Matrix:-

MODULE-05

A set of mn elements written in an array of m rows and n columns embedded in brackets [ ] or ( ) is called a matrix of order m×n

Square matrix, Row matrix, Identify matrix, Diagonal matrix, Null matrix, Symmetric matrix, scalar matrix, upper triangle matrix, lower triangular matrix, unit matrix are few different type of matrix.

Algebra of Matrix:-

Rank of a matrix :-

A rank of a matrix 'A' in its echelon form is equal to the number of non-zero rows. It is denoted by $\rho(A)$

Working procedure:-

1) Make leading entry (first entry in first row) Non-zero, much preferably '1'. (If first entry is zero then interchange with any suitable row to meet the requirement)

2) Row echelon form will be achieved ~~from~~ first and numbers of non-zero rows gives rank of the matrix.

Echelon form:-

All the zero rows are below non-zero rows.

The first non-zero row entry in any non-zero row is 1.

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 6 & 5 \\ 0 & 0 & 3 \\ 0 & 8 & 1 \end{bmatrix}$$ (Its not echelon form)

$R_2 \leftrightarrow R_3$

$$\sim \begin{bmatrix} 2 & 6 & 5 \\ 0 & 8 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

$R_1 \longrightarrow R_1/2$

$R_2 \longrightarrow R_2/8$

$R_3 \longrightarrow R_3/3$

$$\sim \begin{bmatrix} 1 & 3 & 5/2 \\ 0 & 1 & 1/8 \\ 0 & 0 & 1 \end{bmatrix}$$ (Echelon form)

Problems

1) Find rank of the matrix $A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$ by performing row transformation.

sol:- Consider, $A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$

$R_2 \longrightarrow R_2 - 2R_1$

$R_3 \longrightarrow R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2-2 & 3-4 & 5-6 & 1-4 \\ 1-1 & 3-2 & 4-3 & 5-2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 1 & 1 & 3 \end{bmatrix} \quad R_3 \longrightarrow R_3 + R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_2 \longrightarrow R_2/_{-1}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore \rho(A) = 2$ Rank of matrix is 2

2. Find rank of the matrix $A = \begin{bmatrix} 0 & 2 & 3 & 4 \\ 2 & 3 & 5 & 4 \\ 4 & 8 & 13 & 12 \end{bmatrix}$ by performing row transformation.

sol:- Consider,

$$A = \begin{bmatrix} 0 & 2 & 3 & 4 \\ 2 & 3 & 5 & 4 \\ 4 & 8 & 13 & 12 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 & 5 & 4 \\ 0 & 2 & 3 & 4 \\ 4 & 8 & 13 & 12 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$, $R_3 \longrightarrow R_3 - 2R_1$

$$\sim \begin{bmatrix} 2 & 3 & 5 & 4 \\ 0 & 2 & 3 & 4 \\ 4-4 & 8-6 & 13-10 & 12-8 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 & 5 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \end{bmatrix}$$

$R_3 \longrightarrow R_3 - R_2$

$$\sim \begin{bmatrix} 2 & 3 & 5 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 \longrightarrow R_1/2$

$R_2 \longrightarrow R_2/2$

$$\sim \begin{bmatrix} 1 & 3/2 & 5/2 & 2 \\ 0 & 1 & 3/2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore \rho(A) = 2$

3. $A = \begin{bmatrix} 0 & 1 & -3 & 1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & 2 & 0 \end{bmatrix}$

$$A \begin{bmatrix} 0 & 1 & -3 & 1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & 2 & 0 \end{bmatrix} \quad R_1 \longleftrightarrow R_2 \quad \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

$R_3 \longrightarrow R_3 - 3R_1$

$R_4 \longrightarrow R_4 - R_1$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 3 & 1 \\ 3-3 & 1-0 & 0-3 & 2-3 \\ 1-1 & 1-0 & -2-1 & 0-1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix}$$

$R_3 \longrightarrow R_3 - R_2$

$R_4 \longrightarrow R_4 - R_2$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & 1 \\ 0-0 & 1-1 & -3(-3) & -1-1 \\ 0-0 & 1-1 & -3-(-3) & -1-1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & -2 \end{bmatrix} \quad R_4 \longrightarrow R_4 - R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & -2-(-2) \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \rightarrow R_3/-2$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Rank of matrix A is $\rho(A) = 3$

Consistency of system of Linear equation:-

A system of linear equation is said to be consistent if it possess a solution, otherwise the system is said to be inconsistent.

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array}\right\} (1)$$

where a's & b's are constants

The above system can be written in the matrix form $Ax = B$ and is given by

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

To check consistency:-

1) To Find Rank of augmented matrix denoted by $[A:B]$

2) The system of given equation is consistent if $\rho(A) = \rho(A:B)$ (or)

$$\rho[A] = \rho[A:B]$$

If r is rank of matrix and n is no. of unknown

3) If $\rho(A) = \rho(A:B) = r = n$ then system of equation has unique sol

If $\rho(A) = \rho(A:B) = r < n$ then system of equation has infinite sol

If $\rho(A) \neq \rho(A:B)$ then system is inconsistent.

1. Test for consistency and solve $x+y+z = 6$
$x-y+2z = 5$
$3x+y+z = 8$

Sol:- $[A:B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 1 & -1 & 2 & 5 \\ 3 & 1 & 1 & 8 \end{array}\right]$

$R_2 \longrightarrow R_2 - R_1$

$R_3 \longrightarrow R_3 - 3R_1$

$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & -1-1 & 2-1 & 5-6 \\ 3-3 & 1-3 & 1-3 & 8-18 \end{array}\right] \sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & -2 & 1 & -1 \\ 0 & -2 & -2 & -10 \end{array}\right]$

$R_3 \longrightarrow R_3 - (R_2)$

$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & -2 & 1 & -1 \\ 0 & -2-(-2) & -2-1 & -10-(-1) \end{array}\right]$

$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 1 & -2 & 1 & -1 \\ 0 & 0 & -3 & -9 \end{array}\right]$ $R_2 \longrightarrow R_2/-2$
$R_3 \longrightarrow R_3/-3$

$[A:B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 1 & -1/2 & 1/2 \\ 0 & 0 & 1 & 3 \end{array}\right] \longrightarrow (1)$

$\left.\begin{array}{l} \text{Rank of } [A:B] = 3 \\ \text{Rank of } [A] = 3 \end{array}\right\} r = 3$

n - no. of unknown $= 3$

$\therefore \rho[A:B] = \rho(A) = 3 = 3$ (No. of unknown)

$\therefore r = n$ i.e $3 = 3 \Rightarrow$ system is consistent.

... exists unique solution

from matrix eq(1)

$x+y+z=6$

$0.x+y-\frac{1}{2}z=\frac{1}{2}$

$0.x+0.y+z=3 \Rightarrow \boxed{z=3}$

$y-\frac{z}{2}=\frac{1}{2} \Rightarrow y=\frac{1}{2}+\frac{1}{2}(z)$

$=\frac{1}{2}+\frac{1}{2}(3)=\frac{4}{2}=2$

$\boxed{y=2}$

Consider,

$x+y+z=6$

$x+2+3=6$

$x=6-5$

$\boxed{x=1}$

2. Solve $x-4y+7z=14$,

$3x+8y-2z=13$,

$7x-8y+26z=5$

sol:- Consider,

$$[A:B]=\left[\begin{array}{ccc:c} 1 & -4 & 7 & 14 \\ 3 & 8 & -2 & 13 \\ 7 & -8 & 26 & 5 \end{array}\right]$$

$R_2 \rightarrow R_2-R_1$, $R_3 \rightarrow R_3-7R_1$

$$\sim\left[\begin{array}{ccc:c} 1 & -4 & 7 & 14 \\ 3-3 & 8-(-12) & -2(-21) & 13-42 \\ 7-7 & -8-(-28) & 26-(49) & 5-98 \end{array}\right] \sim \left[\begin{array}{ccc:c} 1 & -4 & 7 & 14 \\ 0 & 20 & -23 & -29 \\ 0 & 20 & -23 & -93 \end{array}\right]$$

$R_3 \rightarrow R_3-R_2$

$$\sim \begin{bmatrix} 1 & -4 & 7 & : & 14 \\ 0 & 20 & -23 & : & -29 \\ 0 & 0 & 0 & : & -64 \end{bmatrix} \quad R_2 \to R_2/20$$

$$\sim \begin{bmatrix} 1 & -4 & 7 & : & 14 \\ 0 & 1 & -23/20 & : & -29/20 \\ 0 & 0 & 0 & : & -64 \end{bmatrix}$$

$\rho(A:B) = 3; \ \rho(A) = 2$

$\rho(A:B) \neq \rho(A)$

$\therefore$ System of Equation is inconsistent.

3. Test for consistency and solve.

$5x + 3y + 7z = 4$

$3x + 26y + 2z = 9$

$7x + 2y + 10z = 5$

Sol:- $[A:B] = \begin{bmatrix} 5 & 3 & 7 & : & 4 \\ 3 & 26 & 2 & : & 9 \\ 7 & 2 & 10 & : & 5 \end{bmatrix}$ $\quad R_2 \to 5R_2 - 3R_1$, $\ R_1 \to 5R_3 - 7R_1$

$$\sim \begin{bmatrix} 5 & 3 & 7 & : & 4 \\ 15-15 & 130-9 & 10-21 & : & 45-12 \\ 35-35 & 10-21 & 50-49 & : & 25-28 \end{bmatrix} \sim \begin{bmatrix} 5 & 3 & 7 & : & 4 \\ 0 & 121 & -11 & : & 33 \\ 0 & -11 & 1 & : & -3 \end{bmatrix}$$

$R_2 - R_2/11$

$$\sim \begin{bmatrix} 5 & 3 & 7 & : & 4 \\ 0 & 11 & -1 & : & 3 \\ 0 & -11 & 1 & : & -3 \end{bmatrix} \quad R_3 \to R_3 + R_2$$

$$\longrightarrow \sim \begin{bmatrix} 5 & 3 & 7 & : & 4 \\ 0 & 11 & -1 & : & 3 \\ 0 & 0 & 0 & : & 0 \end{bmatrix} \quad R_1 \to R_1/5, \ R_2 \to R_2/11$$

$$\sim \begin{bmatrix} 5 & 3 & +1/5 & : & 4/5 \\ 0 & 11 & -1/11 & : & 3/11 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

$$\rho[A:B] = 2, \quad \rho[A] = 2$$

$$r = 2, \; n = 3$$

$$r < n, \; 2 < 3$$

$$\rho[A:B] = \rho[A] \text{ but } r < n$$

The system of equation has infinite number of solutions.

$$5x + 3y + 7z = 4$$

$$11y - z = 3$$

Let $z = K$

$$11y - K = 3$$

$$11y = 3 + K$$

$$y = \frac{3+K}{11}$$

$$5x + 3y + 7z = 4$$

$$5x = 4 - 3y - 7z$$

$$5x = 4 - 3\left(\frac{3+K}{11}\right) - 7(K)$$

$$x = \frac{4 - 3\left(\frac{3+K}{11}\right) - 7K}{5}$$

4. Find the value of λ & μ such that the system of equation

$$x + y + z = 6$$

$$x + 2y + 3z = 10$$

$x + 2y + \lambda z = \mu$ may have

i) Unique solution, ii) Infinite solution, iii) No solution.

sol:- $$[A:B] = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 1 & 2 & 3 & : & 10 \\ 1 & 2 & \lambda & : & \mu \end{bmatrix}$$ $R_2 \rightarrow R_2 - R_1$, $R_3 \rightarrow R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 1 & \lambda - 1 & : & \mu - 6 \end{bmatrix}$$ $R_3 \rightarrow R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 0 & \lambda - 1 - 2 & : & \mu - 6 - 4 \end{bmatrix}$$

i) To get unique solution, rank of A:B = Rank of A = r = n

i.e $\rho(A:B) = \rho(A) = n = 3$,

if $\lambda \neq 3$ & $\mu \neq 10$

ii) Infinite number of solution,

$\rho(A:B) = \rho(A) = r \quad r < n$

if $\lambda = 3$ & $\mu = 10$

$r = 2$ & $n = 3$

iii) No solution

$\rightarrow \rho(A:B) \neq \rho(A)$

it is possible if $\lambda = 3$ & $\mu \neq 10$

Gauss - Elimination Method:-

To solve system of linear equation

$$a_{11}x + b_{12}y + c_{13}z = d_{11}$$
$$a_{21}x + b_{22}y + c_{23}z = d_{22}$$
$$a_{31}x + b_{32}y + c_{33}z = d_{33}$$

We have the following methods

1. **Gauss Elimination method:-**

The augumented matrix [A:B] and the coefficient matrix (A) is reduced to upper triangular matrix by performing row transmissions and solution is obtained by back substitution.

2. **Gauss - Jordan method:-**

In the augumented matrix [A:B] the coefficient matrix A is reduced to diagonal matrix and the solution is obtained directly.

Solve the following Equation by

1) Gauss Elimination 2) Gauss jordan method

$$2x+y+3z=1$$
$$4x+4y+7z=1$$
$$2x+5y+9z=3$$

sol:- $[A:B] = \begin{bmatrix} 2 & 1 & 3 & : & 1 \\ 4 & 4 & 7 & : & 1 \\ 2 & 5 & 9 & : & 3 \end{bmatrix}$

Gauss Elimination method

$R_2 \longrightarrow R_2 - 2R_1$, $R_3 \longrightarrow R_3 - R_1$

$$\sim \begin{bmatrix} 2 & 1 & 3 & : & +1 \\ 0 & 2 & 1 & : & -1 \\ 0 & 4 & 6 & : & 2 \end{bmatrix} \quad R_3 \longrightarrow R_3/2$$

$$\sim \left[\begin{array}{ccc:c} 2 & 1 & 3 & 1 \\ 0 & 2 & 1 & -1 \\ 0 & 2 & 3 & 1 \end{array}\right] \quad R_3 \longrightarrow R_3 - R_2$$

$$\sim \left[\begin{array}{ccc:c} 2 & 1 & 3 & 1 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 2 & 2 \end{array}\right]$$

$\Rightarrow 2z = 2$

$\boxed{z = 1}$

$2y + z = -1$

$2y = -1 - 1$

$2y = -2$

$y = \frac{-2}{2}$

$\boxed{y = -1}$

$2x + y + 3z = 1$

$2x - 1 + 3 = 1$

$2x + 2 = 1$

$2x = 1 - 2$

$2x = -1$

$x = -\frac{1}{2}$

$\boxed{x = -0.5}$

Solve $3x+y+2z=3$

$2x-3y-z=-3$

$x+2y+z=4$

Sol:- $[A:B] = \left[\begin{array}{ccc:c} 3 & 1 & 2 & 3 \\ 2 & -3 & -1 & -3 \\ 1 & 2 & 1 & 4 \end{array}\right]$

$R_1 \leftrightarrow R_3$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & 1 & 4 \\ 2 & -3 & -1 & -3 \\ 3 & 1 & 2 & 3 \end{array}\right] \quad \begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - 3R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & 1 & 4 \\ 0 & -3-4 & -1-2 & -3-8 \\ 0 & 1-6 & 2-3 & 3-12 \end{array}\right] \sim \left[\begin{array}{ccc:c} 1 & 2 & 1 & 4 \\ 0 & -7 & -3 & -11 \\ 0 & -5 & -1 & -9 \end{array}\right]$$

$R_2 \rightarrow R_2/-1$

$R_3 \rightarrow R_3/-1$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & 1 & 4 \\ 0 & 7 & 3 & 11 \\ 0 & 5 & 1 & 9 \end{array}\right] R_3 \rightarrow 7R_3 - 5R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & 1 & 4 \\ 0 & 7 & 3 & 11 \\ 0 & 35-35 & 7-15 & 63-55 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & 1 & 4 \\ 0 & 7 & 3 & 11 \\ 0 & 0 & -8 & 8 \end{array}\right]$$

$\rightarrow x+2y+z=4$

$7y+3z=11$

$-8z=8 \rightarrow \boxed{z=-1}$

$7y+3(-1)=11$

$7y=11+3$

$y=\frac{14}{7}$

$\boxed{y=2}$

$\rightarrow x+2y+z=4$

$x+2(2)+(-1)=4$

$x+4-1=4$

$x=4-3 \rightarrow \boxed{x=1}$

$3x - y + 2z = 12$

$2x + y - z = 3$

sol:- $[A:B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 11 \\ 3 & -1 & 2 & 12 \\ 2 & 1 & -1 & 3 \end{array}\right]$ $R_1 \rightarrow R_2 - 3R_1$, $R_3 \rightarrow R_3 - 2R_1$,

$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 11 \\ 0 & -4 & -1 & -21 \\ 0 & -1 & -3 & -19 \end{array}\right]$ $R_2 \rightarrow -R_2$, $R_3 \rightarrow -R_3$

$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 11 \\ 0 & 4 & 1 & 21 \\ 0 & 1 & 3 & 19 \end{array}\right]$ $R_3 \rightarrow 4R_3 - R_2$

$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 11 \\ 0 & 4 & 1 & 21 \\ 0 & 0 & 11 & 55 \end{array}\right]$ $R_3 \rightarrow R_3/11$

$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 11 \\ 0 & 4 & 1 & 21 \\ 0 & 0 & 1 & 5 \end{array}\right]$ $R_1 \rightarrow R_1 - R_3$, $R_2 \rightarrow R_2 - R_3$

$\sim \left[\begin{array}{ccc:c} 1-0 & 1-0 & 1-1 & 11-5 \\ 0-0 & 4-0 & 1-1 & 21-5 \\ 0 & 0 & 1 & 5 \end{array}\right]$

$\sim \left[\begin{array}{ccc:c} 1 & 1 & 0 & 6 \\ 0 & 4 & 0 & 16 \\ 0 & 0 & 1 & 5 \end{array}\right]$ $R_1 \rightarrow 4R_1 - R_2$

$\sim \left[\begin{array}{ccc:c} 4-0 & 4-4 & 0-0 & 24-16 \\ 0 & 4 & 0 & 16 \\ 0 & 0 & 1 & 5 \end{array}\right]$

$\sim \left[\begin{array}{ccc:c} 4 & 0 & 0 & 8 \\ 0 & 4 & 0 & 16 \\ 0 & 0 & 1 & 5 \end{array}\right]$

$4x = 8 \rightarrow \boxed{x = 2}$

$4y = 16 \rightarrow \boxed{y = 4}$

$z = 5 \rightarrow \boxed{z = 5}$

Solve the system of equations by Gauss-Jordan

$x+y+z=10$

$2x-y+3z=19$

$x+2y+3z=22$

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 10 \\ 2 & -1 & 3 & 19 \\ 1 & 2 & 3 & 22 \end{array}\right] \quad \begin{array}{l} R_2 \longrightarrow R_2 - 2R_1 \\ R_3 \longrightarrow R_3 - R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 10 \\ 2-2 & -1-2 & 3-2 & 19-20 \\ 1-1 & 2-1 & 3-1 & 22-10 \end{array}\right] \sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 10 \\ 0 & -3 & 1 & -1 \\ 0 & 1 & 2 & 12 \end{array}\right]$$

$R_3 \longrightarrow 3R_3 + R_2$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 10 \\ 0 & -3 & 1 & -1 \\ 0+0 & 3+(-3) & 6+1 & 36+(-1) \end{array}\right]$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 10 \\ 0 & -3 & 1 & -1 \\ 0 & 0 & 7 & 35 \end{array}\right] \quad R_3 \longrightarrow R_3/7$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 10 \\ 0 & -3 & 1 & -1 \\ 0 & 0 & 1 & 5 \end{array}\right] \quad \begin{array}{l} R_1 \longrightarrow R_1 - R_3 \\ R_2 \longrightarrow R_2 - R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc:c} 1-0 & 1-0 & 1-1 & 10-5 \\ 0-0 & -3-0 & 1-1 & -1-5 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 0 & 5 \\ 0 & -3 & 0 & -6 \\ 0 & 1 & 1 & 5 \end{array}\right] \quad R_2 \longrightarrow R_2/-1$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 0 & 5 \\ 0 & 3 & 0 & 6 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

$R_1 \longrightarrow 3R_1 - R_2$

$$\sim \left[\begin{array}{ccc:c} 3-0 & 3-3 & 0-0 & 15-6 \\ 0 & 3 & 0 & 6 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc:c} 3 & 0 & 0 & 9 \\ 0 & 3 & 0 & 6 \\ 0 & 1 & 1 & 5 \end{array}\right]$$

$3x = 9$ $\qquad$ $3y = 6$ $\qquad$ $\boxed{z = 5}$

$\boxed{x = 3}$ $\qquad$ $\boxed{y = 2}$

Problems:-

1. Solve the system of equation by using Gauss

$$\left.\begin{aligned} 10x+y+z &= 12 \\ x+10y+z &= 12 \\ x+y+10z &= 12 \end{aligned}\right\} \rightarrow (1)$$

$10 > 1+1$
$10 > 1+1$
$10 > 1+1$

The system of equation is diagonally dominant

$10x+y+z = 12 \rightarrow (2) \Rightarrow x = \frac{1}{10}(12-y-z)$

$x+10y+z = 12 \rightarrow (3) \Rightarrow y = \frac{1}{10}(12-x-z)$

$x+y+10z = 12 \rightarrow (4) \Rightarrow z = \frac{1}{10}(12-x-y)$

Let initial conditions be

$x_0 = 0,\ y_0 = 0,\ z_0 = 0$

I iteration:-

$\rightarrow x^{(1)} = \frac{1}{10}(12-y_0-z_0)$

$= \frac{1}{10}(12-0-0) = \frac{12}{10}$

$x^{(1)} = 1.2$

$\rightarrow y^{(1)} = \frac{1}{10}(12-x^{(1)}-z^{0})$

$= \frac{1}{10}(12-1.2-0) = \frac{10.8}{10}$

$y^{(1)} = 1.08$

$\rightarrow z^{(1)} = \frac{1}{10}(12-x^{(1)}-y^{(1)})$

$= \frac{1}{10}(12-1.2-1.08)$

$z^{(1)} = 0.972$

II iteration:-

$\rightarrow (x^{(2)} = \frac{1}{10}(12-y^{(1)}-z^{(1)}) = 0.9948$

$y^{(2)} = \frac{1}{10}(12-x^{(2)}-z^{(1)}) = 1.0033$

$z^{(2)} = \frac{1}{10}(12-x^{(2)}-y^{(2)}) = 1.00019 \simeq 1.0002$

$\underline{\text{III iteration:-}}$

$$\Rightarrow x^{(3)} = \frac{1}{10}\left(12 - z^{(2)} - y^{(2)}\right) = 0.9996$$

$$\Rightarrow y^{(3)} = \frac{1}{10}\left(12 - x^{(3)} - z^{(2)}\right) = 1.0000$$

$$\Rightarrow z^{(3)} = \frac{1}{10}\left(12 - x^{(3)} - y^{(3)}\right) = 1.0000$$

$\underline{\text{IV iteration:-}}$

$$\Rightarrow x^{(4)} = \frac{1}{10}\left(12 - z^{(3)} - y^{(3)}\right) = 0.9999$$

$$\Rightarrow y^{(4)} = \frac{1}{10}\left(12 - x^{(4)} - z^{(3)}\right) = 0.9999$$

$$\Rightarrow z^{(4)} = \frac{1}{10}\left(12 - x^{(4)} - y^{(4)}\right) = 1.0000$$

$\underline{\text{V iteration:-}}$

$$\Rightarrow x^{(5)} = \frac{1}{10}\left(12 - z^{(4)} - y^{(4)}\right) = 1.0000$$

$$\cancel{x}\, y^{(5)} = \frac{1}{10}\left(12 - x^{(5)} - z^{(4)}\right) = 0.9999$$

$$z^{(5)} = \frac{1}{10}\left(12 - x^{(5)} - y^{(5)}\right) = 0.9999$$

$\underline{\text{VI iteration:-}}$

$$\Rightarrow x^{(6)} = \frac{1}{10}\left(12 - z^{(5)} - y^{(5)}\right) = 1$$

$$y^{(6)} = \frac{1}{10}\left(12 - x^{(6)} - z^{(5)}\right) = 1$$

$$z^{(6)} = \frac{1}{10}\left(12 - x^{(6)} - y^{(6)}\right) = 0.9999$$

$\underline{\text{VII iteration:-}}$

$$\Rightarrow x^{(7)} = \frac{1}{10}\left(12 - z^{(6)} - y^{(6)}\right) = 1$$

$$y^{(7)} = \frac{1}{10}\left(12 - x^{(7)} - z^{(6)}\right) = 1$$

$$z^{(7)} = \frac{1}{10}\left(12 - x^{(7)} - y^{(7)}\right) = 1$$

2. Solve the system of equation using Gauss seidal method.

Sol:- $x+y+54z=110$ Take initial approximations of (2, 3, 1.5)

$27x+6y-z=85$

$6x+15y+2z=72$

$1 < 1+54$

$6 < 27+|-1|$

$2 < 6+15$

→ The given system of equation is not diagonally dominant.

⇒ Rearrange the system of equations to get diagonally dominant.

$$27x+6y-z=85 \Rightarrow x=\frac{1}{27}(85-6y+z)$$

$$6x+15y+2z=72 \Rightarrow y=\frac{1}{15}(72-6x-2z)$$

$$x+y+54z=110 \Rightarrow z=\frac{1}{54}(110-x-y)$$

I iteration:-

$$x^{(1)}=\frac{1}{27}(85-6y_0-z_0)=2.4853704$$

$$y^{(1)}=\frac{1}{15}(72-6x^{(1)}-2z_0)=3.58519$$

$$z^{(1)}=\frac{1}{54}(110-x^{(1)}-y^{(1)})=1.9236$$

II - iteration:-

$$x^{(2)}=\frac{1}{27}(85-6y^{(1)}-z^{(1)})=2.42269$$

$$y^{(2)}=\frac{1}{15}(72-6x^{(2)}-2z^{(1)})=3.57444$$

$$z^{(2)}=\frac{1}{54}(110-x^{(2)}-y^{(2)})=1.92598$$

III - iteration:-

$$x^{(3)}=\frac{1}{27}(85-6y^{(2)}-z^{(2)})=2.42516$$

$$y^{(3)}=\cancel{\frac{1}{54}}\ \frac{1}{15}(72-6x^{(3)}-2z^{(2)})=3.57314$$

$$z^{(3)}=\frac{1}{54}(110-x^{(3)}-y^{(3)})=1.92596$$

IV - iterations:-

$$x^{(4)} = \frac{1}{27}(85 - 6y^{(3)} - z^{(3)}) = 2.42545$$

$$y^{(4)} = \frac{1}{15}(72 - 6x^{(4)} - 2z^{(3)}) = 3.57303$$

$$z^{(4)} = \frac{1}{54}(110 - x^{(4)} - y^{(4)}) = 1.92595$$

3. Solve the system of equation using Gauss seidal method.

$2x - 3y + 20z = 25$ $\quad 2 < |-3| + 20$

$20x + y - 2z = 17$ $\quad 1 < 20 + |-2|$

$3x + 20y - z = -18$ $\quad -1 < 3 + 20$

Taking initial approximation as $(0,0,0)$

The system of equation is not diagonally dominant. Rearrange to get diagonally dominant.

$$20x + y - 2z = 17$$
$$3x + 20y - z = -18$$
$$2x - 3y + 20z = 25$$

$\Rightarrow x = 1.$
$y = -1$
$z = 1.$

$$\Rightarrow x = \frac{1}{20}(17 - y + 2z)$$

$$y = \frac{1}{20}(-18 - 3x + z)$$

$$z = \frac{1}{20}(25 + 3y - 2x)$$

I - iteration:-

$$x^{(1)} = \frac{1}{20}(17 - y_0 + 2z_0) = 0.85000$$

$$y^{(1)} = \frac{1}{20}(-18 - 3x^{(1)} + z_0) = -1.02750$$

$$z^{(1)} = \frac{1}{20}(25 + 3y^{(1)} - 2x^{(1)}) = 1.01088$$

4. $83x + 11y - 4z = 95$

$3x + 8y + 29z = 71$

$7x + 52y + 13z = 104$

Not diagonally dominant, Rearrange to get it

$83x + 11y - 4z = 95$

$7x + 52y + 13z = 104$

$3x + 8y + 29z = 71$

$x = 1.29082$

$y = 0.63265$

$z = 4.77433$

$\Rightarrow x = \frac{1}{83}(95 - 11y + 4z)$

$\Rightarrow y = \frac{1}{52}(104 - 7x - 13z)$

$\Rightarrow z = \frac{1}{13}(71 - 3x - 8y)$

I - iteration:- $x^{(1)} = \frac{1}{83}(95 - 11y^{0} + 4z^{0}) = 1.14458$

$y^{(1)} = \frac{1}{52}(104 - 7x^{(1)} - 13z^{0}) = 1.84592$

$z^{(1)} = \frac{1}{13}(71 - 3x^{(1)} - 8y^{(1)}) = 4.06145$

II - iteration:- $x^{(2)} = \frac{1}{83}(95 - 11y^{(1)} + 4z^{(1)}) = 1.09567$

$y^{(2)} = \frac{1}{52}(104 - 7x^{(2)} - 13z^{(1)}) = 0.83714$

$z^{(2)} = \frac{1}{13}(71 - 3x^{(2)} - 8y^{(2)}) = 4.69353$

III - iteration:- $x^{(3)} = \frac{1}{83}(95 - 11y^{(2)} + 4z^{(2)}) = 1.25983$

$y^{(2)} = \frac{1}{52}(104 - 7x^{(3)} - 13z^{(2)}) = 0.65703$

$z^{(2)} = \frac{1}{13}(71 - 3x^{(3)} - 8y^{(3)}) = 4.76649$

II - iteration:-

$$x^{(2)} = \frac{1}{20}\left(17 - y^{(1)} + 2z^{(1)}\right) = 1.28721$$

$$y^{(2)} = \frac{1}{20}\left(-18 - 3x^{(2)} + z^{(1)}\right) = 0.63510$$

$$z^{(2)} = \frac{1}{20}\left(25 + 3y^{(2)} - 2x^{(2)} = 4.77366\right.$$