

LINEAR TRANSFORMATION

Definition:

Let U and V be vector spaces. Let u and v be vectors in U and let c be a scalar.

A transformation $T: U \longrightarrow V$ is said to be linear if

$$T(u+v) = T(u) + T(v)$$

$$T(cu) = cT(u)$$

1. Prove that $T(x, y) = (2x, x+y)$ is linear in $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$.

Solution:

Let (x_1, y_1) and (x_2, y_2) be elements in $\mathbb{R}^2$

$$\begin{aligned} \text{Then } T((x_1, y_1) + (x_2, y_2)) &= T(x_1 + x_2, y_1 + y_2) \\ &= (2x_1 + 2x_2, x_1 + x_2 + y_1 + y_2) \\ &= (2x_1, x_1 + y_1, 2x_2, x_2 + y_2) \\ &= T(x_1, y_1) + T(x_2, y_2) \end{aligned}$$

T preserves vector addition.

$$\begin{aligned} T(c(x_1, y_1)) &= T(cx_1, cy_1) \\ &= (2cx_1, cx_1 + cy_1) \\ &= c(2x_1, x_1 + y_1) \\ &= cT(x_1, y_1) \end{aligned}$$

T preserves vector multiplication.

2. Let P_n be the vector space of real polynomial functions of degree $\leq n$, Show that the transformation $T: P_2 \rightarrow P_1$ is linear.

$$T(ax^2 + bx + c) = (a + b)x + c$$

Solution:

Let $(ax^2 + bx + c, px^2 + qx + r)$ be arbitrary elements of P_2 .

Then

$$\begin{aligned} T((ax^2 + bx + c), (px^2 + qx + r)) &= T((a + p)x^2 + (b + q)x + (c + r)) \\ &= ((a + p) + (b + q))x + (c + r) \\ &= ((a + b)x + c + (p + q)x + r) \\ &= T(ax^2 + bx + c) + T(px^2 + qx + r) \end{aligned}$$

T preserves vector addition

$$\begin{aligned} T(k(ax^2 + bx + c)) &= T(kax^2 + kbx + kc) \\ &= ((ka + kb)x + kc) \\ &= (k(a + b)x + c) \\ &= kT(ax^2 + bx + c) \end{aligned}$$

T preserves vector multiplication.

3. Determine the kernel and the range of the transformation defined by $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 1 \\ 1 & 1 & 4 \end{bmatrix}$

Solution:

Kernel: kernel consists of $x = (x_1, x_2, x_3)$ in R^3 such that $T(x) = 0$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 1 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + 2x_2 + 3x_3 = 0$$

$$-x_2 + x_3 = 0$$

Let $x_3 = k$ then we get $x_2 = k$, and $x_1 = -5k$

Kernel of T is a one dimensional subspace of R^3 with basis $(-5, 1, 1)$

Range:

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 1 \\ 3 & 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The vectors $(1, 0, 1), (0, 1, 1)$ span the range(T).

The L.C $x(1, 0, 1) + y(0, 1, 1)$

$R(T) = \{(x, y, x + y)\}$.

Singular and Nonsingular Linear Transformation:

Determine $F: \mathbb{R}^2$ to $\mathbb{R}^2$ defined by $F(x,y)=(x-y,x-2y)$ is non-singular and hence find F inverse

Solution:

Find the kernel F by setting $F(v)=0$

$$x - y = 0$$

$$x - 2y = 0$$

By simplifying the equation we get $x=0, y=0$.

Hence F is **non- singular and invertible**.

$$\begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$R_2 = R_2 - R_1$$

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ a - b \end{bmatrix}$$
$$y = a - b$$

$x - y = a$ is the first equation

$$\text{Sub } y = a - b$$

$$x - (a - b) = a$$

$$x = a + a - b$$

$$x = 2a - b$$

Consider $a = x$ and $b = y$ we get

$$F^{-1}(x,y) = (2x - y, x - y)$$

Determine $F: \mathbb{R}^2$ to $\mathbb{R}^2$ defined by $F(x,y)=(2x-4y,3x-6y)$ is non-singular.

Solution:

kernel consists of $x = (x_1, x_2)$ in $\mathbb{R}^2$ such that $T(x) = 0$

$$\begin{bmatrix} 2 & -4 \\ 3 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$R_2 = 2R_2 - 3R_1$$

$$\begin{bmatrix} 2 & -4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The system has one free variable.

Hence F is **singular**

Determine $F: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ defined by $F(x, y) = (x + y, x - 2y, 3x + y)$ is non-singular and find F inverse.

Solution:

kernel consists of $x = (x_1, x_2)$ in $\mathbb{R}^2$ such that $T(x) = 0$

$$\begin{bmatrix} 1 & 1 \\ 1 & -2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 = R_2 - R_1, R_3 = R_3 - 3R_1$

$$\begin{bmatrix} 1 & 1 \\ 0 & 3 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 = 3R_3 - 2R_2$

$$\begin{bmatrix} 1 & 1 \\ 0 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We get $Y=0$ and $x=0$ from the above equation.

F is non-singular but not invertible.

We can't find F^{-1} because F is not a square matrix.

Example:1

$F: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ and $G: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ by $F(x, y, z) = (2x, y + z), G(x, y, z) = (x - z, y)$. Find the formulas defining the maps i) $F + G$ ii) $2F - 5G$

Solution:

- i) $F(x, y, z) + G(x, y, z) = (2x, y + z) + (x - z, y) = (2x + x - z, y + z + y) = (3x - z, 2y + z)$
- ii) $2F - 5G = 2(2x, y + z) - 5(x - z, y) = (4x, 2y + 2z) - (5x - 5z, 5y) = (4x - (5x - 5z), 2y + 2z - 5y) = (-x + 5z, -3y + 2z)$

Example:2

$F: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ and $G: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $F(x, y, z) = (2x, y + z), G(x, y, z) = (y, x)$. Find the formulas defining the mapping i) $F \circ G$ ii) $G \circ F$

Solution:

- i) $(G \circ F)(x, y, z) = G(F(x, y, z)) = G(2x, y + z) = (z + y, 2x)$
- ii) The mapping $F \circ G$ is not defined since the image of G is not contained in the domain of F .

Example:3

$F: \mathbb{R}^2$ to $\mathbb{R}^2$ and $G: \mathbb{R}^2$ to $\mathbb{R}^2$ by $F(x, y) = (y, x)$, $G(x, y) = (0, x)$. Find the formulas defining the maps i) $F + G$ ii) $2F - 3G$ iii) FG iv) GF v) F^2

Solution:

- i) $F(x, y) + G(x, y) = (y, x) + (0, x) = (y, 2x)$
- ii) $2F - 3G = 2(y, x) - 3(0, x) = (2y, 2x) - (0, 3x) = (2y, 2x - 3x) = (2y, -x)$
- iii) $FG(x, y) = F(G(x, y)) = F(0, x) = (x, 0)$
- iv) $GF(x, y) = G(F(x, y)) = G(y, x) = (0, y)$
- v) $F^2(x, y) = F(F(x, y)) = F(y, x) = (x, y)$.

Example:4

T is a Linear operator T on $\mathbb{R}^3$ defined by $T(x, y, z) = (2x, 4x - y, 2x + 3y - z)$.

- a) Show that T is invertible. Find formulas for b) T^{-1} c) T^2

Solution:

kernel consists of $x = (x, y, z)$ in $\mathbb{R}^3$ such that $T(x) = 0$

$$\begin{bmatrix} 2 & 0 & 0 \\ 4 & -1 & 0 \\ 2 & 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_2 = R_2 - 2R_1, R_3 = R_3 - R_1$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3 = R_3 + 3R_2$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We get $x = 0$, $y = 0$ and $z = 0$

Hence T is non-singular and so T is invertible.

- b) $T(x, y, z) = (a, b, c)$

$$T^{-1}(a, b, c) = (x, y, z)$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 4 & -1 & 0 \\ 2 & 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$R_2 = R_2 - 2R_1, R_3 = R_3 - R_1$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b - 2a \\ c - a \end{bmatrix}$$

$$R_3 = R_3 + 3R_2$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b - 2a \\ c + 3b - 7a \end{bmatrix}$$

$$T^{-1}(a, b, c) = (a, b - 2a, c + 3b - 7a)$$

Replace a, b, c as x, y, z $T^{-1}(x, y, z) = (x, y - 2x, z + 3y - 7x)$

$$T^2 = T(T(x, y, z)) = T(2x, 4x - y, 2x + 3y - z)$$

$$T(2x, 4x - y, 2x + 3y - z)$$

$$= (2(2x), 4(2x) - (4x - y), 2(2x) + 3(4x - y) - 7(2x + 3y - z))$$

$$T^2 = -((4x), 4x + y, 14x - 6y + z)$$

Example:5

T is a Linear operator T on R^3 defined by $T(x, y, z) = (2x, 4x - y, 2x + 3y - z)$.

a) Show that T is invertible. Find formulas for b) T^{-1}

$$T(x, y, z) = (x + z, x - y, y)$$

Solution:

kernel consists of $x = (x, y, z)$ in R^2 such that $T(x) = 0$

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_2 = R_2 - R_1,$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3 = R_3 + R_2$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We get $x = 0$, $y = 0$ and $z = 0$

Hence T is non-singular and so T is invertible.

b) $T(x, y, z) = (a, b, c)$

$$T^{-1}(a, b, c) = (x, y, z)$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$R_2 = R_2 - R_1$,

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b - a \\ c \end{bmatrix}$$

$R_3 = R_3 + R_2$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b - a \\ c + b - a \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} c + b \\ b - a \\ c + b - a \end{bmatrix}$$

$$T^{-1}(a, b, c) = (c + b, b - a, c + b - a)$$

Replace a, b, c as x, y, z $T^{-1}(x, y, z) = (y + z, y - x, z + y - x)$