

MODULE-2

Taylor's series expansion:-

If $y = f(x)$ is a continuous function then expansion of $f(x)$ at point $x=a$ is given by

$$y = f(x) = y_0 + (x-a)y'(a) + \frac{(x-a)^2}{2!}y''(a) + \frac{(x-a)^3}{3!}y'''(a) + \dots$$

Maclaurin's series:-

If $y = f(x)$ is continuous function then expansion of function at point $x=0$ is given by $y = f(x)$, $x=0$

$$y = f(x) = y_0 + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \frac{x^4}{4!}y^{(4)}(0) + \dots$$

Problems:-

1. Expand $y = e^x$ at $x=0$ or expand e^x as Maclaurin's series.

Sol:- Consider, $y = e^x$

$$y = f(x) = y_0 + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \dots$$

Since given to find M.S $\Rightarrow x=0$

Function

$$y = e^x$$

$$y' = e^x$$

$$y'' = e^x$$

$$y''' = e^x$$

$$y^{(4)} = e^x$$

Value

$$y_0 = y(0) = e^0 = 1$$

$$y'(0) = e^0 = 1$$

$$y''(0) = e^0 = 1$$

$$y'''(0) = e^0 = 1$$

$$y^{(4)}(0) = e^0 = 1$$

$$y = f(x) = e^x = 1 + x(1) + \frac{x^2}{2!}(1) + \frac{x^3}{3!}(1) + \frac{x^4}{4!}(1)$$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$$

2. Expand $\sqrt{1+\sin 2x}$ as M.S

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Sol:- Consider, $\sqrt{1+\sin 2x}$

$$y = f(x) = y_0 + x \cdot y'(0) + \frac{x^2}{2!} y''(0) + \frac{x^3}{3!} y'''(0) + \frac{x^4}{4!} y^{(4)}(0) + \dots$$

Since given to find M.S $\Rightarrow x=0$

$$y = \sqrt{1+\sin 2x} = \sqrt{\sin^2 x + \cos^2 x + 2 \sin x \cdot \cos x}$$

$$= \sqrt{(\sin x + \cos x)^2}$$

$$y = \sin x + \cos x$$

Function

$$y = \sin x + \cos x$$

$$y' = \cos x - \sin x$$

$$y'' = -\sin x - \cos x$$

$$y''' = -\cos x + \sin x$$

$$y^{(4)} = \sin x + \cos x$$

Value

$$y(0) = \sin(0) + \cos(0) = 1$$

$$y'(0) = \cos(0) - \sin(0) = 1$$

$$y''(0) = -\sin(0) - \cos(0) = -1$$

$$y'''(0) = -\cos(0) + \sin(0) = -1$$

$$y^{(4)}(0) = \sin(0) + \cos(0) = 1$$

$$y = \sqrt{1+\sin 2x} = 1 + x(1) + \frac{x^2}{2!}(1) + \frac{x^3}{3!}(-1) + \frac{x^4}{4!}(1)$$

$$y = \sqrt{1+\sin 2x} = 1 + x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24}$$

3. Expand $\log(\sec x)$ as M.S

Sol:- WKT, M.S is given by

$$y = f(x) = y^0 + x \cdot y'(0) + \frac{x^2}{2!} y''(0) + \frac{x^3}{3!} y'''(0) + \dots$$

$$y = \log(\sec x)$$

d.w.r.t x

$$y_1 = \frac{1}{\sec x} \cdot \sec x \cdot \tan x \Rightarrow y_1 \tan x$$

Function

Let $y = \log(\sec x)$
d.w.r.t x

Value at $x=0$

$$y(0) = \log(\sec(0)) = \log 1 = 0$$

$$y_1 = \tan x$$

d.w.r.t x

$$y_1(0) = \tan(0) = 0$$

$$y_2 = \sec^2 x$$

d.w.r.t x

$$y_2(0) = \sec^2(0) = 1$$

$$y_2 = 1 + \tan^2 x$$

$$= 1 + y_1^2$$

$$y_3(0) = 2y_1(0) \cdot y_2(0) = 0$$

$$y_3(0) = 0$$

$$y_3 = 0 + 2y_1 y_2$$

d.w.r.t x

$$y_4 = 2[y_1 y_3 + y_2 y_2]$$

d.w.r.t x

$$y_4(0) = 2[y_1(0) y_3(0) + y_2^2(0)]$$

$$y_4(0) = 2$$

$$y_5 = 2[y_1 y_4 + y_3 y_2 + 2y_2 y_3]$$

$$y_5(0) = 2[y_1(0) y_4(0) + y_3(0) y_2(0) + 2y_2(0) y_3(0)]$$

$$y_5 = 2[y_1 y_4 + 3y_2 y_3]$$

d.w.r.t x

$$y_5(0) = 0$$

$$y_6 = 2[y_1 y_5 + y_4 y_2 + 3(y_2 y_4 + y_3 y_3)]$$

$$y_6(0) = 2[y_1(0) y_5(0) + y_4(0) y_2(0) + 3y_2(0) y_4(0) + 3y_3^2(0)]$$

$$y_6(0) = 2(2 \times 1 + 3 \times 1 \times 2)$$

$$y_6(0) = 16$$

$$\log(\sec x) = 0 + x_1(0) + \frac{x^2}{2!}(1) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(2) + \frac{x^5}{5!}(0) + \frac{x^6}{6!}(16)$$

$$\log(\sec x) = \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45}$$

4. Expand $\tan x$ as MS

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Sol:- function

Value at $x=0$

$$y = \tan x$$

d.w.r.t x

$$y = \tan(0) = 0$$

$$y_1 = \sec^2 x$$

$$= 1 + \tan^2 x$$

$$y_1(0) = \sec^2(0) = 1$$

$$y_1 = 1 + y^2$$

d.w.r.t x

$$y_2 = 0 + 2yy_1$$

d.w.r.t x

$$y_2(0) = 2[y(0) \cdot y_1(0)]$$

$$y_2(0) = 0$$

$$y_3 = 2[y \cdot y_2 + y_1 \cdot y_1]$$

d.w.r.t x

$$y_3(0) = 2[y(0) \cdot y_2(0) + y_1^2(0)]$$

$$y_3(0) = 2$$

$$y_4 = 2[y \cdot y_3 + y_2 \cdot y_1 + y_1 \cdot y_2 + y_1 \cdot y_2]$$

$$y_4(0) = 2[y(0) \cdot y_3(0) + y_2(0) \cdot y_1(0) + y_1(0) \cdot y_2(0) + y_1(0) \cdot y_2(0)]$$

$$y_4(0) = 0$$

$$\tan x = 0 + x_1(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(0)$$

$$\tan x = x_1 + \frac{x^3}{3}$$

5. Expand $\log(1+\sin x)$ as MS

Sol:- function

Value at $x=0$

$$y = \log(1+\sin x)$$

d.w.r.t x

$$y(0) = \log(1+\sin(0))$$

$$y(0) = \log 1 \Rightarrow y(0) = 0$$

$$y_1 = \frac{1}{1+\sin x} \times (0+\cos x)$$

$$y_1(0) = \frac{\cos(0)}{1+\sin(0)} = \frac{1}{1+0}$$

$$y_1 = \cos x$$

d.w.r.t x

$$y_1(0) = 1$$

$$\begin{aligned}
 y_2 &= \frac{(1 + \sin x)(-\sin x) - \cos x \cdot \cos x}{(1 + \sin x)^2} \\
 &= \frac{-\sin x - \sin^2 x - \cos^2 x}{(1 + \sin x)^2} \\
 &= \frac{-(\sin x + \sin^2 x + \cos^2 x)}{(1 + \sin x)^2} \\
 &= \frac{-(\sin x + 1)}{(1 + \sin x)^2} = \frac{-1}{1 + \sin x} \quad \text{d.w.r.t } x
 \end{aligned}$$

$$\begin{aligned}
 y_2(0) &= \frac{-1}{(1 + \sin(0))} = 6 \\
 &= \frac{-1}{1+0} = -1 \\
 y_2(0) &= -1
 \end{aligned}$$

$$\begin{aligned}
 y_3 &= - \left(\frac{-1}{(1 + \sin x)^2} \times \cos x \right) \\
 &= \frac{\cos x}{(1 + \sin x)^2} \quad \text{d.w.r.t } x
 \end{aligned}$$

$$\begin{aligned}
 y_3(0) &= \frac{\cos(0)}{(1 + \sin(0))^2} \\
 y_3(0) &= 1
 \end{aligned}$$

$$y_4 = \frac{(1 + \sin x)^2(-\sin x) - \cos x \cdot 2(1 + \sin x)\cos x}{((1 + \sin x)^2)^2}$$

$$y_4 = \frac{(1 + \sin x)^2(-\sin x) - 2(1 + \sin x)\cos^2 x}{((1 + \sin x)^2)^2}$$

$$y_4 = \frac{-\sin x - \sin^3 x - 2\cos^2 x}{(1 + \sin x)^3}$$

$$\begin{aligned}
 y_4(0) &= \frac{\sin(0) - \sin^3(0) - 2\cos^2(0)}{(1 + \sin(0))^3} \\
 &= \frac{-2}{(1+0)^3} = -2
 \end{aligned}$$

$$\log(1 + \sin x) = 0 + x_1(1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(1) + \frac{x^4}{4!}(-2)$$

$$\log(1 + \sin x) = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{24} \dots$$

6. Expand $\log(1+x)$ as MS

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Sol:- function

$$y = \log(1+x)$$

d.w.r.t x

Value at $x=0$

$$y(0) = \log(1+0)$$

$$= \log 1 = 0$$

$$y(0) = 0$$

$$y_1 = \frac{1}{1+x}$$

d.w.r.t x

$$y_1(0) = \frac{1}{1+0}$$

$$y_1(0) = 1$$

$$y_2 = \frac{-1}{(1+x)^2}$$

d.w.r.t x

$$y_2(0) = \frac{-1}{(1+0)^2} = -1$$

$$y_2(0) = -1$$

$$y_3 = \frac{(-1)(-2)}{(1+x)^3}$$

d.w.r.t x

$$y_3(0) = \frac{2}{(1+0)^3}$$

$$y_3(0) = 2$$

$$y_4 = \frac{(-1)(-2)(-3)}{(1+x)^4}$$

$$y_4 = \frac{2(-3)}{(1+0)^4} = -6$$

$$\log(1+x) = 0 + x(1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(-6)$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

Hence reduce that $\log \sqrt{\frac{1+x}{1-x}} = x + \frac{x^3}{3} + \dots$

$$\log \left(\frac{1+x}{1-x} \right)^{\frac{1}{2}} = \frac{1}{2} (\log(1+x) - \log(1-x))$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

Indeterminate forms

If an expression $f(x)$ at $x=a$ assumes forms like $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \times \infty$, $\infty - \infty$, ∞^0 , 1^∞ do not represent any value are called indeterminate form.

L' Hospital's Rule:-

If $f(x)$ and $g(x)$ are two functions such that

$$i) \lim_{x \rightarrow a} f(x) = 0 \quad \& \quad \lim_{x \rightarrow a} g(x) = 0$$

ii) $f'(x)$ and $g'(x)$ exists and $g'(a)$ is not equals to '0' then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} \quad (g'(a) \neq 0)$$

$$\text{if } f'(a) = 0 \quad \& \quad g'(a) = 0, \text{ then } \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} = \lim_{x \rightarrow a} \frac{f''(x)}{g''(x)}$$

These process is continued until the indeterminate form

This rule is applicable only for $\frac{0}{0}$, $\frac{\infty}{\infty}$ indeterminate form, to all other indeterminate forms rearrange to get $\frac{0}{0}$, $\frac{\infty}{\infty}$ and then apply L'H rule.

Note:- Standard limit:-

$$i) \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 = \lim_{x \rightarrow 0} \frac{x}{\sin x}$$

$$ii) \lim_{x \rightarrow 0} \frac{\tan x}{x} = 1 = \lim_{x \rightarrow 0} \frac{x}{\tan x}$$

1. Evaluate $\lim_{x \rightarrow 0} \frac{a^x - b^x}{x}$

Sol:- Let $K = \lim_{x \rightarrow 0} \frac{a^x - b^x}{x}$

apply L'H rule $\left(\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} \right)$

$$K = \lim_{x \rightarrow 0} \frac{a^x \log a - b^x \log b}{1}$$

Apply limit

$$K = a^0 \log a - b^0 \log b$$

$$K = \log a - \log b$$

$$K = \log\left(\frac{a}{b}\right)$$

Q. Evaluate $\lim_{x \rightarrow \frac{\pi}{2}} \frac{\log(x - \frac{\pi}{2})}{\tan x}$

Sol:- Apply L'H rule

$$\left(\frac{\log\left(\frac{\pi}{2} - \frac{\pi}{2}\right)}{\tan \frac{\pi}{2}} = \frac{-\infty}{\infty} \right)$$

$$K = \lim_{x \rightarrow \frac{\pi}{2}} \frac{\frac{1}{x - \frac{\pi}{2}} (1)}{\sec^2 x} = \lim_{x \rightarrow \frac{\pi}{2}} \frac{\cos^2 x}{x - \frac{\pi}{2}}$$

Apply L'H rule

$$K = \lim_{x \rightarrow \frac{\pi}{2}} \frac{2 \cos x (-\sin x)}{1 - 0}$$

Apply limit

$$K = \frac{-2 \sin \frac{\pi}{2} \cdot \frac{\pi}{2}}{1}$$

$$\boxed{K = 0}$$

Indeterminate of the form 0^0 , ∞^0 , 0^∞ , 1^∞ :-

1. Evaluate $\lim_{x \rightarrow 0} \left(\frac{\tan x}{x} \right)^{\frac{1}{x^2}}$

Sol:- $K = \lim_{x \rightarrow 0} \left(\frac{\tan x}{x} \right)^{\frac{1}{x^2}}$

Take log b.s

$$\log K = \lim_{x \rightarrow 0} \log \left(\frac{\tan x}{x} \right)^{\frac{1}{x^2}}$$

$$= \lim_{x \rightarrow 0} \frac{1}{x^2} \log \left(\frac{\tan x}{x} \right)$$

$$= \lim_{x \rightarrow 0} \frac{\log \left(\frac{\tan x}{x} \right)}{x^2}$$

Apply L'H Rule

$$= \lim_{x \rightarrow 0} \frac{1}{\left(\frac{\tan x}{x}\right)} \times \left(\frac{x \sec^3 x - \tan x(1)}{x^3}\right)$$

$$= \frac{1}{8} \lim_{x \rightarrow 0} \frac{x}{\tan x} \cdot \lim_{x \rightarrow 0} \frac{x \sec^3 x - \tan x}{x^3}$$

Apply L'H rule

$$= \frac{1}{8} \lim_{x \rightarrow 0} \frac{x^3 \sec x \cdot \sec x \cdot \tan x + \sec^3 x(1) - \sec^3 x}{3x^2}$$

$$= \frac{1}{3} \lim_{x \rightarrow 0} \frac{\sec^3 x \cdot \tan x}{x}$$

$$= \frac{1}{3} \lim_{x \rightarrow 0} \frac{\tan x}{x} \cdot \lim_{x \rightarrow 0} \sec^3 x$$

$$\log k = \frac{1}{3} \sec^3(0)$$

$$\log k = \frac{1}{3}$$

$$k = e^{1/3}$$

Q. $\lim_{x \rightarrow \frac{\pi}{2}} (\sin x)^{\tan x}$

Sol:- Let $k = \lim_{x \rightarrow \frac{\pi}{2}} (\sin x)^{\tan x}$

Take $\log$ o.b.s

$$\log k = \lim_{x \rightarrow \frac{\pi}{2}} \log(\sin x)^{\tan x}$$

$$= \lim_{x \rightarrow \frac{\pi}{2}} \tan x \log(\sin x)$$

$$= \lim_{x \rightarrow \frac{\pi}{2}} \frac{\log(\sin x)}{\cot x}$$

Apply L'H Rule

$$= \lim_{x \rightarrow \frac{\pi}{2}} \frac{1}{\sin x} \times \cos x$$

$$= -\lim_{\alpha \rightarrow \frac{\pi}{2}} \frac{\cot \alpha}{\operatorname{cosec}^3 \alpha}$$

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Apply limit

$$= \frac{-\cot\left(\frac{\pi}{2}\right)}{\operatorname{cosec}^3\left(\frac{\pi}{2}\right)} = \frac{0}{1} = 0$$

$$\log k = 0$$

$$k = e^0 \Rightarrow \boxed{k = 1}$$

$$3. \lim_{\alpha \rightarrow \frac{\pi}{2}} (\cos \alpha)^{\frac{\pi}{2} - \alpha}$$

Sol:- $k = \lim_{\alpha \rightarrow \frac{\pi}{2}} (\cos \alpha)^{\frac{\pi}{2} - \alpha}$

Take log

$$\log k = \lim_{\alpha \rightarrow \frac{\pi}{2}} \left(\frac{\pi}{2} - \alpha\right) \log(\cos \alpha)$$

$$= \lim_{\alpha \rightarrow \frac{\pi}{2}} \frac{\log \cos \alpha}{\frac{1}{\left(\frac{\pi}{2} - \alpha\right)}} = \left[\frac{\infty}{\infty}\right]$$

$$= \lim_{\alpha \rightarrow \frac{\pi}{2}} \frac{\frac{1}{\cos \alpha} (-\sin \alpha)}{\frac{-1}{\left(\frac{\pi}{2} - \alpha\right)^2} (-1)} = \lim_{\alpha \rightarrow \frac{\pi}{2}} \frac{-\frac{\sin \alpha}{\cos \alpha}}{\frac{1}{\left(\frac{\pi}{2} - \alpha\right)^2}}$$

$$= \lim_{\alpha \rightarrow \frac{\pi}{2}} \frac{-\tan \alpha}{\frac{1}{\left(\frac{\pi}{2} - \alpha\right)^2}}$$

$$= -\lim_{\alpha \rightarrow \frac{\pi}{2}} \frac{1}{\cot \alpha} \times \left(\frac{\pi}{2} - \alpha\right)^2$$

$$= -\lim_{\alpha \rightarrow \frac{\pi}{2}} \frac{\left(\frac{\pi}{2} - \alpha\right)^2}{\cot \alpha}$$

Apply LH Rule

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{-\frac{1}{2} \left(\frac{\pi}{2} - x \right) (-1)}{\pi \operatorname{cosec}^2 x}$$

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Apply limit

$$\frac{\frac{1}{2} \left(\frac{\pi}{2} - \frac{\pi}{2} \right)}{\pi \operatorname{cosec}^2 \left(\frac{\pi}{2} \right)} = \frac{-0}{1} = 0$$

$$\log k = 0$$

$$k = e^0$$

$$\boxed{k = 1}$$

4. $\lim_{x \rightarrow a} \left(2 - \frac{x}{a} \right) \tan \left(\frac{\pi x}{2a} \right)$

Sol:- Let $k = \lim_{x \rightarrow a} \left(2 - \frac{x}{a} \right) \tan \left(\frac{\pi x}{2a} \right)$

Take log o.b.s

$$\log k = \lim_{x \rightarrow a} \tan \left(\frac{\pi x}{2a} \right) \log \left(2 - \frac{x}{a} \right)$$

$$= \lim_{x \rightarrow a} \frac{\log \left(2 - \frac{x}{a} \right)}{\cot \left(\frac{\pi x}{2a} \right)}$$

Apply LH rule

$$\lim_{x \rightarrow a} \frac{\frac{1}{\left(2 - \frac{x}{a} \right)} \times \frac{-1}{a}}{\pi \operatorname{cosec}^2 \left(\frac{\pi x}{2a} \right) \left(\frac{\pi}{2a} \right)}$$

$$\frac{2}{\pi} \lim_{x \rightarrow a} \frac{1}{\operatorname{cosec}^2 \left(\frac{\pi x}{2a} \right) \left(2 - \frac{x}{a} \right)}$$

$$\frac{2}{\pi} \lim_{x \rightarrow a} \frac{\sin^2 \left(\frac{\pi x}{2a} \right)}{\left(2 - \frac{x}{a} \right)}$$

Apply limit

$$\frac{\frac{2}{\pi} \cdot \sin^2\left(\frac{\pi a}{2a}\right)}{\frac{2-a}{a}} = \frac{\frac{2}{\pi}}{1} \cdot \frac{\sin \frac{\pi}{2}}{1}$$

$$\log k = \frac{2}{\pi} \times 1$$

$$\log k = \frac{2}{\pi}$$

$$k = e^{\frac{2}{\pi}}$$

5. Evaluate $\lim_{x \rightarrow 0} \left(\frac{a^x + b^x}{2} \right)^{\frac{1}{x}}$

Sol:- Let $k = \lim_{x \rightarrow 0} \left(\frac{a^x + b^x}{2} \right)^{\frac{1}{x}}$

Take log o. b. s

$$\log k = \lim_{x \rightarrow 0} \frac{1}{x} \log \left(\frac{a^x + b^x}{2} \right)$$

$$= \lim_{x \rightarrow 0} \frac{\log \left(\frac{a^x + b^x}{2} \right)}{x}$$

Apply LH Rule

$$= \lim_{x \rightarrow 0} \frac{1 \times (a^x \log a + b^x \log b)}{\frac{(a^x + b^x)}{2} \cdot 2}$$

$$= \lim_{x \rightarrow 0} \frac{a^x \log a + b^x \log b}{a^x + b^x}$$

Apply limit

$$= \frac{a^0 \log a + b^0 \log b}{a^0 + b^0}$$

$$= \frac{\log a + \log b}{2}$$

$$\log K = \frac{\log(ab)}{\frac{1}{3}} = \log(ab)^{\frac{1}{3}}$$

$$\log K = \log(ab)^{\frac{1}{3}}$$

$$K = \sqrt[3]{ab}$$

6. Evaluate $\lim_{x \rightarrow 0} \left(\frac{a^x + b^x + c^x}{3} \right)^{\frac{1}{x}}$

Sol:- Let $K = \lim_{x \rightarrow 0} \left(\frac{a^x + b^x + c^x}{3} \right)^{\frac{1}{x}}$

Take log o.b.s

$$\log K = \lim_{x \rightarrow 0} \frac{1}{x} \left\{ \log \left(\frac{a^x + b^x + c^x}{3} \right) \right\}$$

$$\log K = \lim_{x \rightarrow 0} \log \left(\frac{a^x + b^x + c^x}{3} \right)^{\frac{1}{x}}$$

$$\log \left(\frac{m}{n} \right) = \log m - \log n$$

$$= \lim_{x \rightarrow 0} \frac{\log(a^x + b^x + c^x) - \log 3}{x}$$

Apply LH Rule

$$= \lim_{x \rightarrow 0} \frac{1 \times a^x \log a + b^x \log b + c^x \log c}{a^x + b^x + c^x}$$

$$= \lim_{x \rightarrow 0} \frac{a^x \log a + b^x \log b + c^x \log c}{a^x + b^x + c^x}$$

Apply limit

$$\frac{a^0 \log a + b^0 \log b + c^0 \log c}{a^0 + b^0 + c^0} = \frac{1}{3} \cdot \log a + \log b + \log c$$

$$\log K = \frac{1}{3} \log abc$$

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$$\log K = \log (abc)^{\frac{1}{3}}$$

$$K = \sqrt[3]{abc}$$

7. Evaluate $\lim_{x \rightarrow 0} \left(\frac{a^x + b^x + c^x + d^x}{4} \right)^{\frac{1}{x}}$

Sol: Let $K = \lim_{x \rightarrow 0} \left(\frac{a^x + b^x + c^x + d^x}{4} \right)^{\frac{1}{x}}$

Take log o.b.s

$$\log K = \lim_{x \rightarrow 0} \frac{1}{x} \log \left(\frac{a^x + b^x + c^x + d^x}{4} \right)$$

$$= \lim_{x \rightarrow 0} \frac{\log \left(\frac{a^x + b^x + c^x + d^x}{4} \right)}{x}$$

Apply LH Rule

$$\lim_{x \rightarrow 0} \frac{a^x \log a + b^x \log b + c^x \log c + d^x \log d}{a^x + b^x + c^x + d^x} = 0$$

$$\lim_{x \rightarrow 0} \frac{a^x \log a + b^x \log b + c^x \log c + d^x \log d}{a^x + b^x + c^x + d^x}$$

Apply limit

$$\log K = \frac{a^0 \log a + b^0 \log b + c^0 \log c + d^0 \log d}{a^0 + b^0 + c^0 + d^0}$$

$$= \frac{\log a + \log b + \log c + \log d}{4}$$

$$\log k = \log (abcd)^{\frac{1}{4}}$$

$$k = (abcd)^{\frac{1}{4}}$$

8. Evaluate $\lim_{x \rightarrow 0} \left(\frac{\tan x}{x} \right)^{\frac{1}{x}}$

Sol:- Let $k = \lim_{x \rightarrow 0} \left(\frac{\tan x}{x} \right)^{\frac{1}{x}}$

Take log o.b.s

$$\log k = \lim_{x \rightarrow 0} \frac{1}{x} \log \left(\frac{\tan x}{x} \right)$$

$$\log k = \lim_{x \rightarrow 0} \frac{\log \left(\frac{\tan x}{x} \right)}{x} = \lim_{x \rightarrow 0} \frac{\log \tan x - \log x}{x}$$

Apply LH Rule

$$\lim_{x \rightarrow 0} \frac{1 \times \sec^2 x - \frac{1}{x} (1)}{1}$$

$$\lim_{x \rightarrow 0} \frac{x \sec^2 x - \tan x}{x \tan x}$$

Multiply $x \div x$ in denominator

$$= \lim_{x \rightarrow 0} \frac{x \sec^2 x - \tan x}{x^2 \frac{\tan x}{x}}$$

$$= \lim_{x \rightarrow 0} \frac{x \sec^2 x - \tan x}{x^2}$$

Apply LH Rule

$$\lim_{x \rightarrow 0} \frac{x \sec x \cdot \sec x \cdot \tan x + \sec^2 x (1) - \sec^2 x}{2x}$$

$$\lim_{x \rightarrow 0} \frac{2x \cdot \sec^2 x \cdot \tan x}{2x}$$

Apply limit

$$\log K = \sec^0(0) \cdot \tan(0)$$

$$\log K = 0$$

$$K = e^0 = 1$$

$$\boxed{K = 1}$$

$$9. K = \lim_{x \rightarrow \infty} \left(\frac{ax+1}{ax-1} \right)^x$$

Take log o.b.s

$$\log K = \lim_{x \rightarrow \infty} \log \left(\frac{ax+1}{ax-1} \right)^x$$

$$= \lim_{x \rightarrow \infty} x \log \left(\frac{ax+1}{ax-1} \right)$$

$$= \lim_{x \rightarrow \infty} x \log \left(\frac{x(a + \frac{1}{x})}{x(a - \frac{1}{x})} \right)$$

$$\lim_{x \rightarrow \infty} \frac{\log \left(a + \frac{1}{x} \right)}{\frac{\log \left(a - \frac{1}{x} \right)}{\frac{1}{x}}}$$

Apply LH Rule

$$\lim_{x \rightarrow \infty} \frac{\log \left(a + \frac{1}{x} \right) - \log \left(a - \frac{1}{x} \right)}{\frac{1}{x}}$$

$$\lim_{x \rightarrow \infty} \frac{\frac{1}{\left(a + \frac{1}{x} \right)} \left(0 - \frac{1}{x^2} \right) - \frac{1}{\left(a - \frac{1}{x} \right)} \left(0 - \left(-\frac{1}{x^2} \right) \right)}{-\frac{1}{x^2}}$$

$$= \lim_{x \rightarrow \infty} \frac{1}{\left(a + \frac{1}{x}\right)} \left(\frac{-1}{x^2}\right) - \frac{1}{\left(a - \frac{1}{x}\right)} \left(\frac{-1}{x^2}\right)$$

$$\lim_{x \rightarrow \infty} \frac{\left(\frac{-1}{x^2}\right) \left(\frac{1}{a + \frac{1}{x}} - \frac{1}{a - \frac{1}{x}}\right)}{\left(\frac{-1}{x^2}\right)}$$

Apply limit

$$\frac{1}{a + \frac{1}{\infty}} + \frac{1}{a - \frac{1}{\infty}} = \frac{1}{a} + \frac{1}{a}$$

$$\log K = \frac{2}{a}$$

$$K = e^{\frac{2}{a}}$$

$$10. K = \lim_{x \rightarrow 0} (1 - x^2)^{\frac{1}{\log(1-x)}}$$

Sol:- $\log K = \lim_{x \rightarrow 0} \frac{1}{\log(1-x)} \log(1-x^2)$

$$= \lim_{x \rightarrow 0} \frac{\log(1-x^2)}{\log(1-x)}$$

Apply LH Rule

$$\lim_{x \rightarrow 0} \frac{\frac{1}{(1-x^2)} \times (0-2x)}{\frac{1}{(1-x)} \times (0-1)}$$

$$= \frac{\frac{-2x}{(1-x^2)}}{\frac{-1}{(1-x)}}$$

$$= \frac{-2x}{(1-x^2)} \cdot \frac{(1-x)}{-1}$$

$$= \frac{2x - 2x^2}{(1-x^2)}$$

Partial Differentiation:-

Partial differentiation of a function deals with differentiation of function of many independent variables.

Partial derivatives:-

Let U be the function two independent variables then partial derivatives of U is denoted by $\frac{\partial U}{\partial x}$.

Similarly, partial derivative of U with respect to y is denoted by $\frac{\partial U}{\partial y}$ and is defined as,

i.e. if $u = f(x, y)$

$$\Rightarrow \frac{\partial u}{\partial x} = U_x = \lim_{\partial x \rightarrow 0} \frac{f(x + \partial x, y) - f(x, y)}{\partial x}$$

$$\Rightarrow \frac{\partial u}{\partial y} = U_y = \lim_{\partial y \rightarrow 0} \frac{f(x, y + \partial y) - f(x, y)}{\partial y}$$

Higher Order derivatives:-

$$1) \frac{\partial^2 u}{\partial x^2} = U_{xx} \quad 2) \frac{\partial^2 u}{\partial y^2} = U_{yy} \quad 3) \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial x} \right) = U_{xy} \quad 4) \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial y} \right) = U_{yx}$$

Note:- $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

Partial derivative can be extended to the function which contains more than one independent variable.

Problems:-

1. Find $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ $u = x^2 + y^2 \rightarrow (1)$

Sol:-

d.(1) p.w. r. to x

$$\frac{\partial u}{\partial x} = 2x + 0 \quad (\text{Treat all } y \text{ terms as constant})$$

$$u = \frac{\partial u}{\partial x} = 2x$$

d(1) p. w. r. to y (Treat all x terms as constant) $\neq 0$

$$U_y = \frac{\partial u}{\partial y} = 0 + 2y \Rightarrow \frac{\partial u}{\partial y} = 2y$$

2. $Z = 2xy \rightarrow (1)$

Sol: d(1) p. w. r. to x

$$Z_x = \frac{\partial Z}{\partial x} = 2y(1) \text{ (Treat all 'y' terms as constant)}$$

$$Z_x = \frac{\partial Z}{\partial x} = 2y$$

d(1) p. w. r. to y

$$Z_y = \frac{\partial Z}{\partial y} = 2x(1) \text{ (Treat all x terms as constant)}$$

$$Z_y = \frac{\partial Z}{\partial y} = 2x$$

3. $Z = e^{x+y}$

Sol: $Z = e^x \cdot e^y \rightarrow (1)$

d(1) p. w. r. to 'x' (Treat y terms as constant)

$$Z_x = \frac{\partial Z}{\partial x} = e^y \cdot e^x$$

d(1) p. w. r. to 'y' (Treat x terms as constant)

$$Z_y = \frac{\partial Z}{\partial y} = e^x \cdot e^y$$

4. $U = e^x \cdot \sin 2x \cdot \cos y \rightarrow (1)$

Sol: d(1) p. w. r. to 'x' (Treat all 'y' terms as constant)

$$U_x = \frac{\partial u}{\partial x} = \cos y [e^x \cdot \cos 2x(2) + \sin 2x \cdot e^x]$$

d(1) p. w. r. to 'y' (Treat all 'x' terms as constant)

$$U_y = \frac{\partial u}{\partial y} = e^x \cdot \sin 2x [-\sin y] \Rightarrow U_y = \frac{\partial u}{\partial y} = -e^x \cdot \sin 2x \cdot \sin y$$

$$u = \frac{\tan x \cdot \sec y}{x+y} \rightarrow (1)$$

Sol:- $u_x = \frac{\partial u}{\partial x} = \sec y \cdot \frac{\partial}{\partial x} \left(\frac{\tan x}{x+y} \right)$

$$= \sec y \left[\frac{(x+y) \sec^2 x - \tan x (1+0)}{(x+y)^2} \right]$$

$$= \sec y \left[\frac{(x+y) \sec^2 x - \tan x}{(x+y)^2} \right]$$

$$u_y = \frac{\partial u}{\partial y} = \tan x \frac{\partial}{\partial y} \left(\frac{\sec y}{x+y} \right)$$

$$= \tan x \left[\frac{(x+y) \sec y \cdot \tan y - \sec y (1)}{(x+y)^2} \right]$$

6. $u = x^3 - 3xy^2 + x + e^x \cos y + 1$, P.T $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Sol:- $u = x^3 - 3xy^2 + x + e^x \cos y + 1 \rightarrow (1)$

$$u_x = \frac{\partial u}{\partial x} = 3x^2 - 3y^2(1) + 1 + \cos y e^x + 0$$

$$u_{xx} = \frac{\partial^2 u}{\partial x^2} = 6x - 0 + 0 + \cos y e^x$$

$$u_{xx} = 6x + \cos y e^x$$

d (1) p. w. r. to 'y'

$$u_y = \frac{\partial u}{\partial y} = 0 - 3x \cdot 2y + 0 + e^x (-\sin y) + 0$$

$$u_y = -3x \cdot 2y - e^x \sin y$$

$$u_{yy} = \frac{\partial^2 u}{\partial y^2} = -[6x(1) + e^x \cos y] = -6x - e^x \cos y$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x + e^x \cos y - 6x - e^x \cos y$$

$$= 0$$

$$4. \quad u = \tan^{-1}\left(\frac{y}{x}\right)$$

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d.p. w. r. to x, y separately.

$$\frac{\partial u}{\partial x} = y \cdot \frac{1}{1 + \left(\frac{y}{x}\right)^2} \cdot \frac{-1}{x^2}$$

$$= \frac{-y}{x^2} \cdot \frac{1}{\frac{x^2 + y^2}{x^2}} = \frac{-y}{x^2} \cdot \frac{x^2}{x^2 + y^2}$$

$$\frac{\partial u}{\partial x} = \frac{-y}{x^2 + y^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{x} \cdot \frac{1}{1 + \left(\frac{y}{x}\right)^2}$$

$$= \frac{1}{x} \cdot \frac{1}{\frac{x^2 + y^2}{x^2}} = \frac{1}{x} \cdot \frac{x^2}{x^2 + y^2}$$

$$\frac{\partial u}{\partial y} = \frac{x}{x^2 + y^2}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = -y \left[\frac{-1}{(x^2 + y^2)^2} (2x + 0) \right]$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{2xy}{(x^2 + y^2)^2}$$

$$\Rightarrow \frac{\partial^2 u}{\partial y^2} = x \left[\frac{-1}{(x^2 + y^2)^2} (0 + 2y) \right]$$

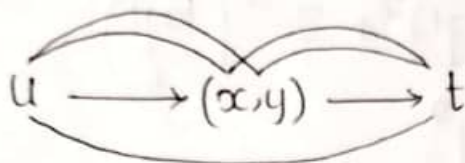
$$\frac{\partial^2 u}{\partial y^2} = \frac{-2xy}{(x^2 + y^2)^2}$$

Total Derivative:-

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If $u = f(x, y)$ where $x = x(t)$ & $y = y(t)$ then we can express u as function of ' t ' alone by substituting the values of x & y in $f(x, y)$. Then we find ordinary derivative differentiation, $\frac{du}{dt}$ is called total derivative.

i.e.



$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$

1. If $u = x^3 y^2$, $x = e^t$, $y = \log t$ then find $\frac{du}{dt}$ and also verify the result by taking direct substitution.

Sol:-

$$u \longrightarrow (x, y) \longrightarrow t$$

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$

Consider, $u = x^3 y^2$ D.P.W. r. to ' x '

$$\frac{\partial u}{\partial x} = y^2 (3x^2)$$

D.P.W. r. to ' y '

$$\frac{\partial u}{\partial y} = x^3 (2y)$$

Consider,

$$x = e^t$$

d.w.r. to t

$$\frac{dx}{dt} = e^t$$

$$y = \log t$$

d.w.r. to t

$$\frac{dy}{dt} = \frac{1}{t}$$

∴ (1) becomes,

$$\frac{du}{dt} = (3x^2y^2)e^t + 2x^3y\left(\frac{1}{t}\right)$$

but $x = e^t$, $y = \log t$

$$\frac{du}{dt} = 3(e^t)^2(\log t)^2 \cdot e^t + 2(e^t)^3 \cdot \log t \left(\frac{1}{t}\right)$$

$$\frac{du}{dt} = 3e^{3t}(\log t)^2 + 2e^{3t}\left(\frac{\log t}{t}\right)$$

$$u = x^3y^2$$

$$u = (e^t)^3(\log t)^2$$

$$u = e^{3t}(\log t)^2$$

d. w. r. to 't'

$$\frac{du}{dt} = e^{3t}(2) \log t \left(\frac{1}{t}\right) + (\log t)^2 e^{3t} \cdot 3$$

∴ Result is verified by direct substitution.

Q. $Z = u^2 + v^2$, $u = at^2$, $v = 2at$ find $\frac{dz}{dt}$

Sol:-

$$Z \longrightarrow (u, v) \longrightarrow t$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial u} \cdot \frac{du}{dt} + \frac{\partial z}{\partial v} \cdot \frac{dv}{dt}$$

Consider, $Z = u^2 + v^2$

d.p. w. r. to u .

$$\frac{\partial z}{\partial u} = 2u + 0 \Rightarrow \frac{\partial z}{\partial u} = 2u$$

d.p. w. r. to v .

$$\frac{\partial z}{\partial v} = 0 + 2v \Rightarrow \frac{\partial z}{\partial v} = 2v$$

Consider, $u = at^2$

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d.w.r.to. t

$$\frac{du}{dt} = 2at \Rightarrow \frac{du}{dt} = 2at$$

$$v = 2at$$

$$\frac{dv}{dt} = 2a$$

$$\therefore (1) \text{ becomes, } \frac{dz}{dt} = 2u \cdot 2at + 2v \cdot 2a$$

$$\text{but } u = at^2, v = 2at$$

$$= 2(at^2) \cdot 2at + 2(2at) \cdot 2a$$

$$= 4a^2t^3 + 8a^2t$$

3. If $u = \sin^{-1}\left(\frac{x}{y}\right)$, $x = e^t$, $y = t^2$ find $\frac{du}{dt}$

$$u \longrightarrow (x, y) \longrightarrow t$$

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$

Consider, $u = \sin^{-1}\left(\frac{x}{y}\right)$ d.p.w.r.to x

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{1}{\sqrt{1 - \left(\frac{x}{y}\right)^2}} \cdot \frac{1}{y} (1) = \frac{1}{\sqrt{\frac{y^2 - x^2}{y^2}}} \cdot \frac{1}{y} \\ &= \frac{y}{\sqrt{y^2 - x^2}} \cdot \frac{1}{y} \end{aligned}$$

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{y^2 - x^2}}$$

Consider, $x = e^t$ d.p.w.r.to y .

$$\begin{aligned} \frac{\partial u}{\partial y} &= \frac{1}{\sqrt{1 - \left(\frac{x}{y}\right)^2}} \cdot x \left(\frac{-1}{y^2}\right) = \frac{1}{\sqrt{\frac{y^2 - x^2}{y^2}}} \cdot \frac{-x}{y^2} \\ &= \frac{y}{\sqrt{y^2 - x^2}} \cdot \frac{-x}{y^2} \end{aligned}$$

$$= \frac{-x}{y\sqrt{y^2-x^2}}$$

Consider, $x = e^t$
d.w.r.to. t

$$\frac{dx}{dt} = e^t$$

$$y = t^2$$

d.w.r.to t

$$\frac{dy}{dt} = 2t$$

$$= \frac{1}{\sqrt{y^2-x^2}} \cdot e^t + \left(-\frac{x}{y}\right) \frac{1}{\sqrt{y^2-x^2}} \cdot 2t$$

$$= \frac{1}{\sqrt{y^2-x^2}} \left(e^t - \frac{x}{y} 2t \right)$$

$$= \frac{1}{\sqrt{(t^2)^2 - (e^t)^2}} \left(e^t - \frac{e^t}{t^2} 2t \right)$$

$$= \frac{e^t}{\sqrt{t^4 - e^{2t}}} \left(1 - \frac{2}{t} \right)$$

4. If $u = f(x, y, z)$ & $x = x(t)$, $y = y(t)$, $z = z(t)$

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt}$$

find $\frac{\partial u}{\partial t}$ at $t=0$ if $u = e^{x^2+y^2+z^2}$ & $x = t^2+1$, $y = t \cos t$, $z = \sin t$

sol:-

$$u \rightarrow (x, y, z) \rightarrow t$$

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt}$$

consider, $u = e^{x^2+y^2+z^2}$

$$u = e^{x^2} \cdot e^{y^2} \cdot e^{z^2}$$

d.p.w.r. to x

$$\frac{\partial u}{\partial x} = e^{y^2} \cdot e^{z^2} \cdot e^{x^2} 2x \rightarrow (1)$$

d.p.w.r. to y

$$\frac{\partial u}{\partial y} = e^{x^2} \cdot e^{z^2} \cdot e^{y^2} 2y \rightarrow (2)$$

d.p.w.r. to z

$$\frac{\partial u}{\partial z} = e^{x^2} \cdot e^{y^2} \cdot e^{z^2} \cdot 2z \rightarrow (3)$$

consider,

$$x = t^2 + 1$$

d.w.r. to t

$$\frac{dx}{dt} = 2t + 0 \Rightarrow \frac{dx}{dt} = 2t$$

$$y = t \cos t$$

d.w.r. to t

$$\frac{dy}{dt} = t(-\sin t) + \cos(1)$$

$$\frac{dy}{dt} = \cos t - t \sin t$$

$$z = \sin t$$

d.w. r to t

$$\frac{dz}{dt} = \cos t$$

$$\frac{du}{dt} = e^{y^2} \cdot e^{z^2} \cdot e^{x^2} (2x) \cdot 2t + e^{z^2} \cdot e^{x^2} \cdot e^{y^2} (2y) \cdot \cos t - t \sin t + e^{x^2} \cdot e^{y^2} \cdot e^{z^2} (2z) \cos t$$

$$= e^{x^2+y^2+z^2} (2t \cdot 2x + 2y(\cos t - t \sin t) + 2z \cos t)$$

$$= 2e^{x^2+y^2+z^2} (2xt + y(\cos t - t \sin t) + z \cos t)$$

$$= 2e^{(t^2+1)+(t \cos t)^2+(\sin t)^2} (0+y+z)$$

$$= 2e^{(0+t \cos t + \sin t)}$$

$$= 2e^{(0)}$$

$$= 2e^{(0)}$$

$$\frac{du}{dt} = 0$$

Total derivative of composite function (two variables)

If u is function of x & y that is $u = f(x, y)$ and x, y are functions of ' r ' and ' s ' then the total derivative of u is given by

$$u \longrightarrow (x, y) \longrightarrow (r, s)$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial r} \quad \& \quad \frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial s}$$

5. $z = f(x, y)$ $x = r \cos \theta$, $y = r \sin \theta$ PT

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$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \left(\frac{1}{r^2} \left(\frac{\partial z}{\partial \theta}\right)^2\right)$$

sol:-

$$z \rightarrow (x, y) \rightarrow (r, \theta)$$

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial r} \quad \& \quad \frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial \theta}$$

$$x = r \cos \theta$$

$$y = r \sin \theta$$

D.P.W. r to 'x'

D.P.W. r to 'y'

$$\frac{\partial x}{\partial r} = \cos \theta (1)$$

$$\frac{\partial y}{\partial r} = \sin \theta (1)$$

D.P.W. r to ' θ '

D.P.W. r to ' θ '

$$\frac{\partial x}{\partial \theta} = -r \sin \theta$$

$$\frac{\partial y}{\partial \theta} = r \cos \theta$$

$$\frac{\partial z}{\partial r} = \left(\frac{\partial z}{\partial x}\right) (\cos \theta) + \frac{\partial z}{\partial y} (\sin \theta)$$

$$\frac{\partial z}{\partial \theta} = \left(\frac{\partial z}{\partial x}\right) (-r \sin \theta) + \frac{\partial z}{\partial y} (r \cos \theta)$$

consider,

$$\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2} \left(\frac{\partial z}{\partial \theta}\right)^2$$

$$= \left(\left(\frac{\partial z}{\partial x}\right) \cos \theta + \left(\frac{\partial z}{\partial y}\right) \sin \theta \right)^2 + \frac{1}{r^2} \left(\left(\frac{\partial z}{\partial x}\right) (-r \sin \theta) + \frac{\partial z}{\partial y} (r \cos \theta) \right)^2$$

$$\text{H.S.} = \left(\frac{\partial z}{\partial x}\right)^2 \cos^2 \theta + \left(\frac{\partial z}{\partial y}\right)^2 \sin^2 \theta + 2 \frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y} \cdot \frac{\partial z}{\partial y} \sin \theta \cdot \cos \theta +$$

$$\frac{1}{r^2} \left[\left(\frac{\partial z}{\partial x}\right)^2 r^2 \sin^2 \theta + \left(\frac{\partial z}{\partial y}\right)^2 r^2 \cos^2 \theta - 2 r r \frac{\partial z}{\partial x} \sin \theta \cdot \frac{\partial z}{\partial y} \cos \theta \right]$$

$$= \left(\frac{\partial z}{\partial x}\right)^2 \cos^2 \theta + \left(\frac{\partial z}{\partial y}\right)^2 \sin^2 \theta + 2 \frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y} \sin \theta \cdot \cos \theta +$$

$$\left(\frac{\partial z}{\partial x}\right)^2 \sin^2 \theta + \left(\frac{\partial z}{\partial y}\right)^2 \cos^2 \theta - 2 \frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y} \sin \theta \cdot \cos \theta$$

6. If $z = f(x, y)$, $x = u - v$, $y = uv$.

$$i) (u+v) \frac{\partial z}{\partial x} = u \frac{\partial z}{\partial u} - v \frac{\partial z}{\partial v}$$

$$ii) (u+v) \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$$

sol:- $z \rightarrow (x, y) \rightarrow (u, v)$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \left(\frac{\partial x}{\partial u} \right) + \frac{\partial z}{\partial y} \left(\frac{\partial y}{\partial u} \right) \rightarrow (i) \quad \& \quad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \left(\frac{\partial x}{\partial v} \right) + \frac{\partial z}{\partial y} \left(\frac{\partial y}{\partial v} \right) \rightarrow (ii)$$

consider: $x = u - v$

diff p.w.r to x & y separately

$$\frac{\partial x}{\partial u} = 1 - 0 \Rightarrow \frac{\partial x}{\partial u} = 1 ; \quad \frac{\partial x}{\partial v} = 0 - 1 \rightarrow \frac{\partial x}{\partial v} = -1$$

$$y = uv$$

diff p.w.r to x & y separately

$$\frac{\partial y}{\partial u} = v(1) \quad \frac{\partial y}{\partial v} = u(1)$$

$\therefore$ (i) & (ii) becomes

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} (1) + \frac{\partial z}{\partial y} (v)$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} (-1) + \frac{\partial z}{\partial y} (u)$$

$$\text{consider: } u \frac{\partial z}{\partial u} - v \frac{\partial z}{\partial v}$$

$$= u \left[\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} v \right] - v \left[-\frac{\partial z}{\partial x} + u \frac{\partial z}{\partial y} \right]$$

$$= u \frac{\partial z}{\partial x} + uv \frac{\partial z}{\partial y} + v \frac{\partial z}{\partial x} - uv \frac{\partial z}{\partial y} = u \frac{\partial z}{\partial x} + v \frac{\partial z}{\partial x} = \frac{\partial z}{\partial x} (u+v)$$

$$\text{ii)} \quad (u+v) \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$$

$$\text{eq-(1)} \quad \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} (v)$$

$$\text{eq-(2)} \quad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial y} u - \frac{\partial z}{\partial x}$$

consider;

$$\begin{aligned} & \text{RHS} \\ &= \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} (v) + \frac{\partial z}{\partial y} (u) - \frac{\partial z}{\partial x} \\ &= \frac{\partial z}{\partial y} (u+v) \\ &= \text{LHS} \end{aligned}$$

$$7. \text{ If } z = f(x, y), \quad x = e^u \cos v, \quad y = e^u \sin v$$

$$\text{PT } \left(\frac{\partial z}{\partial u} \right)^2 + \left(\frac{\partial z}{\partial v} \right)^2 = \left[e^{2u} \left(\left(\frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2 \right) \right]$$

$$\text{sol:-} \quad z \rightarrow (x, y) \rightarrow (u, v)$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u} \rightarrow (1), \quad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v} \rightarrow (2)$$

$$x = e^u \cos v$$

$$y = e^u \sin v$$

d.p.w.r. to u & v separately

d.p.w.r. to u & v separately

$$\frac{\partial x}{\partial u} = \cos v \cdot e^u$$

$$\frac{\partial y}{\partial u} = e^u \sin v$$

$$\frac{\partial x}{\partial v} = -e^u \sin v$$

$$\frac{\partial y}{\partial v} = e^u \cos v$$

eq-(1) becomes

eq-(2)

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} (e^u \cos v) + \frac{\partial z}{\partial y} (e^u \sin v)$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} (e^u \sin v) + \frac{\partial z}{\partial y} (e^u \cos v)$$

$$= \frac{\partial z}{\partial y} (e^u \cos v) - \frac{\partial z}{\partial x} (e^u \sin v)$$

Consider, LHS,

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$$\begin{aligned}
 \left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 &= \left(\frac{\partial z}{\partial x} (e^u \cos v) + \frac{\partial z}{\partial y} (e^u \sin v)\right)^2 + \left(\frac{\partial z}{\partial x} (e^u \cos v) - \frac{\partial z}{\partial y} (e^u \sin v)\right)^2 \\
 &= \left(\frac{\partial z}{\partial x}\right)^2 e^{2u} \cos^2 v + \left(\frac{\partial z}{\partial y}\right)^2 e^{2u} \sin^2 v + 2 \frac{\partial z}{\partial x} e^u \cos u \cdot \frac{\partial z}{\partial y} e^u \sin u + \\
 &\quad \left(\frac{\partial z}{\partial y}\right)^2 e^{2u} \cos^2 v + \left(\frac{\partial z}{\partial x}\right)^2 e^{2u} \sin^2 v - 2 \frac{\partial z}{\partial x} e^u \sin v \cdot \frac{\partial z}{\partial y} e^u \cos v \\
 &= e^{2u} \left[\left(\frac{\partial z}{\partial x}\right)^2 \cos^2 v + \left(\frac{\partial z}{\partial y}\right)^2 \sin^2 v + \left(\frac{\partial z}{\partial x}\right)^2 \cos^2 v + \left(\frac{\partial z}{\partial y}\right)^2 \sin^2 v \right] \\
 &= e^{2u} \left[\sin^2 v \left(\left(\frac{\partial z}{\partial y}\right)^2 + \left(\frac{\partial z}{\partial x}\right)^2 \right) + \cos^2 v \left(\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 \right) \right] \\
 &= e^{2u} \left[\left(\frac{\partial z}{\partial y}\right)^2 + \left(\frac{\partial z}{\partial x}\right)^2 (\sin^2 v + \cos^2 v) \right] \\
 &= e^{2u} \left[\left(\frac{\partial z}{\partial y}\right)^2 + \left(\frac{\partial z}{\partial x}\right)^2 \right] \\
 &= RHS
 \end{aligned}$$

8. PT $x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} + z \cdot \frac{\partial u}{\partial z} = 0$ if $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$

sol: $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$ put $\frac{x}{y} = p, \frac{y}{z} = q, \frac{z}{x} = r$

$$u = f(p, q, r), p = \frac{x}{y}, q = \frac{y}{z}, r = \frac{z}{x}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial x} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial x} + \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x} \rightarrow (1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial y} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial y} + \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial y} \rightarrow (2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial z} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial z} + \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial z} \rightarrow (3)$$

$$p = \frac{x}{y}$$

9. If $u = f(x-y, y-z, z-x)$

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PT $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

sol:- $u = f(x-y, y-z, z-x)$ Put $x-y=a, y-z=b, z-x=c$

$u = f(a, b, c), a = x-y; b = y-z; c = z-x$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial a} \cdot \frac{\partial a}{\partial x} + \frac{\partial u}{\partial b} \cdot \frac{\partial b}{\partial x} + \frac{\partial u}{\partial c} \cdot \frac{\partial c}{\partial x} \rightarrow (1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial a} \cdot \frac{\partial a}{\partial y} + \frac{\partial u}{\partial b} \cdot \frac{\partial b}{\partial y} + \frac{\partial u}{\partial c} \cdot \frac{\partial c}{\partial y} \rightarrow (2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial a} \cdot \frac{\partial a}{\partial z} + \frac{\partial u}{\partial b} \cdot \frac{\partial b}{\partial z} + \frac{\partial u}{\partial c} \cdot \frac{\partial c}{\partial z} \rightarrow (3)$$

$a = x-y$

d.p.w.r to x, y, z separately

$$\frac{\partial a}{\partial x} = 1, \frac{\partial a}{\partial y} = -1, \frac{\partial a}{\partial z} = 0$$

$b = y-z$ d.p.w.r to x, y, z separately

$$\frac{\partial b}{\partial x} = 0, \frac{\partial b}{\partial y} = 1, \frac{\partial b}{\partial z} = -1$$

$c = z-x$ d.p.w.r to x, y, z separately

$$\frac{\partial c}{\partial x} = -1, \frac{\partial c}{\partial y} = 0, \frac{\partial c}{\partial z} = 1$$

eq- (1) becomes

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial a} (1) + \frac{\partial u}{\partial b} (0) + \frac{\partial u}{\partial c} (-1)$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial a} - \frac{\partial u}{\partial c}$$

eq $\rightarrow$ (2) becomes

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial a} (-1) + \frac{\partial u}{\partial b} (1) + \frac{\partial u}{\partial c} (0)$$

D.P.w.r. to x, y, z separately

$$\frac{\partial P}{\partial x} = \frac{1}{y}, \quad \frac{\partial P}{\partial y} = x \left(\frac{-1}{y^2} \right), \quad \frac{\partial P}{\partial z} = 0$$

$$q = \frac{y}{z}$$

D.P.w.r. to x, y, z separately

$$\frac{\partial q}{\partial x} = 0, \quad \frac{\partial q}{\partial y} = \frac{1}{z}, \quad \frac{\partial q}{\partial z} = -\frac{y}{z^2}$$

$$r = \frac{z}{x}$$

D.P.w.r. to x, y, z separately

$$\frac{\partial r}{\partial x} = \frac{-z}{x^2}, \quad \frac{\partial r}{\partial y} = 0, \quad \frac{\partial r}{\partial z} = \frac{1}{x}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial P} \cdot \frac{1}{y} + \frac{\partial u}{\partial q} (0) + \frac{\partial u}{\partial r} \left(\frac{-z}{x^2} \right)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial P} \left(\frac{-x}{y^2} \right) + \frac{\partial u}{\partial q} \left(\frac{1}{z} \right) + \frac{\partial u}{\partial r} (0)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial P} (0) + \frac{\partial u}{\partial q} \left(-\frac{y}{z^2} \right) + \frac{\partial u}{\partial r} \left(\frac{1}{x} \right)$$

$$\text{Consider: } x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z}$$

$$= x \cdot \frac{\partial u}{\partial P} \frac{1}{y} - \frac{z}{x} \cdot \frac{\partial u}{\partial r} - \frac{x}{y} \cdot \frac{\partial u}{\partial P} + \frac{y}{z} \cdot \frac{\partial u}{\partial q} - \frac{y}{z} \cdot \frac{\partial u}{\partial q} + \frac{z}{x} \cdot \frac{\partial u}{\partial r} = 0$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial b} - \frac{\partial u}{\partial a}$$

eq $\rightarrow$ (3) becomes

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial a} (0) + \frac{\partial u}{\partial b} (-1) + \frac{\partial u}{\partial c} (1)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial c} - \frac{\partial u}{\partial b}$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{\cancel{\partial u}}{\cancel{\partial a}} - \frac{\cancel{\partial u}}{\cancel{\partial c}} + \frac{\cancel{\partial u}}{\cancel{\partial b}} - \frac{\cancel{\partial u}}{\cancel{\partial a}} + \frac{\cancel{\partial u}}{\cancel{\partial c}} - \frac{\cancel{\partial u}}{\cancel{\partial b}}$$

$$= 0$$

10. If $u = f(2x-3y, 3y-4z, 4z-2x)$ PT $\frac{1}{2} u_x + \frac{1}{3} u_y + \frac{1}{4} u_z = 0$

soln Put $p = 2x-3y$, $q = 3y-4z$, $r = 4z-2x$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial x} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial x} + \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x} \rightarrow (1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial y} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial y} + \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial y} \rightarrow (2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial z} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial z} + \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial z} \rightarrow (3)$$

$$p = 2x - 3y$$

d.p.w.r. to x, y, z separately.

$$\frac{\partial p}{\partial x} = 2(1) - 0 = \frac{\partial p}{\partial x} = 2, \quad \frac{\partial p}{\partial y} = -3, \quad \frac{\partial p}{\partial z} = 0$$

$$q = 3y - 4z$$

d.p.w.r. to x, y, z separately

$$\frac{\partial q}{\partial x} = 0, \quad \frac{\partial q}{\partial y} = 3, \quad \frac{\partial q}{\partial z} = -4$$

$$r = 4z - 2x$$

dp wrt x, y, z separately

$$\frac{\partial r}{\partial x} = -2, \quad \frac{\partial r}{\partial y} = 0, \quad \frac{\partial r}{\partial z} = 4$$

eq - (i) becomes

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p} \cdot 2 + \frac{\partial u}{\partial q} \cdot 0 + \frac{\partial u}{\partial r} \cdot -2$$

$$\frac{\partial u}{\partial x} = 2 \frac{\partial u}{\partial p} - 2 \frac{\partial u}{\partial r}$$

$$\frac{\partial u}{\partial y} = -3 \frac{\partial u}{\partial p} + 4 \frac{\partial u}{\partial r}$$

consider $\frac{1}{2} u_x + \frac{1}{3} u_y + \frac{1}{4} u_z =$

$$= \frac{1}{2} \left(2 \frac{\partial u}{\partial p} - 2 \frac{\partial u}{\partial r} \right) + \frac{1}{3} \left(-3 \frac{\partial u}{\partial p} + 4 \frac{\partial u}{\partial r} \right) + \frac{1}{4} \left(4 \frac{\partial u}{\partial r} - 4 \frac{\partial u}{\partial q} \right)$$

$$= \frac{\partial u}{\partial p} - \frac{\partial u}{\partial r} + \frac{\partial u}{\partial r} - \frac{\partial u}{\partial p} + \frac{\partial u}{\partial r} - \frac{\partial u}{\partial q}$$

$$= 0$$

Maxima and Minima:-

A function $f(x, y)$ is said to have

- i) Maximum value at the point (a, b) if there exists neighbourhood of a point $(a+h, b+k)$ such that $f(a, b) > f(a+h, b+k)$
- ii) Minimum value at the point (a, b)
if $f(a, b) < f(a+h, b+k)$
- iii) Extreme value: if $f(a, b)$ has maximum value or minimum value then we say that as extreme value.

Note:- $f(a, b)$ is said to be extreme value of $f(x, y)$ if it is a minimum value (or) maximum value.

Working procedure:-

1. Find the stationary points x, y such that $f_x = 0, f_y = 0$.
2. Find the second order partial derivative f_{xx}, f_{xy}, f_{yy} and take $A = f_{xx}, B = f_{xy}, C = f_{yy}$
3. If $AC - B^2$ is greater than '0' zero and A is less than '0' then the function has maximum value. (a) $AC - B^2 > 0, A < 0$
($f_{xx}, f_{yy} - f_{xy}^2 > 0, f_{xx} < 0$)
- $AC - B^2 > 0$ & $A > 0$ then the function has minimum value
4. If $AC - B^2$ is less than '0' or $AC - B^2 = 0$ or $A = 0$ then (x, y) is called saddle point.

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1. Find extreme value of the function $f(x,y) = x^3 + y^3 - 3x - 12y + 20$

Sol:- $f(x,y) = x^3 + y^3 - 3x - 12y + 20 \rightarrow (1)$

D. (1) p.w.r to x, y separately.

$$f(x) = 3x^2 + 0 - 3 - 0 + 0$$

$$f_x = 3x^2 - 3$$

$$f_y = 0 + 3y^2 - 0 - 12 + 0$$

$$f_y = 3y^2 - 12$$

put $f_x = 0$

& $f_y = 0$

$$3x^2 - 3 = 0$$

$$3y^2 - 12 = 0$$

$$3x^2 = 3$$

$$3y^2 = 12$$

$$x^2 = 1$$

$$y^2 = 4$$

$$x = \pm 1$$

$$y = \pm 2$$

w.k.T (x,y) is representation of point

$$(1,2), (1,-2), (-1,2), (-1,-2)$$

• $f_x = 3x^2 - 3$ $f_y = 3y^2 - 12$

$$A = f_{xx} = 6x$$

$$C = f_{yy} = 6y$$

$$B = f_{xy} = 0$$

(f_{xy} = diff of x w.r.to y)

Points

$$AC - B^2 = 6x(6y) - 0 = 36xy$$

$$A = 6x$$

conclusion

$$(1,2)$$

$$72 > 0$$

$$6 > 0$$

Minimum value

$$(1,-2)$$

$$-72 < 0$$

saddle point

$$(-1,2)$$

$$36(-1)(2) = -72 < 0$$

saddle point

$$(-1,-2)$$

$$36(-1)(-2) = 72 > 0$$

$$6(-1) = -6 < 0$$

Maximum value

eq - (1) becomes

$$\begin{aligned}f(1,2) &= (1)^3 + (2)^3 - 3(1) - 12(2) + 20 \\&= 1 + 8 - 3 - 24 + 20 \\&= 2\end{aligned}$$

$$\begin{aligned}f(-1,-2) &= (-1)^3 + (-2)^3 - 3(-1) - 12(-2) + 20 \\&= -1 - 8 + 3 + 24 + 20 \\&= 38\end{aligned}$$

$\therefore$ The function attains maximum value at $(-1,-2)$ and the maximum value is 38.

The function attains minimum value at $(1,2)$ and the minimum value is 2.

3. Find the maxima and minima value of the function.

$$f(x,y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$$

$$\text{sol:- } f(x,y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$$

$$\begin{aligned}f_x \quad f_x &= 3x^2 + 3y^2 - 30x - 0 + 72 \\&= 3x^2 + 3y^2 - 30x + 72\end{aligned}$$

$$\begin{aligned}f_y &= 0 + 6xy - 0 - 30y + 0 \\&= 6xy - 30y\end{aligned}$$

$$\text{Put } f_x = 0$$

$$3x^2 - 3y^2 - 30x + 72 = 0 \longrightarrow (2)$$

$$\text{put } y=0 \text{ in eq---(2)}$$

$$3x^2 - 30x + 72 = 0$$

$$3(x^2 - 10x + 24) = 0$$

$$x^2 - 6x - 4x + 24 = 0$$

$$x(x-6) - 4(x-6) = 0$$

$$f_y = 0$$

$$6xy - 30y = 0$$

$$6y(x-5) = 0$$

$$y = 0 \text{ (i) } x = 5$$

$$x - 4 = 0 ; x - 6 = 0$$

$$\boxed{x = 4}, \boxed{x = 6}$$

$\therefore (4,0)$ & $(6,0)$ are stationary point at $y=0$

at $x=5$ (2) becomes

$$3(5)^2 + 3y^2 - 30(5) + 72 = 0$$

$$75 + 3y^2 - 150 + 72 = 0$$

$$3y^2 - 3 = 0$$

$$y^2 = 1$$

$$y = \pm 1$$

$\therefore$ at $x=5$ stationary points are

$$(5,1) \text{ \& } (5,-1)$$

$$f_x = 3x^2 + 3y^2 - 30 + 72$$

$$A = f_{xx} = 6x + 0 - 30 + 0 = 0$$

$$A = 6x - 30$$

$$B = f_{xy} = 0 + 6y + 0 + 0$$

$$B = 6y$$

$$C = f_{yy} = 6x - 30$$

$$C = 6x - 30$$

Points	$AC - B^2 = (6x-30)(6x-30) - (6y)^2$	$A = 6x-30$	Conclusion
(4,0)	$36 > 0$	$-6 < 0$	Maximum value
(6,0)	$36 > 0$	$6 > 0$	Minimum value
(5,1)	$-36 < 0$		Saddle point
(5,-1)	$-36 < 0$		Saddle point

$$\begin{aligned}
 f(4,0) &= x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x \\
 &= (4)^3 + 3(4)(0) - 15(4)^2 - 0 + 72(4) \\
 &= 64 + 0 - 240 + 288 \\
 &= 352 - 240 \\
 &= 112
 \end{aligned}$$

$$\begin{aligned}
 f(6,0) &= (6)^3 + 0 - 15(6)^2 - 0 + 72(6) \\
 &= 216 - 540 + 432 \\
 &= 648 - 540 \\
 &= 108
 \end{aligned}$$

4 Find the extreme value of the function $f(x,y) = xy(1-x-y)$

sol:- $f = xy(1-x-y)$

$$f = xy - x^2y - xy^2 \rightarrow (1)$$

D. eq - (1) p w.r.to x, y separately

$$f_x = y - 2xy - y^2$$

put $f_x = 0$

$$y^2 - 2xy - y^2 = 0$$

$$y(1-2x-y) = 0$$

$$\boxed{y=0} \quad (i) \quad 1-2x-y=0$$

$$2x+y=1$$

$$+ f_y = 0$$

$$x - x^2 - 2xy = 0$$

$$x(1-x-2y) = 0$$

$$\boxed{x=0} \quad (ii) \quad 1-x-2y=0$$

$$x+2y=1$$

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taking $y=0$ in $x=0$ $(0,0)$

$$y=0 \text{ in } 1-x-2y=0 \rightarrow x=1 \quad (1,0)$$

put $x=0$ in $y=0$ $(0,0)$

$$x=0 \text{ in } 1-2x-y=0 \Rightarrow y=1 \quad (0,1)$$

$\therefore$ stationary points are $(0,0)$ $(1,0)$ $(0,1)$

$$\text{step 2: } f_x = y - 2xy - y^2 \rightarrow (2)$$

D.P. w.r. to x

$$A = f_{xx} = 0 - 2y - 0$$

$$A = -2y$$

D. eq. (2) p.w.r. to y

$$B = f_{xy} = 1 - 2x - 2y$$

$$B = 1 - 2x - 2y$$

$$f_y = x - x^2 - 2xy \rightarrow (3)$$

D. (3) p.w.r. to y

$$C = f_{yy} = 0 - 0 - 2x$$

$$C = -2x$$

points	$AC - B^2 = (-2y)(-2x) - (1 - 2x - 2y)^2$	$A = -2y$	conclusion
$(0,0)$	$-1 < 0$	0	saddle point
$(1,0)$	$-1 < 0$	0	saddle point
$(0,1)$	$0 - (1 - 0 - 2)^2 = -1 < 0$	$-2 < 0$	saddle point
$(\frac{1}{3}, \frac{1}{3})$	$\frac{1}{3} > 0$		Maximum point

conclusion = $\frac{1}{2}$ is maximum value.

5. Find the extreme value of the function $x^3 + y^3 - 3axy, a > 0$

sol: $f(x, y) = x^3 + y^3 - 3axy \rightarrow (1)$

D.(1) p.w.r.to x, y separately

$$f_x = 3x^2 + 0 - 3ay$$

take $f_x = 0$

$$3x^2 - 3ay = 0 \div 3$$

$$x^2 - ay = 0$$

$$\boxed{x^2 = ay}$$

squaring on both sides

$$x^4 = a^2 y^2$$

$$x^4 = a^2 ax$$

$$x^4 - a^3 x = 0$$

$$x(x^3 - a^3) = 0$$

$$x = 0, x^3 - a^3 = 0$$

$$(a-b)(a^2 - ab + b^2) = a^3 - b^3$$

$$(x-a)(x^2 - xa + a^2) = x^3 - a^3$$

$$\boxed{x=a} \quad x^2 - xa + a^2 = 0$$

$$a = 1, b = -a, c = a^2$$

$$f_y = 0 + 3y^2 - 3ax$$

& $f_y = 0$

$$3y^2 - 3ax = 0 \div 3$$

$$y^2 - ax = 0$$

$$\boxed{y^2 = ax}$$

$$x = \frac{-a \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-1 \pm \sqrt{(-a)^2 - 4(1)a^2}}{2(1)}$$

$$x = \frac{-1 \pm \sqrt{-3a^2}}{2} = \frac{-1 \pm ia\sqrt{3}}{2}$$

$x = a$ is one root

take $x=0$ in $y^2 = ax \Rightarrow y = 0(0,0)$

$x=a$ in $y^2 = ax \rightarrow y^2 = a^2$

$y = \pm a$
 $(a, a) (a, -a)$

$\therefore$ Stationary points are $(0,0) (a,a) (a,-a)$

$$f_x = 3x^2 - 3ay \rightarrow (2)$$

D.2.p.w.r.to x

$$A = f_{xx} = 6x - 0$$

D(2).p.w.r.to y

$$B = f_{xy} = 0 - 3a$$

$$f_y = 3y^2 - 3ax$$

$$C = f_{yy} = 6y$$

Points	$AC-B^2 = 36xy - 9a^2$	$A = 61$	conclusion
$(0,0)$	$0 - 9a^2 < 0$	—	Saddle point
(a,a)	$36a^2 - 9a^2 = 27a^2 > 0$	$6a > 0$	Minimum value
$(a,-a)$	$-36a^2 - 9a^2 = -45a^2 < 0$	—	Saddle point

$$f(x,y) = x^3 + y^3 - 3axy$$

$$\begin{aligned} f(a,a) &= a^3 + a^3 - 3a(a)(a) \\ &= 2a^3 - 3a^3 \\ &= -a^3 \end{aligned}$$

∴ Minimum value that the function attain at (a,a) is $-a^3$

Jacobian

If u & v are any two functions of x & y then Jacobian of (u,v) w.r. to (x,y) is denoted by $J\left(\frac{u,v}{x,y}\right)$ (or) $\frac{\partial(u,v)}{\partial(x,y)}$ & is given by

$$J\left(\frac{u,v}{x,y}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial(u,v)}{\partial(x,y)}$$

similarly,

$$J\left(\frac{u,v,w}{x,y,z}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Properties:-

1) If u, v are functions of x, y and x, y are functions of u, v

$$(u, v) \longrightarrow (x, y) \longrightarrow (x, y)$$

then,

$$J\left(\frac{u, v}{x, y}\right) = \frac{\partial(u, v)}{\partial(x, y)} \times \frac{\partial(x, y)}{\partial(x, y)}$$

2) If u, v are functions of x, y and if

$$J = \frac{\partial(u, v)}{\partial(x, y)} \quad \& \quad J' = \frac{\partial(x, y)}{\partial(u, v)}$$

$$\text{then } \boxed{J \cdot J' = 1}$$

Problems:-

1) If $u = e^x \cos y$, $v = e^x \sin y$, find $\frac{\partial(u, v)}{\partial(x, y)}$

sol:- wkt,

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \longrightarrow (1)$$

consider $u = e^x \cos y$

D.P. w.r. to x & y respectively separately,

$$\frac{\partial u}{\partial x} = \cos y e^x, \quad \frac{\partial u}{\partial y} = -e^x \sin y$$

Consider $v = e^x \sin y$

D.P. w.r. to x & y separately

$$\frac{\partial v}{\partial x} = \sin y e^x, \quad \frac{\partial v}{\partial y} = e^x \cos y$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{vmatrix}$$

$$= e^x e^x \begin{vmatrix} \cos y & -\sin y \\ \sin y & \cos y \end{vmatrix}$$

$$= e^{2x} (\cos^2 y + \sin^2 y) = e^{2x}$$

2. $u = x + y + z$

$v = y + z$

$w = z$

find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$

sol:- $\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 1(1-0) - 1(0-0) + 1(0-0)$$

$$= \frac{\partial(u,v,w)}{\partial(x,y,z)} = 1$$

3. If $u = \sqrt{x_1 x_2}$, $v = \sqrt{x_2 x_3}$, $w = \sqrt{x_1 x_3}$ then find $J\left(\frac{u, v, w}{x_1, x_2, x_3}\right)$

sol:- $u = \sqrt{x_1} \cdot \sqrt{x_2}$, $v = \sqrt{x_2} \cdot \sqrt{x_3}$, $w = \sqrt{x_1} \cdot \sqrt{x_3}$

$$\frac{\partial(u, v, w)}{\partial(x_1, x_2, x_3)} = \begin{vmatrix} \frac{\partial u}{\partial x_1} & \frac{\partial u}{\partial x_2} & \frac{\partial u}{\partial x_3} \\ \frac{\partial v}{\partial x_1} & \frac{\partial v}{\partial x_2} & \frac{\partial v}{\partial x_3} \\ \frac{\partial w}{\partial x_1} & \frac{\partial w}{\partial x_2} & \frac{\partial w}{\partial x_3} \end{vmatrix}$$

$$= \begin{vmatrix} \sqrt{x_2} \frac{1}{2\sqrt{x_1}} & \sqrt{x_1} \frac{1}{2\sqrt{x_2}} & 0 \\ 0 & \sqrt{x_3} \frac{1}{2\sqrt{x_2}} & \sqrt{x_2} \frac{1}{2\sqrt{x_3}} \\ \sqrt{x_3} \frac{1}{2\sqrt{x_1}} & 0 & \sqrt{x_1} \frac{1}{2\sqrt{x_3}} \end{vmatrix}$$

$$= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \left[\frac{\sqrt{x_2}}{\sqrt{x_1}} \left(\frac{\sqrt{x_3}}{\sqrt{x_2}} \cdot \frac{\sqrt{x_1}}{\sqrt{x_3}} - 0 \right) - \frac{\sqrt{x_1}}{\sqrt{x_2}} \left(0 - \frac{\sqrt{x_2}}{\sqrt{x_3}} \cdot \frac{\sqrt{x_3}}{\sqrt{x_1}} \right) + 0 \left(0 - \frac{\sqrt{x_3}}{\sqrt{x_2}} \cdot \frac{\sqrt{x_3}}{\sqrt{x_1}} \right) \right]$$

$$= \frac{1}{8} \left[\frac{\sqrt{x_2}}{\sqrt{x_1}} \cdot \frac{\sqrt{x_1}}{\sqrt{x_2}} + \frac{\sqrt{x_1}}{\sqrt{x_2}} \cdot \frac{\sqrt{x_2}}{\sqrt{x_1}} \right]$$

$$= \frac{1}{8} [1+1]$$

$$= \frac{1}{8} \times 2$$

$$= \frac{1}{4}$$

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4. If $x = \rho \cos \phi$, $y = \rho \sin \phi$, $z = z$ find $J \left(\frac{x, y, z}{\rho, \phi, z} \right)$

sol:- $\frac{\partial(x, y, z)}{\partial(\rho, \phi, z)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial z} \end{vmatrix}$

$$= \begin{vmatrix} \cos \phi & -\rho \sin \phi & 0 \\ \sin \phi & \rho \cos \phi & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \cos \phi (\rho \cos \phi - 0) + \rho \sin \phi (\sin \phi - 0) + 0(0-0)$$

$$= \rho \cos^2 \phi + \rho \sin^2 \phi + 0$$

$$= \rho (\sin^2 \phi + \cos^2 \phi)$$

$$= \rho$$

$$\rightarrow \frac{\partial(x, y, z)}{\partial(\rho, \phi, z)} = \rho$$

5. If $x = r \sin \theta \cdot \cos \phi$, $y = r \sin \theta \cdot \sin \phi$, $z = r \cos \theta$

$$PT, J \left(\frac{x, y, z}{r, \theta, \phi} \right) = r^2 \sin \theta$$

sol:- $\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$

$$\begin{vmatrix} \sin\theta \cdot \cos\phi & r\cos\phi\cos\theta & -r\sin\theta\sin\phi \\ \sin\theta \cdot \sin\phi & r\sin\phi\cos\theta & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= \sin\theta \cdot \cos\phi (0 + r^2 \sin^2\theta \cdot \cos\phi) - r\cos\phi\cos\theta (0 - r\sin\theta \cdot \cos\theta \cdot \cos\phi) - r\sin\theta \cdot \sin\phi (-r\sin^2\theta \sin\phi - r\sin\phi \cdot \cos^2\theta)$$

$$= r^2 \sin^3\theta \cos^2\phi + r^2 \cos^2\phi \cdot \cos^2\theta \cdot \sin\theta + r^2 \sin^3\theta \cdot \sin^2\phi + r^2 \sin^2\phi \cdot \sin\theta \cdot \cos^2\theta$$

$$= r^2 \sin\theta (\sin^2\theta \cdot \cos^2\phi + \cos^2\phi \cdot \cos^2\theta + \sin^2\theta \cdot \sin^2\phi + \sin^2\phi \cdot \cos^2\theta)$$

$$= r^2 \sin\theta (\sin^2\theta (\sin^2\phi + \cos^2\phi) + \cos^2\theta (\sin^2\phi + \cos^2\phi))$$

$$= r^2 \sin\theta (\sin^2\theta + \cos^2\theta)$$

$$\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r^2 \sin\theta$$

6. If $x = u^2 - v^2$, $y = v^2 - w^2$, $z = w^2 - u^2$ find $\frac{\partial(x, y, z)}{\partial(u, v, w)}$.

sol:- $\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$

$$= \begin{vmatrix} 2u & -2v & 0 \\ 0 & 2v & -2w \\ -2u & 0 & -2w \end{vmatrix}$$

$$= 2u(2v \cdot 2w - 0) + 2v(0 - (2w - 2u)) - 0$$

$$= 8uvw - 8uvw$$

$$= 0$$

7. If $u = \frac{yz}{x}$, $v = \frac{zx}{y}$, $w = \frac{xy}{z}$ PT $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 4$

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sol:- $\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$

$$= \begin{vmatrix} \frac{-yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & \frac{-zx}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & \frac{-xy}{z} \end{vmatrix}$$

$$= \frac{1}{x^2} \cdot \frac{1}{y^2} \cdot \frac{1}{z^2} \begin{vmatrix} -yz & zx & yx \\ zy & -zx & xy \\ yz & zx & -xy \end{vmatrix}$$

$$= \frac{1}{x^2 y^2 z^2} \left[(-yz)(x^2 yz - x^2 yz) - zx(-xy^2 z - xy^2 z) + xy(xz^2 y + xz^2 y) \right]$$

$$= \frac{1}{x^2 y^2 z^2} \left[0 + zx(2xy^2 z) + xy(2xz^2 y) \right]$$

$$= \frac{2}{x^2 y^2 z^2} xyz (zx y + xy z)$$

$$= \frac{2}{x^2 y^2 z^2} xyz (2zx y)$$

$$= \frac{2}{x^2 y^2 z^2} 2x^2 y^2 z^2$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = 4$$