MODULE - 4

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1. Solution of Higher order differential Equation:-

Solution of Homogeneous differential Equation:-

If the given equation is of the form

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots\cdots + a_n y = 0 \longrightarrow (1)$$

where $a_0, a_1, a_2, a_3, \cdots\cdots,$ constants.

Working procedure:-

1. Let $D = \frac{d}{dx}$, $\rightarrow$ eq-(1) becomes

$$a_0 D^n y + a_1 D^{n-1} y + a_2 D^{n-2} y + \cdots\cdots + a_n y = 0$$

$$(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots\cdots + a_n) y = 0$$

2. Auxially equation $D \longrightarrow m$

$$a_0 m^n + a_1 m^{n-1} + a_2 m^{n-2} + \cdots\cdots + a_n = 0$$

Solving the above equation we get different types of roots. Depending on nature of the roots we write solution for given differential equation.

We solve general quadratic equation in the form of

$$am^2 + bm + C = 0$$

Sl No.	Nature of the root	Solution of D.E
1.	Real & distinct say m_1 & m_2	$y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$
2.	Roots are equal say $m_1 = m_2 = m$	$y = (C_1 + C_2 x) e^{mx}$
3.	Complex pair of root $a \pm ib$ Real pair $= a$ imaginary $= b$	$y = e^{ax}(C_1 \cos bx + C_2 \sin bx)$

1. Solve $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 0$

Sol:-

put $\frac{d}{dx} = D$

$D^2y - 4.Dy + 3y = 0$

$(D^2 - 4D + 3)y = 0$

Auxially equation is given by $D \longrightarrow m$

$m^2 - 4m + 3 = 0$, $m = 1, 3$

$m_1 = 1$, $m_2 = 3$ roots are real and distinct

$\therefore$ Solution is given by

$y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$

$\Rightarrow y = C_1 e^{x} + C_2 e^{3x}$

2. Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$

put $\frac{d}{dx} = D$

$D^2y - 2Dy + y = 0$

$(D^2 - 2D + 1)y = 0$

$m^2 - 2m + 1 = 0$

$a = 1, b = -2, c = 1$, $m = 1, 1,$

roots are equal

$\therefore$ Solution is givenby

$y = (C_1 + C_2 x)e^{x}$

3. Solve $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 7y = 0$

$Dy^2 - 4Dy + 7y = 0 \rightarrow (D^2 - 4D + 7)y = 0$

A.E is given by $D \rightarrow m$

$m^2 - 4m + 7 = 0$

$a = 1,\ b = -4,\ c = 7$

$$m = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(7)(1)}}{2}$$

$$= \frac{4 \pm \sqrt{16 - 28}}{2} = \frac{4 \pm \sqrt{-12}}{2} = \frac{4 \pm \sqrt{-1}\sqrt{12}}{2}$$

$$m = \frac{4 \pm i2\sqrt{3}}{2}$$

$$m = 2 \pm i\sqrt{3}$$

Roots are complex pair

$a = 2,\ b = \sqrt{3}$

$$y = e^{ax}(c_1 \cos bx + c_2 \sin bx)$$

$$y = e^{2x}(c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x)$$

4. $(D^3+6D^2+11D+6)y=0$

Sol:- Auxilary equation is given by

$$m^3+6m^2+11m+6=0$$

Synthetic division

$$\begin{array}{r|rrrr} & 1 & 6 & 11 & 6 \\ -1 & 0 & -1 & -5 & -6 \\ \hline & 1 & 5 & 6 & 0 \end{array}$$

$$m^2+5m+6=0$$

$$m^2+3m+2m+6=0$$

$$m(m+3)+2(m+3)=0$$

$$(m+3)(m+2)=0$$

$$m+3=0 \qquad m+2=0$$

$$m=-3 \qquad m=-2$$

$\therefore$ Roots are real and distinct

$$m_1=-3,\ m_2=-2,\ m_1=-1$$

Complete solution is by

$$y=C_1e^{-3x}+C_2e^{-2x}+C_3e^{-x}$$

5. Solve $(D^5-D^4-D+1)y=0$

Sol:- A.E is given by

$$m^5-m^4-m+1=0$$

$$m^4(m-1)-(m-1)=0$$

$$(m-1)(m^4-1)=0$$

$$m-1=0 \qquad m^4-1=0$$

$$\boxed{m=1} \qquad (m^2)^2-(1^2)^2=0$$

$$(m^2+1)(m^2-1)=0$$

$m^2+1=0$ (or) $m^2-1=0$

$m^2=-1$ $m^2=1$

$m=\pm\sqrt{-1}$ $\boxed{m=\pm 1}$

$\boxed{m=\pm i}$ $\therefore$ Roots are $1, 1, -1, \pm i$

C.S is given by

$$y=(C_1+C_2x)e^{mx}+C_3e^{m_1x}+e^{ax}(C_4\cos bx+C_5\sin bx)$$

$$y=(C_1+C_2x)e^{x}+C_3e^{-x}+e^{0}(C_4\cos x+C_5\sin x)$$

6. $y'''+y=0$

Sol:- $\dfrac{d^3y}{dx^3}+y=0$

Put $\dfrac{d}{dx}=D$

$D^3y+y=0$

$(D^3+1)y=0$

$(m^3+1)=0$

$a=1, b=0, c=0, d=1$

Roots are $m=-1$, $a\pm ib=\frac{1}{2}\pm\frac{i\sqrt{3}}{2}$

$a=\frac{1}{2}$, $b=\frac{\sqrt{3}}{2}$

C.S is given by

$$y=C_1e^{-x}+e^{\frac{x}{2}}\left(C_2\cos\frac{\sqrt{3}}{2}x+C_3\sin\frac{\sqrt{3}}{2}x\right)$$

Non-Homogeneous linear differential equation:-

If the given differential equation is the form

$$a_0\frac{d^n y}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + a_2\frac{d^{n-2}y}{dx^{n-2}} + \dots + a_n y = 0 \rightarrow (1)$$

is called Non-Homogeneous linear D.E.

$$(a_0D^n + a_1D^{n-1} + a_2D^{n-2} + \dots + a_n)y = f(x)$$

Complete solution = Complementary function + Particular Integral

$$C.S = CF + PI$$

$$y = y_c + y_p$$

i) Complementary function (solution of LHS)

ii) Particular Integral (solution of RHS)

Solution of LHS i.e y_c is nothing but finding solution of homogeneous differential equation and we follow the same procedure.

While finding solution of RHS, we need to study different cases.

Case-i) $y_p = \dfrac{e^{ax+b}}{f(D)}$

Replace $D \rightarrow a$ if $f(a) \neq 0$

if $(a) = 0$

Multiply by 'x' to numerator & Differentiate denominator once then

Replace $D \rightarrow a$ if $f'(a) \neq 0$

1. $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = e^x$

Sol: Consider, $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = e^x \longrightarrow (1)$

$$D^2y - 6Dy + 9y) = e^x$$

$$(D^2 - 6D + 9)y = e^x$$

A.E is $m^2 - 6m + 9 = 0$

$$(m-3)^2 = 0$$

$$m = 3, 3$$

$\therefore$ Roots are real and repeated.

$$y_c = (C_1 + C_2 x)e^{3x}$$

To find P.I $y_p = \frac{f(x)}{f(D)} = \frac{e^{ax+b}}{f(D)}$

$$y_p = \frac{e^x}{D^2 - 6D + 9}$$

$$D \to a = 1$$

$$y_p = \frac{e^x}{1^2 - 6 + 9} = \frac{e^x}{4}$$

$\therefore$ Complete solution is

$$y = y_c + y_p$$

$$y = (C_1 + C_2 x)e^{3x} + \frac{e^x}{4}$$

2. $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = e^{3x}$

Sol:- $D^2y - 6Dy + 9y = e^{3x}$

$(D^2 - 6D + 9)y = e^{3x}$

Replace $D \rightarrow m$

$m^2 - 6m + 9 = 0 \Rightarrow (m^2 - 3)^2 = 0$

$m = 3, 3$

$\therefore$ Roots are real and repeated.

$y_c = (C_1 + C_2x)e^{3x}$

To find $y_p = \frac{e^{ax+b}}{f(x)}$

$y_p = \frac{e^{3x}}{D^2 - 6D + 9}$

$D \rightarrow a = 3$

$y_p = \frac{e^{3x}}{9 - 18 + 9} = \frac{e^{3x}}{0}$ since, $f(a) = f(3) = 0$

Multiply x in numerator & diff. denominator once.

$y_p = x \cdot \frac{e^{3x}}{2D - 6}$

$D \rightarrow a = 3$

$x \cdot \frac{6e^{3x}}{6-6} = \frac{e^{3x}}{0}$ $f'(a) = f'(3) = 0$

$\therefore$ Multiply x in num & diff. deno. once

$y_p = x^2 \cdot \frac{e^{3x}}{2 - 0} \Rightarrow y_p = \frac{x^2 \cdot e^{3x}}{2}$

$\therefore$ C.S is given by

$y = (C_1 + C_2x)e^{3x} + \frac{x^2 \cdot e^{3x}}{2}$

3. Solve $(D^2 - D - 2)y = 5^x + \cosh x + \log 2$ (Inverse differential operator method)

Sol: $\rightarrow 5^x = e^{\log 5^x} = e^{x \log 5}$

$\rightarrow \cosh x = \dfrac{e^x + e^{-x}}{2}$

$\rightarrow \log 2 = \log 2 \times 1 = \log 2 \cdot e^{0x}$

$$(D^2 - D - 2)y = e^{x\log 5} + \frac{e^x + e^{-x}}{2} + \log 2 \cdot e^{0x}.$$

Replace $D \longrightarrow m$

$$m^2 - m - 2 = 0$$

$$m = 2, -1$$

$$y_c = C_1 e^{2x} + C_2 e^{-x}$$

$$y_P = \frac{e^{x\log 5} + \frac{e^x + e^{-x}}{2} + \log 2 \cdot e^{0x}}{D^2 - D - 2}$$

$$= \frac{e^{x\log 5}}{D^2 - D - 2} + \frac{1}{2}\left[\frac{e^x}{D^2 - D - 2} + \frac{e^{-x}}{D^2 - D - 2}\right] + \frac{\log 2 \cdot e^{0x}}{D^2 - D - 2}$$

$$= \frac{e^{x\log 5}}{(\log 5)^2 - \log 5 - 2} + \frac{1}{2}\left[\frac{e^x}{\cancel{1} - \cancel{1} - 2} + \frac{e^{-x}}{1 + 1 - 2}\right] + \frac{\log 2 \cdot e^{0x}}{0 - 0 - 2}$$

$$= \frac{e^{x\log 5}}{(\log 5)^2 - \log 5 - 2} + \frac{1}{2}\left[\frac{e^x}{-2} + \frac{x \cdot e^{-x}}{2D - 1}\right] + \frac{\log 2 \cdot e^{0x}}{-2}$$

$$= \frac{e^{x\log 5}}{(\log 5)^2 - \log 5 - 2} + \frac{1}{2}\left[\frac{e^x}{-2} + \frac{x e^{-x}}{-3}\right] + \frac{\log 2}{-2}$$

$$y_P = \frac{5^x}{(\log 5)^2 - \log 5 - 2} + \frac{1}{2}\left[\frac{e^x}{-2} - \frac{x e^{-x}}{3}\right] + \frac{\log 2}{-2}$$

$$y = y_c + y_p$$

$$y = C_1 e^{2x} + C_2 e^{-x} + \frac{5^x}{(\log 5)^2 - \log 5 - 2} + \frac{1}{2}\left[\frac{e^x}{-2} - \frac{e^{-x}}{3}\right] + \frac{\log 2}{-2}$$

<u>Case - ii</u> If $f(x) = \sin(ax+b)$ or $\cos(ax+b)$

$$y_p = \frac{\sin(ax+b)}{f(D)}$$

$D^2 \to -a^2$ if $(-a^2) \neq 0$

if $(-a^2) = 0$

then Multiply by x to numerator & differentiate denominator.

1. Solve $y'' - 4y = \cos 2x$

<u>Sol:-</u> Consider :- $y'' - 4y = \cos 2x$

$$y = y_c + y_p$$

$$\frac{d^2y}{dx^2} - 4y = \cos 2x$$

$$D = \frac{d}{dx}$$

$$D^2 y - 4y = \cos 2x$$

$$(D^2 - 4)y = \cos 2x$$

A.E is $m^2 - 4 = 0$

$m^2 = 4$ $m = \pm 2$

$$y_c = C_1 e^{-2x} + C_2 e^{2x}$$

To find y_p $y_p = \frac{\cos(ax+b)}{f(D)}$

$$y_p = \frac{\cos 2x}{D^2 - 4}$$

$D^2 \to -a^2$ -4

$$y_1 = \frac{\cos 2x}{-4-4} = \frac{-\cos 2x}{8}$$

$\therefore$ C.S is

$$y = C_1 e^{-2x} + C_2 e^{2x} - \frac{\cos 2x}{8}$$

2. Solve $(D^2+1)y = \cos x$

Sol:- $(D^2+1)y = \cos x$

$D \rightarrow m$

$m^2+1=0$

$m^2=-1$, $\boxed{m=\pm i}$ R.P $=0$, I.P $=1$

$y_c = e^{0x}(C_1\cos x + C_2 \sin x)$

To find y_p

$$y_p = \frac{\cos(ax+b)}{f(D)} \qquad D^2 \rightarrow -a^2 = -1$$

$$= \frac{\cos x}{D^2+1} = \frac{\cos x}{-1+1} = \frac{\cos x}{0}$$

Multiply x to num. & diff. deno.

$$y_p = \frac{x.\cos x}{2D}$$

$$y_p = \frac{x}{2}.\int \cos x.dx$$

$$y_p = \frac{x}{2}.\sin x$$

multiply & $\div$ D in RHS

$$= \frac{x}{2}.\frac{D(\cos x)}{D^2} \qquad D^2 \rightarrow -a^2 = -1$$

$$= \frac{x}{2}.\frac{\cancel{-}\sin x}{\cancel{-}1}$$

$$y_p = \frac{x}{2}.\sin x$$

$$y = y_c + y_p$$

$$y = C_1\cos x + C_2\sin x + \frac{x}{2}\cdot\sin x$$

3. $(D^2 - 4D + 3)y = \sin 3x \cdot \cos 2x$

Sol:- $(D^2 - 4D + 3)y = \sin 3x \cos 2x$

$$D \to m$$

$$m^2 - 4m + 3 = 0$$

$$m^2 - 3m - m + 3 = 0$$

$$m(m-3) - (m-3) = 0$$

$$(m-3) = 0 \quad (m-1) = 0$$

$$m = 3 \qquad m = 1$$

$$y_c = C_1 e^{3x} + C_2 e^{x}$$

To find y_p

$$y_p = \frac{\sin 3x \cdot \cos 2x}{(D^2 - 4D + 3)}$$

$$\sin A \cdot \cos B = \frac{1}{2}(\sin(A+B) \cdot \sin(A-B))$$

$$y_p = \frac{1}{2}\left[\frac{\sin 5x + \sin x}{D^2 - 4D + 3}\right]$$

$$= \frac{1}{2}\left[\frac{\sin 5x}{D^2 - 4D + 3} + \frac{\sin x}{D^2 - 4D + 3}\right]$$

$$D^2 \to -a^2 = -25 \quad , \quad D^2 \to -a^2 = -1$$

$$= \frac{1}{2}\left[\frac{\sin 5x}{-25 - 4D + 3} + \frac{\sin x}{-1 - 4D + 3}\right]$$

$$= \frac{1}{2}\left[\frac{\sin 5x}{-4D - 22} + \frac{\sin x}{-4D + 2}\right]$$

$$= \frac{1}{2}\left[\frac{\sin 5x}{-2(2D + 11)} + \frac{\sin x}{-2(2D - 1)}\right]$$

8. $\frac{d^2y}{dx^2} - 4y = \sin^3 x$

Sol:- $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$\sin^3\theta = \frac{3\sin\theta - \sin 3\theta}{4}$

$D^2y - 4y = \frac{3\sin x - \sin 3x}{4}$

$(D^2 - 4)y = \frac{3\sin x - \sin 3x}{4}$

$D \rightarrow m$

$m^2 - 4 = 0 \quad \boxed{m = \pm 2}$

$y_c = C_1 e^{2x} + C_2 e^{-2x}$

To find y_p

$y_p = \frac{\sin(ax+b)}{f(D)}$

$= \frac{\frac{3\sin x - \sin 3x}{4}}{D^2 - 4}$

$= \frac{1}{4}\left[\frac{3\sin x}{D^2-4} - \frac{\sin 3x}{D^2-4}\right]$

$D^2 \rightarrow -a^2 = -1 \quad D^2 \rightarrow -a^2 = -9$

$= \frac{1}{4}\left[\frac{3\sin x}{-5} + \frac{\sin 3x}{13}\right]$

$y_p = \frac{1}{4}\left[\frac{3\sin x}{-5} + \frac{\sin 3x}{13}\right]$

$y = y_c + y_p$

$y = C_1 e^{2x} + C_2 e^{-2x} + \frac{1}{4}\left[\frac{3\sin x}{-5} + \frac{\sin 3x}{13}\right]$

$y = C_1 e^{2x} + C_2 e^{-2x} + \frac{1}{4}\left[\frac{\sin 3x}{13} - \frac{3\sin x}{5}\right]$

Multiply & ÷ 2D-11 Multiply & ÷ 2D+1

$$= -\frac{1}{4}\left[\frac{\sin 5x(2D-11)}{(2D+11)(2D-11)} + \frac{\sin x(2D+1)}{(2D-1)(2D+1)}\right]$$

$$= -\frac{1}{4}\left[\frac{\sin 5x\,(2D-11)}{4D^2-11^2} + \frac{\sin x\,(2D+1)}{4D^2-1}\right]$$

$D^2 \to -a^2 = -25$ $D^2 \to -a^2 = -1$

$$= -\frac{1}{4}\left[\frac{\sin 5x(2D-11)}{4(-25)-121} + \frac{\sin x(2D+1)}{-4-1}\right]$$

$$= -\frac{1}{4}\left[\frac{\sin 5x(2D-11)}{-221} + \frac{\sin x(2D+1)}{-5}\right]$$

$$= \frac{1}{4}\left[\frac{2.\cos 5x.(5)-11\sin 5x}{221} + \frac{2.\cos x+\sin x}{5}\right]$$

$$= \frac{1}{4}\left[\frac{10\cos 5x-11\sin 5x}{221} + \frac{2\cos x+\sin x}{5}\right]$$

$y = y_c + y_p$

$$y = C_1e^{x} + C_2e^{3x} + \frac{1}{4}\left[\frac{10\cos 5x-11\sin 5x}{221} + \frac{2\cos x+\sin x}{5}\right]$$

9) Solve $(D^2+1)y = \sin^2 x \cdot \cos^2 x$

Sol:- $(D^2+1)y = \sin^2 x \cdot \cos^2 x$

A.E is given by

$m^2+1=0$

$m = \pm i$ R.P $=0$
I.P $=1$

$y_c = e^{0x}(C_1 \cos x + C_2 \sin x)$

$$y_p = \frac{\sin^2 x \cdot \cos^2 x}{D^2+1}$$

$$\sin^2 x \cdot \cos^2 x = (\sin x \cdot \cos x)^2$$

$$= \left(\frac{\sin 2x}{2}\right)^2$$

$$= \frac{\sin^2 2x}{4}$$

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

$$\sin^2 2x = \frac{1-\cos 4x}{2\times 4}$$

$$y_p = \frac{1-\cos 4x}{8(D^2+1)}$$

$$= \frac{1}{8}\left[\frac{1\times e^{0x}}{D^2+1} - \frac{\cos 4x}{D^2+1}\right]$$

$D^2 \to -a^2 = 0$ $D^2 \to -a^2 = -16$

$$= \frac{1}{8}\left[\frac{1}{0+1} - \frac{\cos 4x}{-15}\right]$$

$$y = e^{0x}(C_1 \cos x + C_2 \sin x) + \frac{1}{8}\left[\frac{1}{1} + \frac{\cos 4x}{15}\right]$$

Case - iii :- Particular integral of the form $\frac{P(x)}{f(D)}$

where $p(x)$ is polynomial in x

divide numerator using denominator writting $p(x)$ in descending order & $f(D)$ in ascending order. Continue this division until we get remainder '0'. the quotient so obtain is particular integral of the given problem.

1. Solve $(D^2+3D+2)y = 1+3x+x^2$

Sol:- $(D^2+3D+2)y = 1+3x+x^2$

$D \rightarrow m$

$m^2+3m+2=0$

$m=-1,-2$

$y_c = c_1e^{-x} + c_2e^{-2x}$

To find y_p

$$y_p = \frac{1+3x+x^2}{D^2+3D+2}$$

$$2+3D+D^2 \overline{) \; x^2+3x+1 \;} \left(\frac{x^2}{2} \right.$$

$$\underline{x^2+3x+1}$$

$$\underline{0}$$

$\frac{x^2}{2}(2+3D+D^2)$

$= 2.\frac{x^2}{2} + \frac{3}{2}2x + \frac{1}{2}2(1)$

$= x^2+3x+1$

$y_p = \frac{x^2}{2}$

$y = c_1e^{-x} + c_2e^{-2x} + \frac{x^2}{2}$

2. Solve $\frac{d^2y}{dx^2} + 4y = x^2$

Sol: $\frac{d^2y}{dx^2} + 4y = x^2$

$D^2y + 4y = x^2$

$(D^2+4)y = x^2$

A.E is $m^2 + 4 = 0$

$m = \pm 2$

$y_c = C_1 e^{2x} + C_2 e^{-2x}$

To find y_p

$y_p = \frac{x^2}{D^2+4}$

$$4 + D^2 \,)\, x^2 \,\left(\, \frac{x^2}{4} - \frac{1}{8}\right.$$

$$\cancel{x^2} + \tfrac{1}{2}$$

$$-\tfrac{1}{2}$$

$$-\tfrac{1}{2}$$

$$0$$

$\frac{x^2}{4}(4+D^2)$

$4\frac{x^2}{4} + \frac{1}{2}$

$y_p = \frac{x^2}{4} - \frac{1}{8}$

$y = C_1 e^{2x} + C_2 e^{-2x} + \frac{x^2}{4} - \frac{1}{8}$

3. Solve $(D^3+8)y = x^4 + 2x + 1$

Sol:- $(D^3+8)y = x^4 + 2x + 1$

A.E is

$m^3 + 8 = 0$, $m = -2,\ 1 \pm \sqrt{3}i$

$y_c = C_1 e^{2x} + e^{x}(C_2 \cos\sqrt{3}x + C_3 \sin\sqrt{3}x)$

$$8+D^3 \overline{\big)\, x^4 + 2x + 1\,}\left(\frac{x^4}{8} - \frac{x}{8} + \frac{1}{8}\right.$$

$$\begin{array}{r} x^4 + 3x \\ \hline -x+1 \\ -x+0 \\ \hline 1 \\ 1 \\ \hline 0 \\ \hline \end{array}$$

$$y = C_1 e^{-2x} + e^{x}\left(C_2 \cos\sqrt{3}x + C_3 \sin\sqrt{3}\right) + \frac{x^4}{8} - \frac{x}{8} + \frac{1}{8}$$

4. Solve $y'' + 2y' - y = e^{-2x} + x$

Sol:-

$$y'' + 2y' - y = e^{-2x} + x$$

$$(D^2y + 2Dy - y) = e^{-2x} + x$$

$$(D^2 + 2D - 1)y = e^{-2x} + x$$

A.E is given by

$$m^2 + 2m - 1 = 0$$

$$m = -1 \pm \sqrt{2}$$

$$y_c = C_1 e^{-1+\sqrt{2}} + C_2 e^{-1-\sqrt{2}}$$

$$y_p = \frac{e^{-2x} + x}{D^2 + 2D - 1}$$

$$= \frac{e^{-2x}}{D^2+2D-1} + \frac{x}{D^2+2D-1}$$

$$y_p = (y_p)_1 + (y_p)_2$$

$$(y_P)_1 = \frac{e^{-2x}}{D^2+2D-1}$$

$$D \to a = -2$$

$$= \frac{e^{-2x}}{(-2)^2+2(-2)-1}$$

$$= \frac{e^{-2x}}{4-4-1}$$

$$= \frac{e^{-2x}}{-1}$$

$$-1+2D+D^2)\ x\ (-x+2$$

$$\cancel{x}-2$$

$$2$$

$$-2$$

$$0$$

$$y = c_1 e^{(-1+\sqrt{2})x} + c_2 e^{(-1-\sqrt{2})x} - e^{-2x} - x + 2$$

5. Solve $\frac{d^2y}{dx^2} - 3\cdot\frac{dy}{dx} + 2y = x^2 + e^x$

Sol:- $D^2y - 3Dy + 2y = x^2 + e^x$

$$(D^2 - 3D + 2)y = x^2 + e^x$$

$$D \to m$$

$$m^2 - 3m + 2 = 0$$

$$m = 1, 2$$

$$y_c = C_1 e^x + C_2 e^{2x}$$

$$y_P = \frac{x^2 + e^x}{D^2 - 3D + 2}$$

$$= \frac{x^2}{D^2-3D+2} + \frac{e^x}{D^2-3D+2} \Rightarrow y_P = (y_P)_1 + (y_P)_2$$

$(y_P)_1 = \dfrac{x^2}{D^2-3D+2}$

$2-3D+D^2\,)\ \cancel{x^2}\ \left(\dfrac{x^2}{2}+\dfrac{3}{2}x+\dfrac{7}{4}\right.$

$\quad \cancel{x^2}-3x+1$

$\quad 3\cancel{x}-1$

$\quad \cancel{3x}-\frac{9}{2}+0$

$\quad (-) \quad (+)$

$\quad \dfrac{7}{2}$

$\quad \dfrac{7}{2}$

$\quad (-)$

$\quad 0$

$\Rightarrow \dfrac{x^2}{2}(2-3D+D^2)$

$2\cdot\dfrac{x^2}{\cancel{2}} - \dfrac{3}{\cancel{2}}\cancel{2}x + \dfrac{1}{\cancel{2}}\cancel{2}(1)$

$= x^2-3x+1$

$\Rightarrow \dfrac{3}{2}x(2-3D+D^2)$

$= \cancel{2}\cdot\dfrac{3x}{\cancel{2}} - \dfrac{9}{2}(1) + \dfrac{3}{2}(0)$

$\Rightarrow \dfrac{7}{4}(2-3D+D^2)$

$= 2\cdot\dfrac{7}{4} - 0 + 0$

$(y_P)_2 = \dfrac{e^x}{D^2-3D+2}$

$D \to 1$

$= \dfrac{e^x}{1-3+2} = \dfrac{e^x}{0}$

x^{ly} x in nume & diff. deno

$= \dfrac{x\cdot e^x}{2D-3}$

$D \to (1)$

$= \dfrac{x\cdot e^x}{2-3} = \dfrac{x\cdot e^x}{-1}$

$y = -x\cdot e^x + \dfrac{x^2}{2} + \dfrac{3x}{2} + \dfrac{7}{4} + C_1e^x + C_2e^{2x}$

6. $\dfrac{d^2y}{dx^2} + 4y = x^2 + \cos 2x + 2^{-x}$

Sol:- $D^2y + 4y = x^2 + \cos 2x + 2^{-x}$

$(D^2+4)y = x^2 + \cos 2x + 2^{-x}$

$D \rightarrow m$ A.E

$m^2+4 = 0$, $m = \pm 2i$

$y_c = e^{0x}(c_1 \cos 2x - c_2 \sin 2x)$

$y_p = (y_{p_1}) + (y_p)_2 + (y_p)_3$

$$y_p = \frac{x^2}{D^2+4} + \frac{\cos 2x}{D^2+4} + \frac{2^{-x}}{D^2+4}$$

$$(y_p)_1 = \frac{x^2}{D^2+4}$$

$4+D^2)\ x^2\ \left(\frac{x^2}{4} - \frac{1}{8}\right.$

$\quad \cancel{x^2} + \frac{1}{2}$

$\quad (-)\ \ (-)$

$\quad -\frac{1}{2}$

$\quad -\frac{1}{2}$

$\quad 0$

$\Rightarrow \frac{x^2}{4}(4+D^2)$

$\cancel{4}\cdot\frac{x^2}{\cancel{4}} + \frac{1}{\cancel{4}}\cancel{2}(1)$

$\Rightarrow -\frac{1}{8}(4+D^2)$

$= \cancel{4}\cdot -\frac{1}{\cancel{8}_2} + 0$

$$(y_p)_2 = \frac{\cos 2x}{D^2+4}$$

$D^2 \rightarrow -a^2 = -4$

$$\frac{\cos 2x}{-4+4} = \frac{\cos 2x}{0}$$

$$= \frac{x \cos 2x}{2D} = \frac{x \cos 2x}{4}$$

$$(yP)_3 = \frac{2^{-x}}{D^2+4}$$

$$= \frac{e^{\log 2^{-x}}}{D^2+4} = \frac{e^{-x\log 2}}{D^2+4} \longrightarrow D \to a$$

$$= \frac{e^{-x\log 2}}{(\log 2)^2+4}$$

$$yP = \frac{x^2}{4} - \frac{1}{8} + \frac{x\cos 2x}{4} + \frac{e^{-x\log 2}}{(\log 2)^2+4} + e^{0x}(C_1\cos 2x - C_2\sin 2x)$$

7) $\frac{d^2x}{dt^2} - 6.\frac{dx}{dt} + 25x = e^{2t} + \sin t + t$

Sol:- Let $\frac{d}{dt} = D$

$$D^2x - 6Dx + 25x = e^{2t} + \sin t + t$$

$$(D^2 - 6D + 25)x = e^{2t} + \sin t + t$$

A.E is given by

$$m^2 - 6m + 25 = 0$$

$$m = 3 \pm 4i$$

$$xc = e^{3t}(C_1\cos 4t + C_2\sin 4t)$$

$$xp = \frac{e^{2t} + \sin t + t}{D^2 - 6D + 25}$$

$$\left[\frac{e^{2t}}{D^2-6D+25} + \frac{\sin t}{D^2-6D+25} + \frac{t}{D^2-6D+25}\right]$$

$$xp = (xp)_1 + (xp)_2 + (xp)_3$$

$$= \frac{e^{2t}}{D^2 - 6D + 25}$$

$$\therefore D \longrightarrow a = 2$$

$$(x_p)_1 = \frac{e^{2t}}{D^2-6D+25}$$

$$D \to a$$

$$(x_p)_1 = \frac{e^{2t}}{(2)^2-6(2)+25} = \frac{e^{2t}}{4-12+25} = \frac{e^{2t}}{19}$$

$$(x_p)_2 = \frac{\sin t}{D^2-6D+25}$$

$$D^2 \to -a^2 = -1$$

$$= \frac{\sin t}{(-1)-6D+25} = \frac{\sin t}{24-6D}$$

$$= \frac{\sin t\,(24+6D)}{(24-6D)(24+6D)} = \frac{\sin t\,(24+6D)}{(24)^2-(6D)^2}$$

$$= \frac{\sin t\,(24+6D)}{576-36D^2}$$

$$= \frac{1}{36}\left[\frac{\sin t\,(24+6D)}{16-D^2}\right]$$

$$= \frac{1}{36}\left(\frac{-\sin t\,(24+6D)}{D^2-16}\right)$$

$$= -\frac{1}{36}\left[\frac{24\sin t + 6\cos t}{D^2-16}\right]$$

$$= \frac{-\cancel{6}}{\cancel{36}_{6}}\left[\frac{4\sin t + \cos t}{D^2-16}\right]$$

$$= \frac{-1}{6}\left[\frac{4\sin t + \cos t}{D^2-16}\right]$$

$$D^2 \to -a^2 = -1$$

$$= \frac{-1}{6}\left[\frac{4\sin t + \cos t}{-1-16}\right.$$

$$= \frac{e^{2t}}{(2)^2 - 6D + 25} = \frac{e^{2t}}{4 - 6D + 25} = \frac{e^{2t}}{29 - 6(2)} = \frac{e^{2t}}{17D}$$

$$(xP)_2 = \frac{\sin t}{D^2 - 6D + 25} \quad D^2 \to -a^2 = -1 \quad = \frac{\sin t}{(-1)^2 - 6(-1)D + 25} = \frac{\sin t}{-6D + 24}$$

$$= \frac{-\sin t}{6(D-4)}$$

$$\therefore \frac{-\sin t\,(D+4)}{6(D+4)(D-4)} = \frac{-1}{6}\left[\frac{D(\sin t) + 4\sin t}{D^2 - 16}\right] \quad x'^{ly} \;\&\; \div \text{ by } (D+4)$$

$$= -\tfrac{1}{6}\left(\frac{\cos t + 4\sin t}{D^2 - 16}\right) = -\tfrac{1}{6}\left(\frac{\cos t + 4\sin t}{(-1)^2 - 16}\right)$$

$$(xp)_3 = \frac{t}{D^2 - 6D + 25}$$

$$25 - 6D + D^2)\, t \quad \left(\tfrac{t}{25} + \tfrac{6}{625}\right.$$

$$t - \frac{6(1)}{25} + 0$$

$$\tfrac{6}{25} - \tfrac{6}{25} / 0$$

8) $y'' + 4y' - 12y = e^{2x} - 3\sin 2x$

Sol:- $D^2 + 4Dy - 12y = e^{2x} - 3\sin 2x$

$D \to m$ A.E is given

$(m^2 + 4m - 12) = 0$

$m = 2, -6$

$y_c = C_1 e^{2x} + C_2 e^{-6x}$

$$y_p = \frac{e^{2x} - 3\sin 2x}{(D^2 + 4D - 12)}$$

$$y_p = \frac{e^{2x}}{D^2 + 4D - 12} - \frac{3\sin 2x}{D^2 + 4D - 12}$$

$y_p = (y_p)_1 - (y_p)_2$

$$(y_p)_1 = \frac{e^{2x}}{D^2+4D-12}$$

$D \to a$

$$= \frac{e^{2x}}{(2)^2+4(2)-12} = \frac{e^{2x}}{4+8-12} = \frac{e^{2x}}{0}$$

$$= \frac{x\, e^{2x}}{2D+4}$$

$$(y_p)_1 = \frac{x \cdot e^{2x}}{8}$$

$$(y_p)_2 = \frac{3\sin 2x}{D^2+4D-12}$$

$D^2 \to -a^2 = -4$

$$= \frac{3\sin 2x}{-4+4D-12} = \frac{3\sin 2x}{4D-16} = \frac{3}{4}\left[\frac{\sin 2x}{D-4}\right]$$

$$= \frac{3}{4}\left[\frac{\sin 2x\,(D+4)}{(D-4)(D+4)}\right]$$

$$= \frac{3}{4}\left[\frac{\sin 2x\,(D+4)}{D^2-16}\right]$$

$$= \frac{3}{4}\left[\frac{2\cos 2x + 4\sin 2x}{-4-16}\right]$$

$$= \frac{3}{4}\left[\frac{2\cos 2x + 4\sin 2x}{-20}\right]$$

$$= \frac{3\cdot\cancel{2}^{1}}{\cancel{4}_{2}}\left[\frac{\cos 2x + 2\sin 2x}{-20}\right]$$

$$(y_p)_2 = \frac{3}{2}\left[\frac{\cos 2x + 2\sin 2x}{-20}\right]$$

$y = (y_P)_1 + (y_P)_2 + y_c$

$$y = C_1 e^{2x} + C_2 e^{-6x} - \frac{3}{2}\left[\frac{\cos 2x + 2\sin 2x}{20}\right] + \frac{x \cdot e^{2x}}{8}$$

9. $(D^3 - D)y = 2e^x + 4\cos x$

Sol:- $(D^3 - D)y = 2e^x + 4\cos x$

$D \to m$ A.E is

$m^2 - m = 0$

$m = \pm 1, 0$

$y_c = C_1 e^x + C_2 e^{-x} + C_3 e^{0x}$

$$y_P = \frac{2e^x}{D^3 - D} + \frac{4\cos x}{D^3 - D}$$

$y_P = (y_P)_1 + (y_P)_2$

$$(y_P)_1 = \frac{2e^x}{D^3 - D} \quad D \to a$$

$$= \frac{2e^x}{1-1} = \frac{2e^x}{0}$$

x'y' & x in nume & diff deno.

$$= \frac{x \cdot 2e^x}{3D^2 - D}$$

$$= \frac{x \cdot 2e^x}{3-1} = \frac{\cancel{2}x \cdot e^x}{\cancel{2}}$$

$(y_P)_1 = x \cdot e^x$

$$(y_P)_2 = \frac{4\cos x}{D^3 - D}$$

$$= \frac{4\cos x}{D(D^2-1)} = \frac{\overset{2}{\cancel{4}}\cos x}{D(-\cancel{2})}$$

$$= -\frac{2\cos x}{D}$$

$$= -2\int \cos x$$

$$(y_P)_2 = -2\sin x$$

$$y = (y_c) + y_P$$

$$= C_1 e^{x} + C_2 e^{-x} + C_3 e^{0x} + x \cdot e^{x} + -2\sin x$$

10) $$\frac{d^2y}{dx^2} - 4y = \cosh(2x-1) + 3^x$$

Sol: $(D^2-4)y = \cosh(2x-1) + 3^x$

$$(D^2-4)y = 0$$

$$m^2 - 4 = 0$$

$$\therefore m = \pm 2$$

$$y_c = C_1 e^{-2x} + C_2 e^{2x}$$

$$y_P = \frac{\cosh(2x-1) + 3^x}{D^2-4}$$

$$= \frac{\frac{e^{2x-1} + e^{-(2x-1)}}{2} + e^{x\log 3}}{D^2-4}$$

$$\cos = \frac{e^{ax} + e^{-ax}}{2}$$

$$= \frac{1}{2}\left[\frac{e^{2x-1}}{D^2-4} + \frac{e^{-2x+1}}{D^2-4}\right] + \frac{e^{x\log 3}}{D^2-4}$$

$D \to a \to 2$ $\quad$ $D \to a \to -2$ $\quad$ $D \to a = \log 3$

$$= \frac{1}{2}\left[\frac{e^{2x-1}}{0} + \frac{e^{-2x-1}}{0}\right] + \frac{e^{x\log 3}}{(\log 3)^2 - 4}$$

$$= \frac{1}{2}\left[\frac{x \cdot e^{2x-1}}{2D} + \frac{x \cdot e^{-2x-1}}{2D}\right] + \frac{e^{x\log 3}}{(\log 3)^2 - 4}$$

$$y_p = \frac{1}{2}\left[\frac{x \cdot e^{2x-1}}{4} + \frac{x \cdot e^{-2x-1}}{-4}\right] + \frac{e^{x\log 3}}{(\log 3)^2 - 4}$$

$$y = y_c + y_p$$

$$y = c_1 e^{-2x} + c_2 e^{2x} + \frac{1}{2}\left[\frac{x \cdot e^{2x-1}}{4} + \frac{x \cdot e^{-2x-1}}{-4}\right] + \frac{e^{x\log 3}}{(\log 3)^2 - 4}$$

iii) Solve $(D^2 + D)y = x^2 + 2x + 4$.

Sol: Consider $(D^2 + D)y = x^2 + 2x + 4$.

$$(D^2 + D)y = 0$$

A.E $m^2 + m = 0$

$$\therefore m = -1, \; m = 0$$

$$y_c = c_1 e^0 + c_2 e^{-1x}$$

$$\therefore y_c = c_1 e^0 + c_2 e^{-x}.$$

$$y_p = \frac{x^2 + 2x + 4}{D^2 + D.}$$

$$D + D^2) \; x^2 \cancel{+ 2x} + 4 \; (x^3/3 + 4x.$$

$$\underline{x^2 + \cancel{2x}}$$

$$4$$

$$\underline{-4}$$

$$\underline{0}$$

$$(D + D^2)\left(x^3/3\right)$$

$$= D.x^3/3 + D^2.x^3/3$$

$$= 1/3 \cdot 3x^2 + 1/3 \cdot 3(2x)$$

$$= x^2 + 2x$$

$$(D + D)^2 (4x)$$

$$= D(4x) + D^2(4x) = 4 + 0$$

$y = y_c + y_p$

$= c_1 e^0 + c_2 e^{-x} + x^3/3 + 4x.$

12) $(D^3 + 3D^2 + D)y = x^2 + x.$

Sol $(D^3 + 3D^2 + D)y = x^2 + x.$

$(D^3 + 3D^2 + D)y = 0$

$m^3 + 3m^2 + m = 0$

$\therefore m = 0 \quad m = \frac{-3 \pm \sqrt{5}}{2}$

$y_c = c_1 e^{0.x} + c_2 e^{\frac{-3+\sqrt{5}}{2}} + c_3 e^{\frac{-3-\sqrt{5}}{2}}$

$y_p = \frac{x^2 + x}{D^3 + 3D^2 + D}$

$D + 3D^2 + D^3) \; \cancel{x^2} + x \; (x^3/3 - \frac{5x^2}{2} + 13x$

$\cancel{x^2} + 6x + 2$

$(-) \quad (-) \quad (-)$

$- \cancel{5}x - 2$

$- \cancel{5x} + 15$

13

$13 + 0$

0

$x^3/3 \, (D + 3D^2 + D^3)$

$= 1/\cancel{3} \cdot 3x^2 + \cancel{3} \cdot 1/\cancel{3}(6x) + 6/3$

$= x^2 + 6x + \cancel{6}^{2}/\cancel{3}$

$= x^2 + 6x + 2$

$-\frac{5x^2}{2} \, (D + 3D^2 + D^3)$

$-5/\cancel{2} \times \cancel{2}x + 3(-5/\cancel{2})(-2)$

$-5 + 15$

$y_p = \frac{x^3}{3} - \frac{5x^2}{2} + 13x$

$y = y_c + y_p$

$y = c_1 e^{0x} + c_2 e^{-3+\sqrt{5}/2} + c_3 e^{-3-\sqrt{5}/2}$

$+ x^3/3 - \frac{5x^2}{2} + 13x.$

Method of variation of parameter:-

Consider second order linear differential equation.

$$a_0 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = \phi(x) \longrightarrow (1)$$

Working procedure:-

1. Write auxilary equation, then according to nature of the root

2. Complete solution is assumed as $y = Ay_1 + By_2$ using complementary function (y_c) where we need to find the

3. values of A & B where

$$A = \int -\frac{y_2(x)\phi(x)}{w}\, dx \qquad B = \int \frac{y_1(x)\phi(x)}{w}\, dx$$

where, w is $w = y_1 y_2' - y_2 y_1'$

It is called woronskain

1. Solve $\frac{d^2y}{dx^2} + y = \sec x . \tan x$ by using variation of parameter method.

Sol:- $\frac{d^2y}{dx^2} + y = \sec x . \tan x$

$D = \frac{d}{dx}$, $\phi(x) = \sec x . \tan x$

$D^2y + y = \sec x . \tan x \Rightarrow (D^2+1)y = \sec x . \tan x$

A.E is given by

$m^2 + 1 = 0$

$\boxed{m = \pm i}$ R.P : 0 , I.P : 1

$y_c = e^{0x}(C_1\cos x + C_2\sin x)$

$y_c = C_1\cos x + C_2\sin x$

Assume C.S as $y = Ay_1 + By_2$

$y = A\cos x + B\sin x \longrightarrow (1)$

$y_1 = \cos x \qquad y_2 = \sin x$

$y_1' = -\sin x \qquad y_2' = \cos x$

$W = y_1y_2' - y_2y_1'$

$W = \cos x(\cos x) - \sin x(-\sin x)$

$W = \cos^2 x + \sin^2 x$

$\boxed{W = 1}$

$$A = \int \frac{-y_2\,\phi(x)}{W}\,.dx$$

$$A = \int \frac{-\sin x}{1}\,.\sec x.\tan x\,.dx$$

$$A = -\int \sin x.\frac{1}{\cos x}.\tan x.dx$$

$$A = -\int \tan^2 x.dx$$

$$= -\int(\sec^2 x - 1)\,dx$$

$$= -\int \sec^2 x.dx + \int dx$$

$$\boxed{A = -\tan x + x + K_1}$$

$$B = \int \frac{y_1(x)\,\phi(x)}{W}\,dx$$

$$= \int \frac{\cos x \cdot \sec x \, \tan x \, dx}{1}$$

$$= \int \tan x \, dx$$

$$B = \log \sec x + k_2$$

$\therefore$ (1) becomes

$$y = (-\tan x + x + k_1)\cos x + (\log \sec x + k_2)\sin x$$

2. $y'' + y = \tan x$

Sol:- $D^2 y + y = \tan x$

$$(D^2+1)y = \tan x$$

A.E is given by $m^2+1=0$

$$\boxed{m = \pm i}$$

$$y_c = e^{0x}(C_1 \cos x + C_2 \sin x)$$

$$y_c = C_1 \cos x + C_2 \sin x$$

Assume C.S as $y = Ay_1 + By_2$

$$y = A\cos x + B\sin x \longrightarrow (1)$$

$y_1 = \cos x$ $\quad$ $y_2 = \sin x$

$y_1' = -\sin x$ $\quad$ $y_2' = \cos x$

$$w = y_1 y_2' - y_2 y_1'$$

$$w = \cos x(\cos x) - \sin x(-\sin x)$$

$$w = \cos^2 x + \sin^2 x$$

$$\boxed{w = 1}$$

$$A = \int \frac{-y_2(x)\,\phi(x)}{\omega}\,dx$$

$$= \int \frac{-\sin x \cdot \tan x \cdot dx}{1}$$

$$= -\int \sin x \cdot \tan x \cdot dx$$

$$= -\int \frac{\sin^2 x}{\cos x}\,dx = \int \frac{1-\cos^2 x}{\cos x}\,dx$$

$$= -\int \frac{1}{\cos x}\,dx + \int \cos x\,dx$$

$$= -\int \sec x\,dx + \int \cos x\,dx$$

$$= -\log(\sec x + \tan x) + \sin x + K_1$$

$$B = \int \frac{y_1(x)\,\phi(x)}{\omega}\,dx$$

$$= \int \frac{\cos x \cdot \tan x}{1}\,dx = \int \cancel{\cos x} \cdot \frac{\sin x}{\cancel{\cos x}}\,dx$$

$$= \int \sin x\,dx$$

$$B = -\cos x + K_2$$

$$y = (-\log(\sec x + \tan x) + K_1)\cos x + (-\cos x + K_2)\sin x$$

3. $y'' - 2y' + 2y = e^x \tan x$

Sol:- $D^2y - 2Dy + 2y = e^x \tan x$

$(D^2 - 2D + 2)y = e^x \tan x$

A.E is given by $m^2 - 2m + 2 = 0$

$m = 1 \pm i$ R.P = 1, I.P = 1

$y_c = e^x(C_1 \cos x + C_2 \sin x) \Rightarrow C_1 e^x \cos x + C_2 e^x \sin x$

Assume C.S as $Ay_1 + By_2$

$y_1 = e^x \cos x$ $\quad$ $y_2 = e^x \sin x$

$y_1' = e^x(-\sin x) + \cos x . e^x$ $\quad$ $y_2' = e^x \cos x + \sin x . e^x$

$y_1' = \cos x . e^x - \sin x . e^x$

$w = y_1 y_2' - y_2 y_1'$

$= e^x \cos x(\cos x . e^x + \sin x . e^x) - e^x \sin x(\cos x . e^x - \sin x . e^x)$

$= e^{2x}(\cos^2 x + \cancel{\cos x . \sin x} - \cancel{\sin x . \cos x} + \sin^2 x)$

$w = e^{2x}(1)$

$\boxed{w = e^{2x}}$

$A = \int \frac{-y_2(x)\phi(x)}{w} dx$

$= \int -\frac{e^x \sin x . e^x . \tan x}{e^{2x}} dx = -\int \frac{\cancel{e^{2x}} . \sin x . \tan x}{\cancel{e^{2x}}} . dx$

$= -\int \frac{\sin^2 x}{\cos x} dx = -\int \frac{1 - \cos^2 x}{\cos x} dx$

$$= -\int \frac{1}{\cos x} dx + \int \cos x . dx$$

$$= -\int \sec x . dx + \int \cos x . dx$$

$$A = -\log(\sec x + \tan x) + \sin x + K_1$$

$$B = \int \frac{y_1(x)\,\phi(x)}{\omega} dx = \int \frac{e^x \cos x . e^x \tan x}{e^{2x}} dx$$

$$= \int \frac{\cancel{e^{2x}} . \cancel{\cos x}}{\cancel{e^{2x}}} . \frac{\sin x}{\cancel{\cos x}} dx$$

$$B = \int \sin x . dx = -\cos x + K_2$$

$$y = (-\log(\sec x + \tan x) + \sin x + K_1)\cos x + (-\cos x + K_2)\sin x$$

4. $y'' - 6y' + 9y = \frac{e^{3x}}{x^2}$

Sol:- $(D^2y - 6Dy + 9y) = \frac{e^{3x}}{x^2}$

A.E is given by

$$m^2 - 6m + 9 = 0$$

$$(m-3)^2 = 0$$

$$m = 3, 3$$

$$y_c = (C_1 + C_2 x)e^{3x} \Rightarrow y_c = C_1 e^{3x} + C_2 e^{3x} . x$$

C.S is given by $y = Ay_1 + By_2$

$$y = Ae^{3x} + Be^{3x} . x$$

$y_1 = e^{3x}$ $\quad$ $y_2 = xe^{3x}$

$y_1' = 3e^{3x}$ $\quad$ $y_2' = x . 3e^{3x} + e^{3x}$

$$w = y_1 y_2' - y_2 y_1'$$

$$W = e^{3x}\left(3x \cdot e^{3x} + e^{3x}\right) - x\, e^{3x}\left(3e^{3x}\right)$$

$$= e^{3x} \cdot e^{3x}\left(\cancel{3x} + 1 - \cancel{3x}\right)$$

$$\boxed{W = e^{6x}}$$

5. $y'' + 2y' + y = e^{-x} \log x$

Sol: $D^2y + 2Dy + y = e^{-x} \log x$

$(D^2 + 2D + 1)y = e^{-x} \log x$

A.E is given by $m^2 + 2m + 1 = 0$

$m = -1, -1$

$y_c = (c_1 + c_2 x)e^{-x} \Rightarrow e^{-x}c_1 + c_2 e^{-x} x$

C.S is $y = Ay_1 + By_2$

$y = Ae^{-x} + Be^{-x}$

$y_1 = e^{-x} \qquad y_2 = e^{-x} x$

$y_1' = -e^{-x} \qquad y_2' = -e^{-x}(x) + e^{-x}$

$W = y_1 y_2' - y_2 y_1'$

$= e^{-x}(-xe^{-x} + e^{-x}) - xe^{-x}(-e^{-x})$

$= e^{-2x}(-\cancel{x} + 1 + \cancel{x})$

$\boxed{W = e^{-2x}}$

$A = \int \frac{-y_2(x)\phi(x)}{W} dx = \int \frac{-x \cdot e^{-x} \cdot e^{-x}}{e^{-2x}} \cdot d \log x \cdot dx$

$= -\int \underset{\text{II}}{x} \cdot \underset{\text{I}}{\log x} \cdot dx$

$= -\left[\log x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{1}{x} \cdot dx\right] + K_1$

$A = -\log x \cdot \frac{x^2}{2} - \frac{x^2}{4} + K_1$

$$B = \int \frac{y_1(x)\,\phi(x)}{W}\,dx$$

$$= \int \frac{e^{-x}.e^{-x}.\log x}{e^{-2x}}.dx = \int \underset{\mathrm{II}}{(1)}.\underset{\mathrm{I}}{\log x}.dx$$

$$= \log x . x - \int x . \frac{1}{x} . dx + k_2$$

$$B = x.\log x - x + k_2$$

$$y = \left(-\log x . \frac{x^2}{2} - \frac{x^2}{4} + k_1\right)e^{-x} + (x.\log x - x + k_2)\,e^{-x}.x$$

6. $(D^2+3D+2)y = e^{e^x}$

<u>Sol:-</u> $(D^2+3D+2)y = e^{e^x}$

A.E is given by $m^2+3m+2=0$

$$m = -1, -2$$

$$y_c = C_1 e^{-x} + C_2 e^{-2x}$$

C.S is given by $y = Ay_1 + By_2$

$$y_1 = e^{-x} \qquad y_2 = e^{-2x}$$

$$y_1' = -e^{-x} \qquad y_2' = -2e^{-2x}$$

$$W = y_1 y_2' - y_2 y_1'$$

$$= e^{-x}(-2e^{-2x}) - e^{-2x}(-e^{-x})$$

$$= -2e^{-3x} + e^{-3x} = e^{-3x}(-2+1)$$

$$\boxed{W = -e^{-3x}}$$

$$A = \int \frac{-y_2(x)\,\phi(x)}{W}\,dx$$

$$A = \int \frac{\cancel{+} e^{-2x} \cdot e^{e^x}}{\cancel{+} e^{-3x}} dx + K_1$$

$$= \int e^{-2x} \cdot e^{3x} \cdot e^{e^x} \cdot dx = \int e^x \cdot e^{e^x} \cdot dx$$

put $e^x = z$

$e^x \cdot dx = dz$

$$A = \int e^z dz + K_1$$

$$A = e^{e^x} + K_1$$

$$B = \int \frac{e^{-x} \cdot e^{e^x}}{-e^{-3x}} dx + K_2$$

$$= -\int e^{-x} \cdot e^{3x} \cdot e^{e^x} \cdot dx + K_2$$

$$= -\int e^{2x} \cdot e^{e^x} \cdot dx + K_2 = -\int e^x \cdot e^{e^x} \cdot e^x dx + K_2$$

put $e^x = z$

$e^x \cdot dx = dz$

$$= -\int \underset{\mathrm{I}}{z} \cdot \underset{\mathrm{II}}{e^z} \cdot dz$$

$$= -\left[z \int e^z \cdot dz - \int \int e^z \frac{d}{dz}(z) dz \right] + K_2$$

$$B = -[z \cdot e^z - e^z] + K_2$$

$$B = -[e^x e^{e^x} - e^{e^x}] + K_2$$

$$y = (e^{e^x} + K_1) e^{-x} + (-e^x e^{e^x} + e^{e^x} + K_2) e^{-2x}$$

$\Rightarrow (D^2+1)y = \operatorname{cosec}x \cdot \cot x$

Sol:- $(D^2+1)y = \operatorname{cosec}x \cdot \cot x$

A.E is given by $m^2+1=0$

$\boxed{m = \pm i}$

$y_c = e^{0x}(C_1\cos x + C_2\sin x)$

$y_c = C_1\cos x + C_2\sin x$

$y = Ay_1 + By_2$ $\quad y_1 = \cos x \quad y_2 = \sin x$

$y_1' = -\sin x \quad y_2' = \cos x$

$w = y_1y_2' - y_2y_1'$

$= \cos x(\cos x) - \sin x(-\sin x)$

$= \cos^2 x + \sin^2 x$

$\boxed{w = 1}$

$A = \int \frac{-y_2(x)\phi(x)}{w}dx$

$= -\int \frac{\sin x \cdot \operatorname{cosec}x \cdot \cot x \cdot dx}{1}$ $\quad = -\int \cancel{\sin x} \cdot \frac{1}{\cancel{\sin x}} \cdot \cot x \cdot dx$

$= -\int \cot x \cdot dx$

$A = -\log \sin x + k_1$

$B = \int \frac{\cos x \cdot \operatorname{cosec}x \cdot \cot x \cdot dx}{1}$

$= \int \cos x \cdot \frac{1}{\sin x} \cdot \frac{\cos x}{\sin x} \cdot dx$

$= \int \frac{\cos^2 x}{\sin^2 x} dx$

$= \int \frac{1-\sin^2 x}{\sin^2 x} dx$

$$= \int \frac{1}{\sin^2 x} dx - \int 1 \cdot dx$$

$$= \int \operatorname{cosec}^2 x \cdot dx - \int dx$$

$$B = -\cot x - x + k_2$$

$$y = (-\log \sin x + k_1) \cos x + (-\cot x - x + k_2) \sin x$$

Differential Equation reducable to D.E constant co-efficient:

Here, we study the D.E involving the variable co-efficient. This can be reduced to equation with constant co-efficient by specific substitution.

Here we study two types of equations.

a) *Legendre's linear second order D.E:-*

If the given D.E is of the form

$$(ax+b)^2 \frac{d^2y}{dx^2} + (ax+b)\frac{dy}{dx} + y = \phi(x) \longrightarrow (1)$$

$(ax+b)^2 D^2y + (ax+b)Dy + y = \phi(x)$. Then the D.E is called Legendres linear second order D.E which contains variable where $\phi(x)$ is function of x

put $\log(ax+b) = t$

$$(ax+b)Dy = aDy$$

$$(ax+b)^2 D^2y = a^2 D(D-1)y$$

with this substitution eq-(1) reduces to differential equation with constant variable and is know method to solve it.

b) Cauchy's linear second order D.E

if the given D.E is of the form

$$x^2y'' + xy' + y = \phi(x)$$

put $\log x = t \Rightarrow x = e^t$

$$xy' = Dy$$

$$x^2y'' = D(D-1)y$$

1. Solve $(3x+2)^2y'' + 3(3x+2)y' - 36y = 8x^2+4x+1$

Sol:- put $\log(3x+2) = t$ $\Rightarrow 3x+2 = e^t$

$(3x+2)y' = 3Dy$ $\qquad x = \dfrac{e^t-2}{3}$

$(3x+2)^2y'' = 9D(D-1)y$

$\therefore$ (1) becomes

$$9D(D-1)y + 3(3Dy) - 36y = 8\left(\frac{e^t-2}{3}\right)^2 + 4\left(\frac{e^t-2}{3}\right) + 1$$

$$9(D^2 - \cancel{D} + \cancel{D} - 4)y = \frac{8}{9}\left(e^{2t} + 4 - 4e^t\right) + \frac{4}{3}\left(e^t - 2\right) + 1$$

$$9(D^2-4)y = \frac{1}{9}\left[\frac{8}{9}e^{2t} + \frac{32}{9} - \frac{32e^t}{9} + \frac{4e^t}{3} - \frac{8}{3} + 1\right]$$

$$(D^2-4)y = \frac{1}{9}\left[\frac{8}{9}e^{2t} - \frac{32e^t}{9} + \frac{4e^t}{3} + \frac{32}{9} - \frac{5}{3}\right]$$

$$(D^2-4)y = \frac{1}{9}\left[\frac{8}{9}e^{2t} - \frac{20}{9}e^t + \frac{17}{9}\right]$$

A.E is $m^2-4=0$, $\boxed{m=\pm 2}$

$$y_c = C_1e^{2t} + C_2e^{-2t}$$

$$y_p = \frac{1}{9}\left[\frac{8}{9}e^{2t} - \frac{20}{9}e^t + \frac{17}{9}\right]$$

$$y_p = \frac{1}{9}\left[\frac{8}{9}\frac{e^{2t}}{(D^2-4)} - \frac{20}{9}\frac{e^t}{(D^2-4)} + \frac{17}{9}\frac{1\times e^{0t}}{(D^2-4)}\right]$$

$$y_p = \frac{1}{9}\left[\frac{8}{9}\frac{e^{2t}}{0} - \frac{20}{9}\frac{e^t}{(-3)} + \frac{17}{9}\frac{1}{(-4)}\right]$$

$$y_p = \frac{1}{9}\left[\frac{8}{9} + \frac{e^{2t}}{2D} + \frac{20e^t}{27} - \frac{17}{36}\right]$$

$$y_p = \frac{1}{9}\left[\frac{8}{9} + \frac{e^{2t}}{4} + \frac{20e^t}{27} - \frac{17}{36}\right]$$

$y = y_c + y_p$

$y = C_1 e^{2t} + C_2 e^{-2t} + \frac{1}{9}\left[\frac{2}{9} t e^{2t} + \frac{20}{27} e^t - \frac{17}{36}\right]$

$y = C_1 (3x+2)^2 + C_2 (3x+2)^{-2} + \frac{1}{9}\left[\frac{2}{9}\log(3x+2)(3x+2)^{-2} + \frac{20}{27}(3x+2) - \frac{17}{36}\right]$

2. Solve $(x+2)^2 y'' - (x+2)y' + y = 3x+4 \longrightarrow (1)$

put $\log(x+2) = t$ $\qquad x+2 = e^t$

$\boxed{x = e^t - 2}$

$(x+2)y' = Dy$

$(x+2)^2 y'' = D(D-1)y$

$\therefore$ (1) becomes

$D(D-1)y - Dy + y = 3x+4$

$(D^2 - D)y - Dy + y = 3(e^t - 2) + 4$

$D^2 y - Dy - Dy + y = 3e^t - 6 + 4$

$D^2 y - 2Dy + y = 3e^t - 2$

$(D^2 - 2D + 1)y = 3e^t - 2$

A.E is $m^2 - 2m + 1 = 0$

$(m-1)^2 = 0$

$m = 1, 1$

$y_c = (C_1 + C_2 t)e^t$

$y_p = \dfrac{3x+4}{(D^2 - 2D + 1)}$

$y_p = \dfrac{3(e^t - 2) + 4}{D^2 - 2D + 1}$

$$y_P = \frac{3e^t - 2}{D^2 - 2D + 1}$$

$$y_P = \frac{3e^t}{D^2 - 2D + 1} - \frac{2}{D^2 - 2D + 1}$$

Since 1 is root of A.E, Replacing $D \to a$ deno. of 1^{st} term becomes 0 two times.

$$y_P = \frac{3t^2 . e^t}{2} - \frac{2e^{0t}}{0 - 0 + 1}$$

$$y_P = \frac{3t^2 e^t}{2} - 2$$

$$y = y_c - y_P$$

$$y = (C_1 + C_2 t)e^t + \frac{3t^2 e^t}{2} - 2$$

$$y = (C_1 + C_2 \log(x+2)) + \frac{3}{2}(\log(x+2))^2(x+2) - 2$$

3. Solve $x^2y'' - 3xy' + 4y = (1+x)^2$

Sol: put $\log x = t$, $x = e^t$

$$xy' = Dy$$

$$x^2y'' = D(D-1)y$$

$$D(D-1)y - 3Dy + 4y = (1+e^t)^2$$

$$(D^2 - D - 3D + 4)y = 1 + e^{2t} + 2e^t$$

$$(D^2 - 4D + 4)y = 1 + e^{2t} + 2e^t$$

A.E is $m^2 - 4m + 4 = 0$

$$(m-2)^2 = 0$$

$$m = 2, 2$$

$$y_c = (C_1 + C_2t)e^{2t}$$

$$P.I = y_p = \frac{1 + e^{2t} + 2e^t}{D^2 - 4D + 4}$$

$$y_p = \frac{1 \times e^{0t}}{D^2 - 4D + 4} + \frac{e^{2t}}{D^2 - 4D + 4} + 2 \cdot \frac{e^t}{D^2 - 4D + 4}$$

$D \to a = 0$ $\quad$ $D \to a = 2$ $\quad$ $D \to a = 1$

$$y_p = \frac{1}{0 - 0 + 4} + \frac{e^{2t}}{4 - 8 + 4} + 2 \cdot \frac{e^t}{1 - \cancel{4} + \cancel{4}}$$

$$= \frac{1}{4} + \frac{t \cdot e^{2t}}{2D - 4} + 2e^t$$

$D \to 2$

$$y_p = \frac{1}{4} + \frac{t^2 e^{2t}}{2} + 2e^t$$

$$y = y_c + y_p$$

$$y = (C_1 + C_2t)e^{2t} + \frac{1}{4} + \frac{t^2 e^{2t}}{2} + 2e^t$$

$$y = (C_1 + C_2t)e^{2t} + \frac{1}{4} + \frac{t^2 x^2}{2} + 2x$$

4) Solve $x^2y'' - xy' + 4y = \cos(\log x)$

Sol: put $\log x = t$ $x = e^t$

$xy' = Dy$

$x^2y'' = D(D-1)y$

$D(D-1)y - Dy + 4y = \cos(t)$

$(D^2 - D - D + 4)y = \cos t$

$(D^2 - 2D + 4)y = \cos t$

A.E is $m^2 - 2m + 4 = 0$

$m = 1 \pm \sqrt{3}i$

$y_c = e^t(C_1 \cos\sqrt{3}t + C_2 \sin\sqrt{3}t)$

$y_p = \dfrac{\cos t}{D^2 - 2D + 4}$

$D^2 \to -a^2 = -1$

$= \dfrac{\cos t}{-1-2D+4} = \dfrac{\cos t}{-2D+3} = -\dfrac{\cos t}{(2D-3)}\dfrac{(2D+3)}{(2D+3)}$

$= -\left[\dfrac{2(-\sin t) + 3\cos t}{4D^2 - 9}\right]$ $D^2 \to -a^2 \to -1$

$y_p = \dfrac{2\sin t}{-13} + \dfrac{(-3\cos t)}{-13}$

$y = e^t(C_1 \cos\sqrt{3}t + C_2 \sin\sqrt{3}t) - \frac{2}{13}\sin t + \dfrac{3\cos t}{13}$

$y = x(C_1 \cos(\sqrt{3}\log x) + C_2 \sin(\sqrt{3}\log x) - \frac{2}{13}\sin(\log x) + \frac{3}{13}\cos(\log x))$

5. $x^2 \frac{d^2y}{dx^2} + x \cdot \frac{dy}{dx} + y = \sin(\log x^2)$

Sol:- $x^2y'' + xy' + y = \sin(\log x^2)$

put $\log x = t$, $x = e^t$

$xy' = Dy$

$x^2y'' = D(D-1)y$

$D(D-1)y + Dy + y = \sin 2t$

$(D^2 - \cancel{D} + \cancel{D} + 1)y = \sin 2t$

$(D^2+1)y = \sin 2t$

A.E is $m^2 + 1 = 0$

$m = \pm i$

$y_c = e^{0t}(C_1 \cos t + C_2 \sin t)$

$$y_p = \frac{\sin 2t}{D^2+1}$$

$D^2 \rightarrow -a^2 = -4$

$$y_p = \frac{\sin 2t}{-4+1} = \frac{\sin 2t}{-3}$$

$$y_p = \frac{\sin 2t}{-3}$$

$y = y_c + y_p$

$$= e^{0t}(C_1 \cos t + C_2 \sin t) - \frac{\sin 2t}{3}$$

$$y = C_1 \cos(\log x) + C_2 \sin(\log x) - \frac{\sin(2 \log x)}{3}$$

6. $x^2y'' + xy' + 9y = 3x^2 + \sin(3\log x)$

Sol: put $\log x = t$, $e^t = x$

$$xy' = Dy$$

$$x^2y'' = D(D-1)y$$

$$D(D-1)y + Dy + 9y = 3e^{2t} + \sin(3t)$$

$$(D^2 - \cancel{D} + \cancel{D} + 9)y = 3e^{2t} + \sin 3t$$

$$(D^2-9)y = 3e^{2t} + \sin 3t$$

A.E is $m^2 - 9 = 0$

$$m = \pm 3$$

$$y_c = (c_1e^{3t} + c_2e^{-3t})$$

$$P.I = y_p = \frac{3e^{2t} + \sin 3t}{D^2-9}$$

$$y_p = \frac{3e^{2t}}{D^2-9} + \frac{\sin 3t}{D^2-9}$$

$$= \frac{3e^{2t}}{-4-9} + \frac{\sin 3t}{9-9}$$

$$= \frac{3e^{2t}}{-5} + \frac{t\sin 3t \times (2D+9)}{(2D-9)\times(2D+9)}$$

$$= \frac{3e^{2t}}{-5} + \frac{t\sin 3t\,(2D+9)}{4D^2-81}$$

$D^2 - 2 - 9$

$$y_p = \frac{3e^{2t}}{-5} + \frac{t.\,2(3)\cos 3t + 9t\sin 3t}{4(-9)-81}$$

$$y_p = \frac{3e^{2t}}{-5} \cancel{+} - \left[\frac{6t\cos 3t + 9t\sin 3t}{-117}\right]$$

$$y = c_1x^3 + c_2x^{-3} - \frac{3x^2}{5} + \frac{6\log x\cos(3\log x) + 9\log x\sin(3\log x)}{117}$$