_1 \quad \text{or} \quad A_2 V_2$$

$$= 0.0314 \times 4.0 = 0.1256 \text{ m}^3/\text{s}$$

$$= \mathbf{125.6 \text{ litres/s. Ans.}}$$

$$\{ \because 1 \text{ m}^3 = 1000 \text{ litres} \}$$

Problem 6.3 State Bernoulli's theorem for steady flow of an incompressible fluid. Derive an expression for Bernoulli's equation from first principle and state the assumptions made for such a derivation.

Solution. Statement of Bernoulli's Theorem. It states that in a steady, ideal flow of an incompressible fluid, the total energy at any point of the fluid is constant. The total energy consists of pressure energy, kinetic energy and potential energy or datum energy. These energies per unit weight of the fluid are :

$$\text{Pressure energy} = \frac{p}{\rho g}$$

$$\text{Kinetic energy} = \frac{v^2}{2g}$$

$$\text{Datum energy} = z$$

Thus mathematically, Bernoulli's theorem is written as

$$\frac{p}{\rho g} + \frac{v^2}{2g} + z = \text{Constant.}$$

Derivation of Bernoulli's theorem. For derivation of Bernoulli's theorem, Articles 6.3 and 6.4 should be written.

Assumptions are given in Article 6.5.

Problem 6.4 The water is flowing through a pipe having diameters 20 cm and 10 cm at sections 1 and 2 respectively. The rate of flow through pipe is 35 litres/s. The section 1 is 6 m above datum and section 2 is 4 m above datum. If the pressure at section 1 is 39.24 N/cm^2 , find the intensity of pressure at section 2.

Solution. Given :

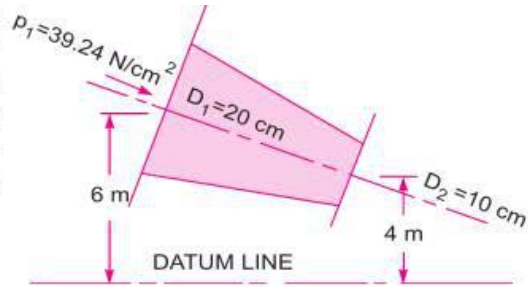


Fig. 6.3

At section 1,

$$D_1 = 20 \text{ cm} = 0.2 \text{ m}$$

$$A_1 = \frac{\pi}{4} (.2)^2 = .0314 \text{ m}^2$$

$$p_1 = 39.24 \text{ N/cm}^2 \\ = 39.24 \times 10^4 \text{ N/m}^2$$

$$z_1 = 6.0 \text{ m}$$

At section 2,

$$D_2 = 0.10 \text{ m}$$

$$A_2 = \frac{\pi}{4} (0.1)^2 = .00785 \text{ m}^2$$

$$z_2 = 4 \text{ m}$$

$$p_2 = ?$$

Rate of flow,

$$Q = 35 \text{ lit/s} = \frac{35}{1000} = .035 \text{ m}^3/\text{s}$$

Now

$$Q = A_1 V_1 = A_2 V_2$$

$\therefore$

$$V_1 = \frac{Q}{A_1} = \frac{.035}{.0314} = 1.114 \text{ m/s}$$

and

$$V_2 = \frac{Q}{A_2} = \frac{.035}{.00785} = 4.456 \text{ m/s}$$

Applying Bernoulli's equation at sections 1 and 2, we get

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2$$

$$\text{or } \frac{39.24 \times 10^4}{1000 \times 9.81} + \frac{(1.114)^2}{2 \times 9.81} + 6.0 = \frac{p_2}{1000 \times 9.81} + \frac{(4.456)^2}{2 \times 9.81} + 4.0$$

$$\text{or } 40 + 0.063 + 6.0 = \frac{p_2}{9810} + 1.012 + 4.0$$

$$\text{or } 46.063 = \frac{p_2}{9810} + 5.012$$

$$\therefore \frac{p_2}{9810} = 46.063 - 5.012 = 41.051$$

$$\therefore p_2 = 41.051 \times 9810 \text{ N/m}^2 \\ = \frac{41.051 \times 9810}{10^4} \text{ N/cm}^2 = 40.27 \text{ N/cm}^2. \text{ Ans.}$$

Problem 6.5 Water is flowing through a pipe having diameter 300 mm and 200 mm at the bottom and upper end respectively. The intensity of pressure at the bottom end is 24.525 N/cm^2 and the pressure at the upper end is 9.81 N/cm^2 . Determine the difference in datum head if the rate of flow through pipe is 40 lit/s.

Solution. Given :

Section 1, $D_1 = 300 \text{ mm} = 0.3 \text{ m}$
 $p_1 = 24.525 \text{ N/cm}^2 = 24.525 \times 10^4 \text{ N/m}^2$

Section 2, $D_2 = 200 \text{ mm} = 0.2 \text{ m}$
 $p_2 = 9.81 \text{ N/cm}^2 = 9.81 \times 10^4 \text{ N/m}^2$

Rate of flow = 40 lit/s

or $Q = \frac{40}{1000} = 0.04 \text{ m}^3/\text{s}$

Now $A_1 V_1 = A_2 V_2 = \text{rate of flow} = 0.04$

$\therefore V_1 = \frac{.04}{A_1} = \frac{.04}{\frac{\pi}{4} D_1^2} = \frac{0.04}{\frac{\pi}{4} (0.3)^2} = 0.5658 \text{ m/s}$
 $\approx 0.566 \text{ m/s}$

$V_2 = \frac{.04}{A_2} = \frac{.04}{\frac{\pi}{4} (D_2)^2} = \frac{0.04}{\frac{\pi}{4} (0.2)^2} = 1.274 \text{ m/s}$

Applying Bernoulli's equation at sections (1) and (2), we get

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2$$

or $\frac{24.525 \times 10^4}{1000 \times 9.81} + \frac{.566 \times .566}{2 \times 9.81} + z_1 = \frac{9.81 \times 10^4}{1000 \times 9.81} + \frac{(1.274)^2}{2 \times 9.81} + z_2$

or $25 + .32 + z_1 = 10 + 1.623 + z_2$

or $25.32 + z_1 = 11.623 + z_2$

$\therefore z_2 - z_1 = 25.32 - 11.623 = 13.697 = 13.70 \text{ m}$

$\therefore$ Difference in datum head $= z_2 - z_1 = \mathbf{13.70 \text{ m. Ans.}}$

Problem 6.6 The water is flowing through a taper pipe of length 100 m having diameters 600 mm at the upper end and 300 mm at the lower end, at the rate of 50 litres/s. The pipe has a slope of 1 in 30. Find the pressure at the lower end if the pressure at the higher level is 19.62 N/cm^2 .

Solution. Given :

Length of pipe, $L = 100 \text{ m}$

Dia. at the upper end, $D_1 = 600 \text{ mm} = 0.6 \text{ m}$

$\therefore$ Area, $A_1 = \frac{\pi}{4} D_1^2 = \frac{\pi}{4} \times (.6)^2$
 $= 0.2827 \text{ m}^2$

$p_1 = \text{pressure at upper end}$
 $= 19.62 \text{ N/cm}^2$

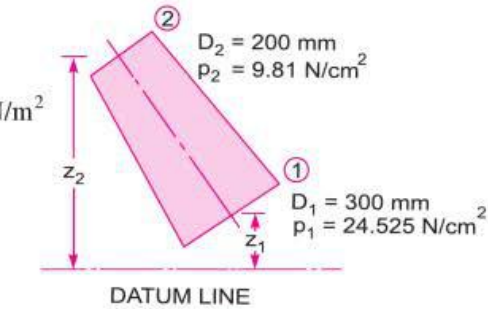


Fig. 6.4

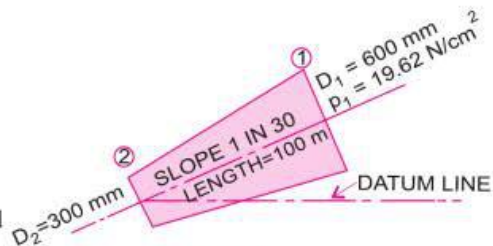


Fig. 6.5

$$= 19.62 \times 10^4 \text{ N/m}^2$$

Dia. at lower end, $D_2 = 300 \text{ mm} = 0.3 \text{ m}$

$\therefore$ Area, $A_2 = \frac{\pi}{4} D_2^2 = \frac{\pi}{4} (.3)^2 = 0.07068 \text{ m}^2$

$Q = \text{rate of flow} = 50 \text{ litres/s} = \frac{50}{1000} = 0.05 \text{ m}^3/\text{s}$

Let the datum line passes through the centre of the lower end.

Then $z_2 = 0$

As slope is 1 in 30 means $z_1 = \frac{1}{30} \times 100 = \frac{10}{3} \text{ m}$

Also we know $Q = A_1 V_1 = A_2 V_2$

$\therefore V_1 = \frac{Q}{A} = \frac{0.05}{.2827} = 0.1768 \text{ m/sec} = 0.177 \text{ m/s}$

and $V_2 = \frac{Q}{A_2} = \frac{0.05}{.07068} = 0.7074 \text{ m/sec} = 0.707 \text{ m/s}$

Applying Bernoulli's equation at sections (1) and (2), we get

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2$$

or $\frac{19.62 \times 10^4}{1000 \times 9.81} + \frac{.177^2}{2 \times 9.81} + \frac{10}{3} = \frac{p_2}{\rho g} + \frac{.707^2}{2 \times 9.81} + 0$

or $20 + 0.001596 + 3.334 = \frac{p_2}{\rho g} + 0.0254$

or $23.335 - 0.0254 = \frac{p_2}{1000 \times 9.81}$

or $p_2 = 23.3 \times 9810 \text{ N/m}^2 = 228573 \text{ N/m}^2 = 22.857 \text{ N/cm}^2. \text{ Ans.}$

The Bernoulli's equation was derived on the assumption that fluid is inviscid (non-viscous) and therefore frictionless. But all the real fluids are viscous and hence offer resistance to flow. Thus there are always some losses in fluid flows and hence in the application of Bernoulli's equation, these losses have to be taken into consideration. Thus the Bernoulli's equation for real fluids between points 1 and 2 is given as

$$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + z_2 + h_L \quad \dots(6.5)$$

where h_L is loss of energy between points 1 and 2.

Problem 6.7 A pipe of diameter 400 mm carries water at a velocity of 25 m/s. The pressures at the points A and B are given as 29.43 N/cm^2 and 22.563 N/cm^2 respectively while the datum head at A and B are 28 m and 30 m. Find the loss of head between A and B.

Solution. Given :

Dia. of pipe, $D = 400 \text{ mm} = 0.4 \text{ m}$

Velocity, $V = 25 \text{ m/s}$

At point A,

$$p_A = 29.43 \text{ N/cm}^2 = 29.43 \times 10^4 \text{ N/m}^2$$

$$z_A = 28 \text{ m}$$

$$v_A = v = 25 \text{ m/s}$$

$\therefore$ Total energy at A,

$$\begin{aligned} E_A &= \frac{p_A}{\rho g} + \frac{v_A^2}{2g} + z_A \\ &= \frac{29.43 \times 10^4}{1000 \times 9.81} + \frac{25^2}{2 \times 9.81} + 28 \\ &= 30 + 31.85 + 28 = 89.85 \text{ m} \end{aligned}$$

At point B,

$$p_B = 22.563 \text{ N/cm}^2 = 22.563 \times 10^4 \text{ N/m}^2$$

$$z_B = 30 \text{ m}$$

$$v_B = v = v_A = 25 \text{ m/s}$$

$\therefore$ Total energy at B,

$$\begin{aligned} E_B &= \frac{p_B}{\rho g} + \frac{v_B^2}{2g} + z_B \\ &= \frac{22.563 \times 10^4}{1000 \times 9.81} + \frac{25^2}{2 \times 9.81} + 30 = 23 + 31.85 + 30 = 84.85 \text{ m} \end{aligned}$$

$\therefore$ Loss of energy $= E_A - E_B = 89.85 - 84.85 = 5.0 \text{ m}$. **Ans.**

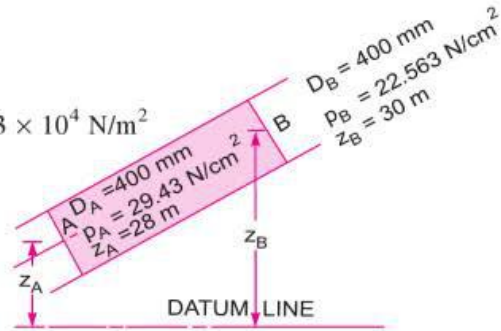


Fig. 6.6

Problem 6.8 A conical tube of length 2.0 m is fixed vertically with its smaller end upwards. The velocity of flow at the smaller end is 5 m/s while at the lower end it is 2 m/s. The pressure head at the smaller end is 2.5 m of liquid. The loss of head in the tube is $\frac{0.35(v_1 - v_2)^2}{2g}$, where v_1 is the velocity at the smaller end and v_2 at the lower end respectively. Determine the pressure head at the lower end. Flow takes place in the downward direction.

Solution. Let the smaller end is represented by (1) and lower end by (2)

Given :

Length of tube, $L = 2.0 \text{ m}$

$$v_1 = 5 \text{ m/s}$$

$$p_1/\rho g = 2.5 \text{ m of liquid}$$

$$v_2 = 2 \text{ m/s}$$

Loss of head

$$= h_L = \frac{0.35(v_1 - v_2)^2}{2g}$$

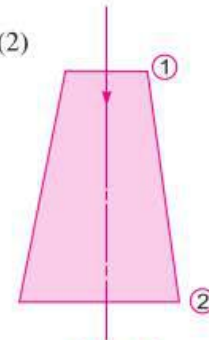


Fig. 6.7

$$= \frac{0.35 [5 - 2]^2}{2g} = \frac{0.35 \times 9}{2 \times 9.81} = 0.16 \text{ m}$$

Pressure head, $\frac{p_2}{\rho g} = ?$

Applying Bernoulli's equation at sections (1) and (2), we get

$$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + z_2 + h_L$$

Let the datum line passes through section (2). Then $z_2 = 0$, $z_1 = 2.0$

$$\therefore 2.5 + \frac{5^2}{2 \times 9.81} + 2.0 = \frac{p_2}{\rho g} + \frac{2^2}{2 \times 9.81} + 0 + 0.16$$

$$2.5 + 1.27 + 2.0 = \frac{p_2}{\rho g} + 0.203 + .16$$

or $\frac{p_2}{\rho g} = (2.5 + 1.27 + 2.0) - (.203 + .16)$

$$= 5.77 - .363 = 5.407 \text{ m of fluid. Ans.}$$

Problem 6.9 A pipeline carrying oil of specific gravity 0.87, changes in diameter from 200 mm diameter at a position A to 500 mm diameter at a position B which is 4 metres at a higher level. If the pressures at A and B are 9.81 N/cm² and 5.886 N/cm² respectively and the discharge is 200 litres/s determine the loss of head and direction of flow.

Solution. Discharge, $Q = 200 \text{ lit/s} = 0.2 \text{ m}^3/\text{s}$

Sp. gr. of oil $= 0.87$

$\therefore \rho$ for oil $= .87 \times 1000 = 870 \frac{\text{kg}}{\text{m}^3}$

Given : At section A, $D_A = 200 \text{ mm} = 0.2 \text{ m}$

Area, $A_A = \frac{\pi}{4} (D_A)^2 = \frac{\pi}{4} (.2)^2$

$$= 0.0314 \text{ m}^2$$

$$p_A = 9.81 \text{ N/cm}^2$$

$$= 9.81 \times 10^4 \text{ N/m}^2$$

If datum line is passing through A, then

$$Z_A = 0$$

$$V_A = \frac{Q}{A_A} = \frac{0.2}{0.0314} = 6.369 \text{ m/s}$$

At section B, $D_B = 500 \text{ mm} = 0.50 \text{ m}$

Area, $A_B = \frac{\pi}{4} D_B^2 = \frac{\pi}{4} (.5)^2 = 0.1963 \text{ m}^2$

$$p_B = 5.886 \text{ N/cm}^2 = 5.886 \times 10^4 \text{ N/m}^2$$

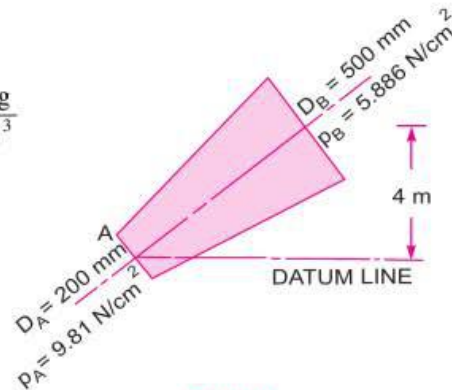


Fig. 6.8

$$Z_B = 4.0 \text{ m}$$

$$V_B = \frac{Q}{\text{Area}} = \frac{0.2}{.1963} = 1.018 \text{ m/s}$$

Total energy at A

$$\begin{aligned} E_A &= \frac{p_A}{\rho g} + \frac{V_A^2}{2g} + Z_A \\ &= \frac{9.81 \times 10^4}{870 \times 9.81} + \frac{(6.369)^2}{2 \times 9.81} + 0 = 11.49 + 2.067 = 13.557 \text{ m} \end{aligned}$$

Total energy at B

$$\begin{aligned} E_B &= \frac{p_B}{\rho g} + \frac{V_B^2}{2g} + Z_B \\ &= \frac{5.886 \times 10^4}{870 \times 9.81} + \frac{(1.018)^2}{2 \times 9.81} + 4.0 = 6.896 + 0.052 + 4.0 = 10.948 \text{ m} \end{aligned}$$

- (i) **Direction of flow.** As E_A is more than E_B and hence flow is taking place from A to B. **Ans.**
(ii) **Loss of head** $= h_L = E_A - E_B = 13.557 - 10.948 = 2.609 \text{ m}$. **Ans.**

PRACTICAL APPLICATIONS OF BERNOULLI'S EQUATION

Bernoulli's equation is applied in all problems of incompressible fluid flow where energy considerations are involved. But we shall consider its application to the following measuring devices :

1. Venturimeter.
2. Orifice meter.
3. Pitot-tube.

Venturimeter. A venturimeter is a device used for measuring the rate of a flow of a fluid flowing through a pipe. It consists of three parts :

(i) A short converging part, (ii) Throat, and (iii) Diverging part. It is based on the Principle of Bernoulli's equation.

Expression for rate of flow through venturimeter

Consider a venturimeter fitted in a horizontal pipe through which a fluid is flowing (say water), as shown in Fig. 6.9.

Let d_1 = diameter at inlet or at section (1),

p_1 = pressure at section (1)

v_1 = velocity of fluid at section (1),

$$a = \text{area at section (1)} = \frac{\pi}{4} d_1^2$$

and d_2, p_2, v_2, a_2 are corresponding values at section (2).

Applying Bernoulli's equation at sections (1) and (2), we get

$$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + z_2$$

As pipe is horizontal, hence $z_1 = z_2$

$$\therefore \frac{p_1}{\rho g} + \frac{v_1^2}{2g} = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} \quad \text{or} \quad \frac{p_1 - p_2}{\rho g} = \frac{v_2^2}{2g} - \frac{v_1^2}{2g}$$

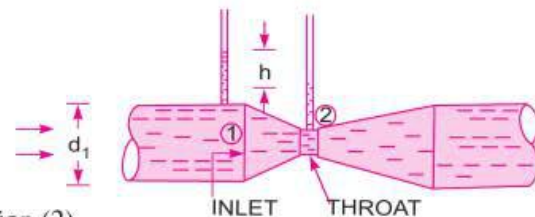


Fig. 6.9 Venturimeter.