MODULE-3

MODULE - 03

LINEAR DIFFERENTIAL EQUATIONS

An equation of the form $\frac{dy}{dx} + P(x)y = Q(x) \longrightarrow (1)$ is called linear differential equation.

The solution of equation (1) is given by

$$I.F = e^{\int P(x)dx}$$

Complete solution is given by

$$y \cdot (I.F) = \int IF \; Q(x)\, dx + C$$

where $P(x)$ & $Q(x)$ is function of x

1. solve $\frac{dy}{dx} + \frac{y}{x} = x$

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$P(x) = \frac{1}{x} \qquad Q(x) = x$$

$$I.F = e^{\int P(x)dx} = e^{\int \frac{1}{x}dx} = e^{\log x} = x$$

Complete solution is given by

$$y \cdot (I.F) = \int IF \; Q(x)\, dx + c$$

$$y(x) = \int x\,(x)\, dx + c$$

$$xy = \int x^2 dx + c$$

$$xy = \frac{x^3}{3} + c$$

$$\boxed{y = \frac{x^2}{3} + \frac{c}{x}}$$

2. solve $\frac{dy}{dx} + \tan x \cdot y = \sec^2 x$

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$P(x) = \tan x \qquad Q(x) = \sec^2 x$

$$I.F = e^{\int p(x)dx} = e^{\int \tan x\, dx} = e^{\log \sec x} = \sec x$$

Complete solution is given by $I.F(y) = \int I.F\, Q(x)dx + c$

$$y(\sec x) = \int \sec x \cdot \sec^2 x + c$$

$$y(\sec x) = \int \sec^2 x\, dx + c$$

$$y(\sec x) = \tan x + c$$

$$\boxed{y = \frac{\tan x}{\sec x} + \frac{c}{\sec x}}$$

BERNAULLI DIFFERENTIAL EQUATION [REDUCIABLE TO LINEAR DIFFERENTIAL EQUATION]

If given differential equation is of the form $\frac{dy}{dx} + p(x,y) = y^n \longrightarrow (1)$ is called bernaulli differential equation. The above equation can be reducible to linear differential equation by taking substitution then we solve the resulting differential equation as we solve linear differential equation.

1. Solve $x\frac{dy}{dx} + y = x^3y^6$

$\div$ by xy^6

$$\frac{1}{y^6}\frac{dy}{dx} + \frac{1}{xy^5} = x^2 \longrightarrow (1)$$

put $\frac{1}{y^5} = t$

$$-\frac{5}{y^6}\frac{dy}{dx} = \frac{dt}{dx} \quad x^{ly} \text{ by } -\frac{1}{5}$$

$$\frac{1}{y^6}\frac{dy}{dx} = -\frac{1}{5}\frac{dt}{dx}$$

$\therefore$ 1 becomes

$$-\frac{1}{5}\frac{dt}{dx}+\frac{1}{x}(t)=x^2 \qquad x^{ly}\ by(-5)$$

$$\frac{dt}{dx}-\frac{5}{x}t=-5x^2$$

$$\left(\frac{dy}{dx}+P(x)y=Q(x)\right)$$

$$P(x)=-\frac{5}{x} \qquad Q(x)=-5x^2$$

$$I.F=e^{\int P(x)dx}=e^{\int -\frac{5}{x}dx}=e^{-5\log x}=\cancel{e}^{\cancel{\log} x^{-5}}=x^{-5}$$

C.S is $t(IF)=\int IF\ Q(x)\,dx+c$

$$t(x^{-5})=\int x^{-5}(-5x^2)\,dx+c$$

$$t(x^{-5})=-5\int x^{-3}\,dx+c$$

$$t(x^{-5})=+5.\frac{x^{-2}}{+2}+c$$

$$\boxed{\frac{1}{y^5x^5}=\frac{5}{2x^2}+c}$$

2. $$\frac{dy}{dx}-y\tan x=\frac{\sin x\cos^2x}{y^2} \qquad x^{ly}\ by\ y^2$$

$$y^2\frac{dy}{dx}-y^3\tan x=\sin x\cos^2x \quad \text{——} (1)$$

put $y^3=t$

$$3y^2\frac{dy}{dx}=\frac{dt}{dx}$$

$$y^2\frac{dy}{dx}=\frac{1}{3}\frac{dt}{dx}$$

$\therefore$ (1) becomes

$$\frac{1}{3}\frac{dt}{dx}-t\tan x=\sin x\cos^2x \qquad x^{ly}\ by\ 3$$

$$\frac{dt}{dx}-3t\tan x=3\sin x\cos^2x$$

$$\frac{dt}{dx} - 3\tan x\, t = 3\sin x\cos^2 x$$

$$P(x) = -3\tan x \qquad Q(x) = 3\sin x\cos^2 x$$

$$I.F = e^{\int P(x)dx} = e^{\int -3\tan x\, dx} = e^{-3.\log \sec x}$$

$$= e^{\log \sec x^{-3}} = \frac{1}{\sec^3 x}$$

C.S is $t\,(IF) = \int IF\; Q(x)\, dx + c$

$$t\left(\frac{1}{\sec^3 x}\right) = \int \frac{1}{\sec^3 x}\, 3\sin x\cos^2 x\, dx + c$$

$$t\left(\frac{1}{\sec^3 x}\right) = 3\int \sin x\cos^5 x\, dx + c$$

put $\cos x = z$

$-\sin x\, dx = dz$

$\sin x\, dx = -dz$

$$t\left(\frac{1}{\sec^3 x}\right) = 3\int z^5(-dz) + c$$

$$t\left(\frac{1}{\sec^3 x}\right) = -3\left(\frac{z^6}{6}\right) + c$$

$$\boxed{\frac{y^3}{\sec^3 x} = \frac{-1}{2}\cos^6 x + C}$$

3. $xy(1+xy^2)\frac{dy}{dx} = 1$

$$\frac{dx}{dy} = xy(1+xy^2)$$

$$\frac{dx}{dy} = xy + x^2y^3$$

$$\frac{dx}{dy} - xy = x^2y^3 \quad \div \text{ by } x^2$$

$$\frac{1}{x^2}\frac{dx}{dy} - \frac{y}{x} = y^3$$

put $\frac{1}{x} = t$

$$-\frac{1}{x^2}\frac{dx}{dy} = \frac{dt}{dy}$$

$$-\frac{dt}{dy} - yt = y^3 \quad \times^{ly} \text{ by } (-1)$$

$$\frac{dt}{dy} + yt = -y^3$$

$$P(y) = y \quad Q(y) = -y^3$$

$$IF = e^{\int P(y)dy} = e^{\int y\,dy} = e^{y^2/2}$$

C.S is $t(IF) = \int IF\, Q(y)dy + c$

$$t(e^{y^2/2}) = \int e^{y^2/2}(-y^3)\,dy + c$$

$$t(e^{y^2/2}) = -\int e^{y^2/2}\, y^2\, y \,.\, dy + c$$

put $y^2 = z$

$$2y\,dy = dz$$

$$y\,dy = \frac{dz}{2}$$

$$t(e^{y^2/2}) = -\int e^{z/2}\, z\, \frac{dz}{2} + c$$

$$t(e^{y^2/2}) = -\frac{1}{2}\left[z\, 2e^{z/2} - \int 2e^{z/2}\frac{d(z)}{dz}\, dz\right] + c$$

$$t(e^{y^2/2}) = -\frac{1}{2}\left[2ze^{z/2} - 2\int e^{z/2}(1)\,dz\right] + c$$

$$t(e^{y^2/2}) = -\left[ze^{z/2} - 2e^{z/2}\right] + c$$

$$\boxed{\frac{e^{y^2/2}}{x} = -y^2 e^{y^2/2} + 2e^{y^2/2} + c}$$

4. solve $r\sin\theta - \cos\theta \frac{dr}{d\theta} = r^2$

$$r\sin\theta - r^2 = \cos\theta \frac{dr}{d\theta}$$

$$\cos\theta \frac{dr}{d\theta} - r\sin\theta = -r^2 \quad \div r^2\cos\theta$$

$$\frac{1}{r^2}\frac{dr}{d\theta} - \frac{\tan\theta}{r} = -\frac{1}{\cos\theta} \longrightarrow (1)$$

put $\frac{1}{r} = t$

$$-\frac{1}{r^2}\frac{dr}{d\theta} = \frac{dt}{d\theta}$$

$\therefore$ (1) becomes

$$-\frac{dt}{d\theta} - \tan\theta\, t = -\sec\theta \quad \text{x}^{ly} \text{ by} (-1)$$

$$\frac{dt}{d\theta} + \tan\theta\, t = \sec\theta$$

$$P(\theta) = \tan\theta \qquad Q(\theta) = \sec\theta$$

$$IF = e^{\int P(\theta)d\theta} = e^{\int \tan\theta\, d\theta} = e^{\log\sec\theta} = \sec\theta$$

C.S is given by

$$t(I.F) = \int IF\, Q(\theta)\, d\theta + c$$

$$t(\sec\theta) = \int \sec\theta(\sec\theta)d\theta + c$$

$$t(\sec\theta) = \int \sec^2\theta\, d\theta + c$$

$$t(\sec\theta) = \tan\theta + c$$

$$\boxed{\frac{\sec\theta}{r} = \tan\theta + c}$$

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5. $\frac{dy}{dx} + xy = xy^3$

$\div$ by y^3

$\frac{1}{y^3}\frac{dy}{dx} + \frac{x}{y^2} = x$ —— ①

$\frac{1}{y^3}\frac{dy}{dx} + \frac{x}{y^2} = x$

put $\frac{1}{y^2} = t$

$-\frac{2}{y^3}\frac{dy}{dx} = \frac{dt}{dx}$ $\times$ ly $(-\frac{1}{2})$

$\frac{1}{y^3}\frac{dy}{dx} = -\frac{1}{2}\frac{dt}{dx}$

$\therefore$ ① becomes

$-\frac{1}{2}\frac{dt}{dx} + xt = x$ $\times$ ly (-2)

$\frac{dt}{dx} - 2xt = -2x$

$\left[\frac{dy}{dx} + P(x)y = Q(x)\right]$ $P(x) = -2x$ $Q(x) = -2x$

$I.F = e^{\int P(x)dx} = e^{\int -2x\,dx} = e^{-2\left(\frac{x^2}{2}\right)} = e^{-x^2}$

$y(IF) = \int IF\, Q(x)dx + C$

$y(e^{-x^2}) = \int e^{-x^2}(-2x)dx + C$ put $x^2 = z$

$= -2\int e^{-x^2} x\,dx + C$ $2x\,dx = dz$

$= -2\int e^{-z}\frac{dz}{2} + C$ $x\,dx = \frac{dz}{2}$

$= -\frac{2}{2}\int e^{-z}dz + C$

$= -e^{-z} + C$

$\boxed{\frac{1}{y^2}e^{-x^2/2} = -\frac{1}{2}e^{-x^2} + C}$

6. $\frac{dy}{dx} + y\tan x = y^3 \sec x$

$\div$ by y^3

$$\frac{1}{y^3}\frac{dy}{dx} + \frac{\tan x}{y^2} = \sec x \quad \text{——} ①$$

put $\frac{1}{y^2} = t$

$$\frac{-2}{y^3}\frac{dy}{dx} = \frac{dt}{dx} \quad \div(-2)$$

$$\frac{1}{y^3}\frac{dy}{dx} = -\frac{1}{2}\frac{dt}{dx}$$

$$① \Rightarrow -\frac{1}{2}\frac{dt}{dx} + \tan x \, t = \sec x$$

$\times$ by (-2)

$$\frac{dt}{dx} - 2t\tan x = -2\sec x$$

$$p(x) = -2\tan x \quad Q(x) = -2\sec x$$

$$I.F = e^{\int p(x)dx}$$

$$= e^{\int -2\tan x\,dx} = e^{-2\log \sec x}$$

$$= \cancel{e}^{\log \sec x^{-2}} = \frac{1}{\sec^2 x}$$

$$y(IF) = \int IF\, Q(x)\,dx + c$$

$$y\left(\frac{1}{\sec^2 x}\right) = \int \frac{1}{\sec^{\cancel{2}} x}(-2\cancel{\sec x})\,dx + c$$

$$\frac{y}{\sec^2 x} = -2\int \cos x\,dx + c$$

$$\boxed{\frac{y^2}{\sec^2 x} = -2\sin x + c}$$

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EXACT DIFFERENTIAL EQUATION

An equation of the form $M(x,y)dx + N(x,y)dy = 0 \rightarrow (1)$ is called exact differential equation iff $\frac{\partial m}{\partial y} = \frac{\partial n}{\partial x}$

complete solution is given by $\int m(x,y)dx + \int n(y)dy = C$

1. Solve $(2x+y+1)dx + (x+2y+1)dy = 0$

$m = 2x+y+1$ $n = x+2y+1$

D P wrt y D P wrt x

$\frac{\partial m}{\partial y} = 0+1+0$ $\frac{\partial n}{\partial x} = 1+0+0$

$$\therefore \frac{\partial m}{\partial y} = \frac{\partial n}{\partial x}$$

$\therefore$ The given equation is exact differential equation

complete solution is given by

$$\int m(x,y)dx + \int n(y)dy = c$$

$$\int(2x+y+1)dx + \int(2y+1)dy = c$$

$$2\int x\,dx + y\int dx + \int dx + 2\int y\,dy + \int dy = c$$

$$\cancel{2}\left(\frac{x^2}{\cancel{2}} + yx + x\right) + \cancel{2}\left(\frac{y^2}{\cancel{2}} + y\right) = c$$

$$x^2 + xy + x + y^2 + y = c$$

2. Solve $\frac{dy}{dx} + \frac{x+3y-4}{3x+9y-2} = 0$

$$\frac{dy(3x+9y-2) + (x+3y-4)dx}{dx(3x+9y-2)} = 0$$

$$(x+3y-4)dx+(3x+9y-2)dy=0 \longrightarrow (1)$$

$m = x+3y-4$ $\qquad$ $n = 3x+9y-2$

D.P. wrt y $\qquad$ D.P wrt x

$\frac{\partial m}{\partial y} = 3$ $\qquad$ $\frac{\partial n}{\partial x} = 3$

$\therefore \frac{\partial m}{\partial y} = \frac{\partial n}{\partial x}$ This is exact differential equation

Complete solution is $\int m(x,y)\,dx + \int n(y)\,dy = C$

$$\int (x+3y-4)\,dx + \int (9y-2)\,dy = C$$

$$\int x\,dx + 3y\int dx - 4\int dx + 9\int y\,dy - 2\int dy = C$$

$$\frac{x^2}{2} + 3yx - 4x + 9\frac{y^2}{2} - 2y = C$$

3. $(y^3-3x^2y)\,dx - (x^3-3xy^2)\,dy = 0$

$m = y^3 - 3x^2y$ $\qquad$ $n = -x^3+3xy^2$

D.P wrt y $\qquad$ D.P wrt x

$\frac{\partial m}{\partial y} = 3y^2 - 3x^2(1)$ $\qquad$ $\frac{\partial n}{\partial x} = -3x^2+3y^2$

$n(y) = 0$

$\therefore \frac{\partial m}{\partial y} = \frac{\partial n}{\partial x}$

This is exact differential equatio

$$\int m(x,y)\,dx + \int n(y)\,dy = C$$

$$\int (y^3-3x^2y)\,dx + \int 0 \cdot dy = C$$

$$\int y^3\,dx - \int 3x^2y\,dx = C$$

$$y^3x - \cancel{3}y\frac{x^3}{\cancel{3}} = C$$

$$xy^3 - x^3y = C$$

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4. solve $(1+e^{x/y})\,dx + e^{x/y}\left(1-\frac{x}{y}\right)dy = 0$

$m = (1+e^{x/y})$ $\qquad$ $n = e^{x/y}\left(1-\frac{x}{y}\right)$

D.p w.r.t y $\qquad$ D.p w.r.t x

$\frac{\partial m}{\partial y} = 0 + e^{x/y}\,x\left(-\frac{1}{y^2}\right)$ $\qquad$ $\frac{\partial n}{\partial x} = e^{x/y}\left(\frac{1}{y}\right) - e^{x/y}\left(\frac{1}{y}\right) + \frac{x}{y}e^{x/y}\left(\frac{1}{y}\right)$

$\frac{\partial m}{\partial y} = -\frac{x e^{x/y}}{y^2}$ $\qquad$ $= \cancel{\frac{e^{x/y}}{y}} - \cancel{\frac{1}{y}e^{x/y}} - \frac{x e^{x/y}}{y^2}$

$\frac{\partial n}{\partial x} = -\frac{x e^{x/y}}{y^2}$

$\frac{\partial m}{\partial y} = \frac{\partial n}{\partial x}$ $\therefore$ This is exact D.E

$\int m(x,y)\,dx + \int n(y)\,dy = C$

$\int (1+e^{x/y})\,dx + \int 0\,dy = C$

$\int 1\,dx + \int e^{x/y}\,dx = C$

$x + \frac{e^{x/y}}{1/y} = C$ $\qquad$ $\boxed{x + y e^{x/y} = C}$

5. solve $y\sin 2x\,dx - (1-y^2+\cos^2 x)\,dy = 0$

$m = y\sin 2x$ $\qquad$ $n = -1+y^2-\cos^2 x$

$\frac{\partial m}{\partial y} = \sin 2x$ $\qquad$ $\frac{\partial n}{\partial x} = -2\cos x(-\sin 2x)$

$\frac{\partial n}{\partial x} = 2\cos x\sin x = \sin 2x$

$\therefore \frac{\partial m}{\partial y} = \frac{\partial n}{\partial x}$

This is exact differential equation

C.S is given by $\int m(x,y)\,dx + \int n(y)\,dy = C$

$\int (y \sin 2x)\,dx + \int (y^2-1)\,dy = C$

$y\left(-\frac{\cos 2x}{2}\right) + \frac{y^3}{3} - y = C$

$$\boxed{-\frac{y}{2}\cos 2x + \frac{y^3}{3} - y = C}$$

6. $(x^2+y)\,dx + (y^3+x)\,dy = 0$

$m = x^2+y$ $\qquad$ $n = y^3+x$

d p wrt y $\qquad$ d p wrt x

$\frac{\partial m}{\partial y} = 1$ $\qquad$ $\frac{\partial n}{\partial x} = 1$

$\therefore \frac{\partial m}{\partial y} = \frac{\partial n}{\partial x}$ This is exact D.E

$\int m(x,y)\,dx + \int n(y)\,dy = C$

$\int (x^2+y)\,dx + \int y^3\,dy = C$

$$\boxed{\frac{x^3}{3} + xy + \frac{y^4}{4} = C}$$

7. $(3x^2+6xy^2)\,dx + (6x^2y+4y^3)\,dy = 0$

$m = 3x^2+6xy^2$ $\qquad$ $n = 6x^2y+4y^3$

$\frac{\partial m}{\partial y} = 0 + 6x(2y)$ $\qquad$ $\frac{\partial n}{\partial x} = 6y(2x) + 0$

$= 12xy$ $\qquad$ $= 12xy$

$\therefore \frac{\partial m}{\partial y} = \frac{\partial n}{\partial x}$

This is exact differential equatio

$\int m(x,y)\,dx + \int n(y)\,dy = C$

$\int (3x^2+6xy^2)\,dx + \int (4y^3)\,dy = C$

$3\int x^2\,dx + 6y^2\int x\,dx + 4\int y^3\,dy = C$

$\cancel{3}\cdot\frac{x^3}{\cancel{3}} + \overset{3}{\cancel{6}}y^2\left(\frac{x^2}{\cancel{2}}\right) + \cancel{4}\left(\frac{y^4}{\cancel{4}}\right) = C$

$$\boxed{x^3 + 3y^2x^2 + y^4 = C}$$

Equations Reduciable To Exact Differential Equations

If the given differential equation is of the form $m\,dx + n\,dy = 0$, and $\frac{\partial m}{\partial y} \neq \frac{\partial n}{\partial x}$ then to reduce equation as exact differential equation. we followe the following procedure

Case-1; If $\frac{\partial m}{\partial y} - \frac{\partial n}{\partial x} \simeq M$ then find $\frac{1}{M}\left(\frac{\partial m}{\partial y} - \frac{\partial n}{\partial x}\right) = g(y$

then $IF = e^{-\int g(y)\,dy}$, multiply this IF to equati it becomes exact differential equation and w following the procedure of solving exact differential equation.

Case-2; If $\frac{\partial m}{\partial y} - \frac{\partial n}{\partial x} \simeq N$ then find $\frac{1}{n}\left(\frac{\partial m}{\partial y} - \frac{\partial n}{\partial x}\right) = f($

then $IF = e^{\int f(x)\,dx}$, multiplying this IF to given equation, it becomes exact and we follow th procedure of solving exact D.E

1. Solve $(4xy+3y^2-x)\,dx + x(x+2y)\,dy = 0$.

$m = 4xy + 3y^2 - x$ $\qquad$ $n = x^2 + 2xy$

$\frac{\partial m}{\partial y} = 4x + 6y - 0$ $\qquad$ $\frac{\partial n}{\partial x} = 2x + 2y$

$\frac{\partial m}{\partial y} \neq \frac{\partial n}{\partial x}$ This is not exact D.E

Consider

$$\frac{\partial m}{\partial y} - \frac{\partial n}{\partial x} = 4x + 6y - (2x+2y)$$

$$= 2x + 4y = 2(x+2y) \simeq N$$

$$\frac{1}{N}\left(\frac{\partial m}{\partial y} - \frac{\partial n}{\partial x}\right) = \frac{1}{x\cancel{(x+2y)}} 2\cancel{(x+2y)} = \frac{2}{x} = f(x)$$

$$I.F = e^{\int p(x)dx} = e^{\int \frac{2}{x} dx} = e^{\log x^2} = x^2$$

x^{ly} by x^2

$$x^2(4xy + 3y^2 - x)\, dx + x^3(x+2y)\, dy = 0$$

$m = x^2(4xy + 3y^2 - x)$ $\quad$ $n = x^3(x+2y)$

$m = 4x^3y + 3x^2y^2 - x^3$ $\quad$ $n = x^4 + 2x^3y$

$\frac{\partial m}{\partial y} = 4x^3 + 6x^2y - 0$ $\quad$ $\frac{\partial n}{\partial x} = 4x^3 + 6x^2y$

$\frac{\partial m}{\partial y} = \frac{\partial n}{\partial x}$ This is exact D. E

C. S is given by $\int m(x.y)dx + \int n(y)\, dy = C$

$$\int (4x^3y + 3x^2y^2 - x^3)\, dx + \int 0\, dy = C$$

$$\cancel{4}\frac{x^4}{\cancel{4}}(y) + \cancel{3}y^2\frac{x^3}{\cancel{3}} - \frac{x^4}{4} = C$$

$$\boxed{x^4y + x^3y^2 - \frac{x^4}{4} = C}$$

2. Solve $(x^2+y^2+x)\,dx + xy\,dy = 0$.

$m = x^2+y^2+x$ $\qquad$ $n = xy$

$\frac{\partial m}{\partial y} = 2y$ $\qquad$ $\frac{\partial n}{\partial x} = y$

$\frac{\partial m}{\partial y} \neq \frac{\partial n}{\partial x}$ This is not exact D.E

Consider,

$\frac{\partial m}{\partial y} - \frac{\partial n}{\partial x} = (2y-y) = y \simeq N$

$\frac{1}{N}\left(\frac{\partial m}{\partial y} - \frac{\partial n}{\partial x}\right) = \frac{1}{x\cancel{y}}(\cancel{y}) = \frac{1}{x} = f(x)$

$I.F = e^{\int f(x)\,dx} = e^{\int \frac{1}{x}\,dx} = \cancel{e}^{\cancel{\log} x} = x$

x^ly x in equation

$x(x^2+y^2+x)\,dx + x(xy)\,dy = 0$

$m = x^3+xy^2+x^2$ $\qquad$ $n = x^2y$ $\qquad$ $n(y) = 0$

$\frac{\partial m}{\partial y} = 2xy$ $\qquad$ $\frac{\partial n}{\partial x} = y2x = 2xy$

$\frac{\partial m}{\partial y} = \frac{\partial n}{\partial x}$ This is exact D. E

C. S is given by $\int m(x,y)\,dx + \int n(y)\,dy = C$

$\int (x^3+xy^2+x^2)\,dx + \int 0.\,dy = C$

$\int x^3\,dx + \int xy^2\,dx + \int x^2\,dx = C$

$$\boxed{\frac{x^4}{4} + y^2.\frac{x^2}{2} + \frac{x^3}{3} = C}$$

3. $(x^2+y^3+6x)\,dx + (y^2x)\,dy = 0$

$m = x^2+y^3+6x \qquad n = xy^2$

$\frac{\partial m}{\partial y} = 3y^2 \qquad \frac{\partial n}{\partial x} = y^2$

$\frac{\partial m}{\partial y} \neq \frac{\partial n}{\partial x}$ ∴ This is not exact D.E

$\frac{\partial m}{\partial y} - \frac{\partial n}{\partial x} = 3y^2 - y^2 = 2y^2 \simeq N$

$\frac{1}{N}\left(\frac{\partial m}{\partial y} - \frac{\partial n}{\partial x}\right) = \frac{1}{xy^2}(2y^2) = \frac{2}{x} \simeq f(x)$

$I.F = e^{\int p(x)dx} = e^{\int \frac{2}{x}dx} = e^{2\log x} = e^{\log x^2} = x^2$

$\times$ly x^2 in eq^n 1

$x^2(x^2+y^3+6x)\,dx + x^2(y^2x)\,dy = 0$

$(x^4+x^2y^3+6x^3)\,dx + (x^3y^2)\,dy = 0$

$m = x^4+x^2y^3+6x^3 \qquad n = x^3y^2$

$\frac{\partial m}{\partial y} = 3x^2y^2 \qquad \frac{\partial n}{\partial x} = 3x^2y^2 \qquad n(y) = 0$

$\frac{\partial m}{\partial y} = \frac{\partial n}{\partial x}$ ∴ This is exact D.E

C.S is $\int m(x,y)\,dx + \int n(y)\,dy = C$

$\int (x^4+x^2y^3+6x^3)\,dx + \int 0\,dy = C$

$\frac{x^5}{5} + y^3\frac{x^3}{3} + 6\frac{x^4}{4} = C$

$$\boxed{\frac{x^5}{5} + y^3\frac{x^3}{3} + \frac{3x^2}{2} = C}$$

4. solve $y(2x-y+1)dx + x(3x-4y+3)dy = 0$ 10

$m = 2xy - y^2 + y$ $\quad$ $n = 3x^2 - 4xy + 3x$

$\frac{\partial m}{\partial y} = 2x - 2y + 1$ $\quad$ $\frac{\partial n}{\partial x} = 6x - 4y + 3$

$\frac{\partial m}{\partial y} \neq \frac{\partial n}{\partial x}$ this is not exact D.E

$\frac{\partial m}{\partial y} - \frac{\partial n}{\partial x} = (2x-2y+1) - (6x-4y+3)$

$= 2x - 2y + 1 - 6x + 4y - 3$

$= -4x + 2y - 2$

$= 2(-2x + y - 1)$

$= -2(2x - y + 1) \simeq M$

$\frac{1}{M}\left(\frac{\partial m}{\partial y} - \frac{\partial n}{\partial x}\right) = \frac{1}{y\cancel{(2x-y+1)}} - 2\cancel{(2x-y+1)} = -\frac{2}{y} = g(y)$

$I.F = e^{-\int g(y)dy} = e^{-\int -\frac{2}{y} dy} = e^{\int \frac{2}{y} dy}$

$= e^{2\log y} = \cancel{e}^{\cancel{\log} y^2} = y^2$

x^{ly} y^2 in the eq^n.

$y^3(2x-y+1)dx + xy^2(3x-4y+3)dy = 0$

$m = 2xy^3 - y^4 + y^3$ $\quad$ $n = 3x^2y^2 - 4xy^3 + 3xy^2$

$\frac{\partial m}{\partial y} = 2x(3y^2) - 4y^3 + 3y^2$ $\quad$ $\frac{\partial n}{\partial x} = 3y^2(2x) - 4y^3 + 3y^2$

$= 6xy^2 - 4y^3 + 3y^2$ $\quad$ $= 6xy^2 - 4y^3 + 3y^2$

$\frac{\partial m}{\partial y} = \frac{\partial n}{\partial x}$ This is exact D.E

C.S is $\int m(x,y)dx + \int n(y)dy = C$

$\int (2xy^3 - y^4 + y^3)dx + \int 0\, dy = C$

$\cancel{2}y^3\frac{x^2}{\cancel{2}} - y^4x + y^3x = C$

$$\boxed{x^2y^3 - xy^4 + xy^3 = C}$$

5. $y(2xy+1)dx - x\,dy = 0$

$m = 2xy^2 + y \qquad n = -x$

$\frac{\partial m}{\partial y} = 4xy + 1 \qquad \frac{\partial n}{\partial x} = -1$

$\frac{\partial m}{\partial y} \neq \frac{\partial n}{\partial x}$ This is not exact D.E

$$\frac{\partial m}{\partial y} - \frac{\partial n}{\partial x} = (4xy+1) - (-1)$$

$$= 4xy + 2$$

$$= 2(2xy+1) \simeq M$$

$$\frac{1}{M}\left(\frac{\partial m}{\partial y} - \frac{\partial n}{\partial x}\right) = \frac{1}{2xy^2+y}\, 2(2xy+1)$$

$$= \frac{1}{y\cancel{(2xy+1)}}\, 2\cancel{(2xy+1)} = \frac{2}{y} = g(y)$$

$$I.F = e^{-\int g(y)dy} = e^{-\int \frac{2}{y}dy} = e^{-2\log y} = \cancel{e}^{\log y^{-2}}$$

$$= \frac{1}{y^2}$$

x'ly $\frac{1}{y^2}$ in equation

$$\frac{1}{y^{\cancel{2}}}\cancel{y}(2xy+1)\,dx - \frac{x}{y^2}dy = 0$$

$$\left(\frac{2xy}{\cancel{y}} + \frac{1}{y}\right)dx - \frac{x}{y^2}dy = 0$$

$m = \left(2x + \frac{1}{y}\right) \qquad n = -\frac{x}{y^2}$

$\frac{\partial m}{\partial y} = -\frac{1}{y^2} \qquad \frac{\partial n}{\partial x} = -\frac{1}{y^2}$

$\frac{\partial m}{\partial y} = \frac{\partial n}{\partial x}$ This is exact D.E

C.S is $\int m(x,y)\,dx + \int n(y)\,dy = C$

$\int \left(2x + \frac{1}{y}\right) dx + \int 0\,dy = C$

$\cancel{2}\frac{x^2}{\cancel{2}} + \frac{x}{y} = C$

$\boxed{x^2 + \frac{x}{y} = C}$

6. $y(x+y)\,dx + (x+2y-1)\,dy = 0$

$m = xy + y^2 \qquad n = x + 2y - 1$

$\frac{\partial m}{\partial y} = x + 2y \qquad \frac{\partial n}{\partial x} = 1.$

$\frac{\partial m}{\partial y} \neq \frac{\partial n}{\partial x}$ This is not exact D.E

$\frac{\partial m}{\partial y} - \frac{\partial n}{\partial x} = (x+2y-1) \simeq N$

$\frac{1}{N}\left(\frac{\partial m}{\partial y} - \frac{\partial n}{\partial x}\right) = \frac{1}{\cancel{(x+2y-1)}} \cancel{(x+2y-1)} = 1 = f(x)$

$I.F = e^{\int f(x)\,dx} = e^{\int dx} = e^{x}$

x^{ly} e^x in equation.

$e^x y(x+y)\,dx + e^x(x+2y-1)\,dy = 0$

$m = e^x xy + e^x y^2 \qquad n = e^x x + 2e^x y - e^x$

$\frac{\partial m}{\partial y} = e^x x + 2ye^x \qquad \frac{\partial n}{\partial x} = e^x x + 2e^x y$

$\frac{\partial m}{\partial y} = \frac{\partial n}{\partial x}$ This is exact D.E

$$\int m(x,y)\,dx + \int n(y)\,dy = C$$

$$\int (e^x xy + e^x y^2)\,dx + \int 0\,dy = C$$

$$y\int e^x x\,dx + y^2\int e^x\,dx = C$$

$$y\int e^x - xe^x\,dx + y^2\int e^x\,dx = C$$

$$y\int e^x(1-x)\,dx + y^2\int e^x\,dx = C$$

$$ye^x(x) + y^2e^x - y = C$$

$$\boxed{e^x(xy + y^2 - y) = C}$$

ORTHOGONAL TRAJECTORIES

If two family of curves are such that every member of one family intersepts with every member of another family at right angle then they are said to be orthogonal trajectories of each other.

CARTESIAN FORM

Working procedure:

step 1; Given $f(x,y) = C$ differentiate wrt x and eliminate the parameter

step 2; Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ and solve the resulting differential equation which gives orthogonal trajectories of given family.

SELF ORTHOGONAL FAMILY:

If the differential equation of the given famil remains unaltered after replacing dy/dx by $-dx/dy$ then the given family of curve is said to be self orthogonal.

1. find orthogonal trajectories of the circles [12]

$x^2+y^2=a^2$

$x^2+y^2=a^2 \longrightarrow (1)$

diff wrt x

$2x+2y\dfrac{dy}{dx}=0$ (parameter $\underline{a}$ is eliminated)

replace $\dfrac{dy}{dx}$ by $-\dfrac{dx}{dy}$

$2x+2y\dfrac{dy}{dx}=0 \quad \div 2$

$x+y\left(-\dfrac{dx}{dy}\right)=0$

$x=y\dfrac{dx}{dy}$

$\dfrac{dy}{y}=\dfrac{dx}{x}$

$\int\dfrac{dy}{y}=\int\dfrac{dx}{x}$

$\log y=\log x+\log m$

$\cancel{\log} y=\cancel{\log}(mx)$

$\boxed{y=mx}$

2. $x^{2/3}+y^{2/3}=a^{2/3}$

$x^{2/3}+y^{2/3}=a^{2/3} \longrightarrow (1)$

diff wrt x

$\dfrac{2}{3}x^{-1/3}+\dfrac{2}{3}y^{-1/3}\dfrac{dy}{dx}=0$ (parameter is eliminated) replace $\frac{dy}{dx}$ as $-\frac{dx}{dy}$

$$x^{-1/3} + y^{-1/3}\left(-\frac{dx}{dy}\right) = 0$$

$$x^{-1/3} = y^{-1/3}\frac{dx}{dy}$$

$$x^{-1/3}\,dy = y^{-1/3}\,dx$$

$$y^{1/3}\,dy = x^{1/3}\,dx$$

$$\int y^{1/3}\,dy - \int x^{1/3}\,dx = C$$

$$\frac{y^{4/3}}{4/3} - \frac{x^{4/3}}{4/3} = C$$

$$\frac{3y^{4/3}}{4} - \frac{3x^{4/3}}{4} = C$$

$$\boxed{x^{4/3} - y^{4/3} = C}$$ where. $\frac{4}{3} = K$

3. $\frac{x^2}{a^2} + \frac{y^2}{b^2+\lambda} = 1$ where λ is parameter.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2+\lambda} = 1 \longrightarrow (1)$$

diff (1) wrt x

$$\frac{2x}{a^2} + \frac{2y}{b^2+\lambda}\frac{dy}{dx} = 0$$

$\div 2$

$$\frac{x}{a^2} + \frac{y}{b^2+\lambda}y_1 = 0$$

$$\frac{x}{a^2} = -\frac{yy_1}{b^2+\lambda}$$

$$\frac{-x}{a^2 y_1} = \frac{y}{b^2+\lambda} \quad \text{—— (2)}$$

from (1) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2+\lambda} = 1$

$$\frac{x^2}{a^2} - 1 = -\frac{y^2}{b^2+\lambda}$$

$$\frac{x^2-a^2}{a^2} = \frac{-yy}{b^2+\lambda}$$

$$\frac{x^2-a^2}{-ya^2} = \frac{y}{b^2+\lambda}$$

$\therefore$ (2) becomes

$$\frac{+x}{a^2 y_1} = \frac{x^2-a^2}{+ya^2} \quad (\lambda \text{ eliminated})$$

Replace $\dfrac{dy}{dx} \rightarrow -\dfrac{dx}{dy}$ $\left[y_1 \rightarrow -\dfrac{1}{y_1}\right]$

$$\frac{x}{-1/y_1} = \frac{x^2-a^2}{y}$$

$$-xy_1 = \frac{x^2-a^2}{y}$$

$$-x.\frac{dy}{dx} = \frac{x^2-a^2}{y}$$

$$y\,dy = \frac{(x^2-a^2)}{-x}\,dx$$

$$\int y\,dy = \int -x\,dx + \int \frac{a^2}{x}\,dx$$

$$\frac{y^2}{2} = -\frac{x^2}{2} + a^2 \log x + c$$

$$\boxed{\frac{x^2}{2} + \frac{y^2}{2} = a^2 \log x + c}$$

4. prove that family of parabola $y^2 = 4a(x+a)$ is self orthogonal.

$$y^2 = 4a(x+a) \longrightarrow (1)$$

$$y^2 = 4ax + 4a^2$$

D wrt x.

$$\cancel{2}yy_1 = \cancel{4}^{2}a$$

$$\boxed{a = \frac{yy_1}{2}}$$

① becomes $y^2 = \cancel{4}^{2}\left(\frac{yy_1}{\cancel{2}}\right)\cdot\left(x + \frac{yy_1}{2}\right)$

$$y^2 = \cancel{2}yy_1\left(\frac{2x+yy_1}{\cancel{2}}\right)$$

$$\cancel{y^2} = \cancel{y}y_1(2x+yy_1)$$

$$y = y_1(2x+yy_1) \longrightarrow (2)$$

(parameter is eliminated)

Replace y_1 by $-\frac{1}{y_1}$

$$y = -\frac{1}{y_1}\left(2x + y\left(-\frac{1}{y_1}\right)\right)$$

$$y = -\frac{1}{y_1}\left(2x - \frac{y}{y_1}\right)$$

$$y = -\frac{1}{y_1}\left(\frac{2xy_1 - y}{y_1}\right)$$

$$y_1^2 y = -2xy_1 + y$$

$$y_1^2 y + 2xy_1 = y$$

$$y = y_1(yy_1 + 2x) \longrightarrow (3)$$

Since ② & ③ are same

$\therefore$ ① is self orthogonal.

5. find orthogonal trajectory for $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1 \longrightarrow (1)$$

D wrt x

$$\frac{2x}{a^2+\lambda} + \frac{2y}{b^2+\lambda}\frac{dy}{dx} = 0$$

$\div 2$

$$\frac{x}{a^2+\lambda} = \frac{-yy_1}{b^2+\lambda}$$

$$\frac{a}{b} = \frac{c}{d}$$

$$\frac{a-c}{b-d} = \frac{c}{d}$$

$$\frac{a-c}{b-d} = \frac{a}{b}$$

$$\left(\frac{a-c}{b-d}\right) = \left(\frac{a}{b}\right)$$

Consider

$$\frac{x-(yy_1)}{a^2+\lambda-(b^2+\lambda)} = \frac{x}{a^2+\lambda}$$

$$\frac{x+yy_1}{a^2+\cancel{\lambda}-b^2-\cancel{\lambda}} = \frac{x}{a^2+\lambda}$$

$$\boxed{\frac{x+yy_1}{a^2-b^2} = \frac{x}{a^2+\lambda}} \longrightarrow (2)$$

Consider $\left(\frac{a-c}{b-d} = \frac{c}{d}\right)$

$$\frac{x+yy_1}{a^2-b^2} = \frac{-yy_1}{b^2+\lambda}$$

$$\boxed{\frac{x+yy_1}{-y_1(a^2-b^2)} = \frac{y}{b^2+\lambda}} \longrightarrow (3)$$

2 & 3 in 1

$$x\left(\frac{x}{a^2+\lambda}\right) + y\left(\frac{y}{b^2+\lambda}\right) = 1$$

$$\frac{x(x+yy_1)}{a^2-b^2} + \frac{y(x+yy_1)}{-y_1(a^2-b^2)} = 1$$

$$(x+yy_1)\frac{(xy_1-y)}{y_1(a^2-b^2)}=1$$

$$(x+yy_1)(xy_1-y)=y_1(a^2-b^2) \longrightarrow (4)$$

Replace y_1 by $-\frac{1}{y_1}$

$$\left(x+y\left(-\frac{1}{y_1}\right)\right)\left(x\left(-\frac{1}{y_1}\right)-y\right)=-\frac{1}{y_1}(a^2-b^2)$$

$$\left(\frac{xy_1-y}{y_1}\right)(-1)\left(\frac{x+yy_1}{y_1}\right)=-\frac{1}{y_1}(a^2-b^2)$$

$$(x+yy_1)(xy_1-y)=y(a^2-b^2) \longrightarrow (5)$$

$\therefore$ 1 is self orthogonal.

ORTHOGONAL TRAJECTORIES IN POLAR FORM

working procedure;

- Given polar family occur $f(r,\theta)=C$ take log on both sides and diff wrt θ and eliminate the parameter
- Replace $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ and solve resulting differential equation which gives orthogonal trajectories of given family of curve

1. find orthogonal trajectories of following curves $r=a(1+\sin\theta)$

$$r=a(1+\sin\theta)$$

$$\log r=\log a+\log(1+\sin\theta)$$

$$\frac{1}{r}\frac{dr}{d\theta}=0+\frac{\cos\theta}{1+\sin\theta}$$ (parameter a is eliminated

Replace $\frac{dr}{d\theta}\rightarrow -r^2\frac{d\theta}{dr}$

$$-\frac{1}{r}r^2\frac{d\theta}{dr}=\frac{\cos\theta}{1+\sin\theta}$$

$$-r\frac{d\theta}{dr}=\frac{\cos\theta}{(1+\sin\theta)}$$

$$\left(\frac{1+\sin\theta}{\cos\theta}\right)d\theta=-\frac{1}{r}dr$$

$$\left(\frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}\right) d\theta = -\frac{1}{r} dr$$

$\int$ obs

$$\int \sec\theta \, d\theta + \int \tan\theta \, d\theta = -\int \frac{1}{r} dr$$

$$\log(\sec\theta + \tan\theta) + \log(\sec\theta) = -\log r + \log c$$

$$\log((\sec\theta + \tan\theta)\sec\theta) = \log\left(\frac{c}{r}\right)$$

$$\sec^2\theta + \sec\theta \cdot \tan\theta = \frac{c}{r}$$

$$\boxed{r(\sec^2\theta + \sec\theta \cdot \tan\theta) = c}$$

2. $r^n \cos n\theta = a^n$

$$\log r^n + \log(\cos n\theta) = \log a^n$$

$$n \cdot \log r + \log(\cos n\theta) = n \cdot \log a$$

$$n \frac{1}{r} \frac{dr}{d\theta} + \frac{n(-\sin n\theta)}{\cos n\theta} = n(0)$$

$\cancel{n} \cdot \frac{1}{r} \frac{dr}{d\theta} = \cancel{n} \tan n\theta$ (parameter a is eliminated)

Replace $\frac{dr}{d\theta} \rightarrow -r^2 \frac{d\theta}{dr}$

$$\frac{1}{\cancel{r}} - r^{\cancel{2}} \frac{d\theta}{dr} = \tan n\theta$$

$$-r \, d\theta = dr \tan n\theta$$

$$\frac{d\theta}{\tan n\theta} = -\frac{dr}{r}$$

$$\int \frac{1}{\tan n\theta} d\theta = -\int \frac{1}{r} dr$$

$$\int \cot n\theta \, d\theta = -\int \frac{1}{r} dr$$

$$\frac{\log(\sin n\theta)}{n} = -\log r + \log c$$

$$-\log r = \frac{\log(\sin n\theta)}{n} + \log c$$

$$-\log r = \log(\sin n\theta)^{1/n} + \log c$$

$$-(\log r + \log c) = \log(\sin n\theta)^{1/n}$$

$$(\log rc)^{-1} = \log(\sin n\theta)^{1/n}$$

$$(rc)^{-1} = (\sin n\theta)^{1/n}$$

$$\frac{1}{r} c^{-1} = \sin^{1/n} n\theta \qquad c^{1} = b$$

$$\frac{b}{r} = \sin^{1/n} n\theta$$

$$\boxed{b = r \sin^{1/n} n\theta}$$

3. $r = a(1-\cos\theta)$

$$\log r = \log a + \log(1-\cos\theta)$$

$$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{\sin\theta}{(1-\cos\theta)}$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{\sin\theta}{(1-\cos\theta)}$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{\cancel{2}\,\sin\theta/2 \cos\theta/2}{\cancel{2}\,\sin^2\theta/2}$$

$$\frac{1}{r}\frac{dr}{d\theta} = \cot\theta/2$$

$$\frac{1}{r} - r^2\frac{d\theta}{dr} = \cot\theta/2$$

$$-r\,d\theta = \cot\theta/2\,dr$$

$$\frac{d\theta}{\cot\theta/2} = -\frac{dr}{r}$$

$$\int \tan\theta/2\,d\theta = -\int \frac{1}{r}\,dr$$

$$\frac{\log(\sec\theta/2)}{1/2} = -\log r + \log c$$

$$\log(\sec\theta/_2)^2 = \log\left(\frac{c}{r}\right)$$

$$\boxed{r\sec^2\theta/_2 = C}$$

4. $r = 4a\sec\theta\cdot\tan\theta$

$$\log r = \log 4a + \log(\sec\theta) + \log(\tan\theta)$$

$$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{\cancel{\sec\theta}\cdot\tan\theta}{\cancel{\sec\theta}} + \frac{\sec^2\theta}{\tan\theta}$$

$$\frac{1}{r}\frac{dr}{d\theta} = \tan\theta + \frac{\sec^2\theta}{\tan\theta}$$ (parameter is eliminated)

Replace $\frac{dr}{d\theta} \rightarrow -r^2\frac{d\theta}{dr}$

$$-r\frac{d\theta}{dr} = \frac{\tan^2\theta + \sec^2\theta}{\tan\theta}$$

$$\frac{\tan\theta}{\tan^2\theta + \sec^2\theta}\, d\theta = -\frac{1}{r}\, dr$$

$$\int \frac{\frac{\sin\theta}{\cos\theta}}{\frac{\sin^2\theta}{\cos^2\theta} + \frac{1}{\cos^2\theta}}\, d\theta = -\int \frac{1}{r}\, dr$$

$$\int \frac{\sin\theta}{\cancel{\cos\theta}} \cdot \frac{\cos^{\cancel{2}}\theta}{(1+\sin^2\theta)}\, d\theta = -\int \frac{1}{r}\, dr$$

$$\int \frac{\sin\theta\cos\theta}{(1+\sin^2\theta)}\, d\theta = -\int \frac{1}{r}\, dr$$

put $1 + \sin^2\theta = t$

$2\sin\theta\cos\theta\, d\theta = dt$

$\sin\theta\cos\theta\, d\theta = \frac{dt}{2}$

$$\int \frac{dt}{2t} = -\int \frac{1}{r}\, dr$$

$$\frac{1}{2}\log t = -\log r + \log c$$

$$\frac{1}{2}\log(1+\sin^2\theta) = \log\left(\frac{c}{r}\right)$$

$$\log(1+\sin^2\theta) = 2\log\left(\frac{c}{r}\right)$$

$$\log(1+\sin^2\theta) = \log\left(\frac{c}{r}\right)^2$$

$$(1+\sin^2\theta) = \frac{c^2}{r^2}$$

$$\boxed{r^2(1+\sin^2\theta) = C}$$

5. $r^2 = a^2\cos 2\theta$

$$\log r^2 = \log a^2 + \log(\cos 2\theta)$$

$$2\log r = 2\log a + \log(\cos 2\theta)$$

$$\cancel{2}\frac{1}{r}\frac{dr}{d\theta} = -\frac{\sin 2\theta}{\cos 2\theta}(\cancel{2})$$

$$\frac{1}{r}\frac{dr}{d\theta} = -\tan 2\theta$$

$$\frac{1}{\cancel{r}} + r^{\cancel{2}}\frac{d\theta}{dr} = \cancel{+}\tan 2\theta$$

$$r\,d\theta = \tan 2\theta\,dr$$

$$\frac{d\theta}{\tan 2\theta} = \frac{dr}{r}$$

$$\int\cot 2\theta\,d\theta = \int\frac{1}{r}\,dr$$

$$\frac{\log(\sin 2\theta)}{2} = \log r + \log c$$

$$\log(\sin 2\theta) = 2\log(rc)$$

$$\log(\sin 2\theta) = \log(rc)^2$$

$$\boxed{\frac{\sin 2\theta}{c^2} = r^2}$$

6. $r(1+\cos\theta) = 2a$

$$\log r + \log(1+\cos\theta) = \log 2a$$

$$\frac{1}{r}\frac{dr}{d\theta} + \frac{(-\sin\theta)}{(1+\cos\theta)} = 0$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{\sin\theta}{(1+\cos\theta)}$$

$$\frac{1}{\cancel{r}} - r^{\cancel{2}}\frac{d\theta}{dr} = \frac{\sin\theta}{(1+\cos\theta)}$$

$$-r\frac{d\theta}{dr} = \frac{\sin\theta}{(1+\cos\theta)}$$

$$\left(\frac{1+\cos\theta}{\sin\theta}\right)d\theta = -\frac{dr}{r}$$

$$\int \operatorname{cosec}\theta\, d\theta + \int \cot\theta\, d\theta = -\int \frac{1}{r}\, dr$$

$$\log(\operatorname{cosec}\theta - \cot\theta) + \log(\sin\theta) = -\log r + \log c$$

$$\log((\operatorname{cosec}\theta - \cot\theta)\sin\theta) = \log\left(\frac{c}{r}\right)$$

$$\log(\sin\theta\operatorname{cosec}\theta - \sin\theta\cot\theta) = \log\left(\frac{c}{r}\right)$$

$$\log\left(\cancel{\sin\theta}\frac{1}{\cancel{\sin\theta}} - \frac{\cos\theta}{\cancel{\sin\theta}}\cancel{\sin\theta}\right) = \log\left(\frac{c}{r}\right)$$

$$\log(1-\cos\theta) = \log\left(\frac{c}{r}\right)$$

$$\boxed{r(1-\cos\theta) = c}$$

NEWTONS LAW OF COOLING.

Acc to law, the temperature of body changes at a rate proportional to the difference in temperature between that of surroundingmedium and the body itself.

If θ_0 is the temperature of the surroundings and θ is the temperature of the body at any time t, then $\frac{d\theta}{dt} = -K(\theta-\theta_0)$ where K is constant therefore, solving this differential equation we get $\theta = \theta_0 + (\theta - \theta_0) e^{-kt}$

1. A body originally at 80°C cools down to 60°C in 20 minutes. the temperature of the air being 40°C. what will be temperature of body after 40 minutes from original.

If θ is temperature of body at any time t, then $\theta = 80°C$

temperature of surrounding $\theta_0 = 40°$.

$$\frac{d\theta}{dt} = -k(\theta - \theta_0)$$

$$\frac{d\theta}{dt} = -k(\theta - 40)$$

$$\int \frac{d\theta}{(\theta - 40)} = -k \int dt$$

$$\log(\theta - 40) = -kt + c$$

$$\theta - 40 = e^{-kt + c}$$

$$\boxed{\theta = 40 + e^{-kt + c}}$$

where $a = e^c$

$$\theta = 40 + e^{-kt} a \rightarrow (1)$$

at $t = 0$, $\theta = 80$

$$80 = 40 + e^{-k(0)} a$$

$$\boxed{a = 40}$$

at $t = 20$, $\theta = 60$

$$60 = 40 + e^{-k(20)} 40$$

$$\frac{20}{40} = e^{-20k}$$

$$\frac{1}{2} = e^{-20k}$$

$$\log\left(\frac{1}{2}\right) = \log e^{-20k}$$

$$\log \frac{1}{2} = -20K$$

$$K = -\frac{1}{20} \log \frac{1}{2} = 0.0346$$

$$\boxed{K = 0.0346}$$

when $t = 40$ $\theta = ?$ $① \Rightarrow \theta = 40 + e^{-(0.0346)(40)}(40)$

$$\boxed{\theta \simeq 50^\circ}$$

2. A body in air at 25°C cools from 100°C to 75°C in 1 minute. find temperature of body at the end of 3 minutes.

temperature of surrounding medium $\theta_0 = 25°C$

at $t = 0$ $\theta = 100°C$

at $t = 1$ $\theta = 75°C$

at $t = 3$ $\theta = ?$

$$\frac{d\theta}{dt} = -K(\theta - \theta_0)$$

$$\frac{d\theta}{dt} = -K(\theta - 25)$$

$$\int \frac{d\theta}{(\theta - 25)} = -K \int dt$$

$$\log(\theta - 25) = -Kt + c$$

$$\theta - 25 = e^{-Kt} a$$

$$\boxed{\theta = 25 + e^{-Kt} a}$$

at $t = 0 \Rightarrow 100 = 25 + e^{0} a$

$$100 - 25 = a$$

$$\boxed{a = 75}$$

at $t = 1$, $\theta = 75$

$$75 = 25 + e^{-K}(75)$$

$$50 = 75 e^{-K}$$

$$e^{-K} = \frac{2}{3}$$

$$e^{K} = \frac{3}{2}$$

$$K = \ln\left(\frac{3}{2}\right)$$

$$\boxed{K = 0.4054}$$

at $t = 3$, $\theta = 25 + e^{(-0.4054)(3)} 75$

$$\boxed{\theta = 47.22}$$

$\therefore$ temperature of body at $t = 3$ min is $47.22°C$

NON-LINEAR DIFFERENTIAL EQUATION

Here we study differential equation of first order and n^{th} degree. Generally equation can be written in the form of

$$A_0\left(\frac{dy}{dx}\right)^n + A_1\left(\frac{dy}{dx}\right)^{n-1} + A_2\left(\frac{dy}{dx}\right)^{n-2} + \cdots \longrightarrow (1)$$

where A_0, A_1, A_2 are functions of x & y

let $\frac{dy}{dx} = P \Rightarrow A_0 p^n + A_1 p^{n-1} + A_2 p^{n-2} + \cdots = 0$

1. solve $p^2 - 2p + 1 = 0$

$(p-1)^2 = 0$

$p - 1 = 0 \quad p - 1 = 0$

$\frac{dy}{dx} = 1 \quad \frac{dy}{dx} = 1$

$dy = dx \quad dy = dx$

$\int$ obs

$y = x + c \quad y = x + c$

$y - x - c = 0 \quad y - x - c = 0$

solution of equation is $(y - x - c)(y - x - c) = 0$

2. solve $p^2 - 5p - 6 = 0$

$p^2 - 6p + 1p - 6 = 0$

$p(p-6) + 1(p-6) = 0$

$(p-6) = 0 \quad (p+1) = 0$

$\frac{dy}{dx} = 6 \qquad \frac{dy}{dx} = -1$

$dy = 6\,dx \qquad dy = -dx$

$\int dy = 6\int dx \qquad \int dy = -\int dx$

$y = 6x + c \qquad y = -x + c$

$\boxed{y - 6x - c = 0} \qquad \boxed{y + x - c = 0}$

$(y - 6x - c)(y + x - c) = 0$

3. find general solution of equation.

$yp^2 + (x-y)p - x = 0$

$yp^2 + (x-y)p - x = 0$

$yp^2 + px - yp - x = 0$

$yp(p-1) + x(p-1) = 0$

$(yp + x) = 0 \qquad (p-1) = 0$

$yp = -x \qquad \frac{dy}{dx} - 1 = 0$

$y \cdot \frac{dy}{dx} = -x \qquad \frac{dy}{dx} = 1$

$y\,dy = -x\,dx \qquad dy = dx$

$\int y\,dy = -\int x\,dx \qquad \int dy = \int dx$

$\frac{y^2}{2} = -\frac{x^2}{2} + c \qquad y = x + c$

$\boxed{\frac{x^2}{2} + \frac{y^2}{2} - c = 0} \qquad \boxed{y - x - c = 0}$

$\therefore$ General solution is $\left(\frac{x^2}{2} + \frac{y^2}{2} - c\right)(y - x - c) = 0$

4. Solve $p^2 + 2py \cot x = y^2$

$$p^2 + 2py\cot x - y^2 = 0$$

$$ax^2 + bx + c = 0$$

$$[ap^2 + bp + c = 0]$$

$a = 1 \quad b = 2y\cot x \quad c = -y^2$

$$p = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$p = \frac{-2y\cot x \pm \sqrt{(2y\cot x)^2 - 4(-y^2)}}{2(1)}$$

$$p = \frac{-2y\cot x \pm \sqrt{4y^2\cot^2 x + 4y^2}}{2}$$

$$p = \frac{-2y\cot x \pm \sqrt{4y^2}\left(\sqrt{\cot^2 x + 1}\right)}{2}$$

$$p = \frac{-2y\cot x \pm 2y\left(\sqrt{\operatorname{cosec}^2 x}\right)}{2}$$

$$p = \frac{-2y\cot x \pm 2y\operatorname{cosec} x}{2}$$

$$p = -y\cot x \pm y\operatorname{cosec} x$$

$p = -y\cot x + y\operatorname{cosec} x$

$\frac{dy}{dx} = y(\operatorname{cosec} x - \cot x)$

$\int \frac{dy}{y} = \int (\operatorname{cosec} x - \cot x)\,dx$

$\log y = \log(\operatorname{cosec} x - \cot x) - \log(\sin x) + \log c$

$\log y - \log c = \log(\operatorname{cosec} x - \cot x) - \log(\sin x)$

$p = -y\cot x - y\operatorname{cosec} x$

$\frac{dy}{dx} = -\cot x - \operatorname{cosec} x\,dx$

$\int \frac{dy}{y} = -\int \cot x\,dx - \int \operatorname{cosec} x\,dx$

$\log y = -\log(\sin x) - \log(\operatorname{cosec} x - \cot x) + \log c$

$\log y - \log c = -[\log(\sin x) + \log(\operatorname{cosec} x - \cot x)]$

$\log\left(\frac{y}{c}\right) = -[\log(\sin x)(\operatorname{cosec} x - \cot x)]$

$$\log\left(\frac{y}{c}\right) = \log\left(\frac{\operatorname{cosec}x - \cot x}{\sin x}\right)$$

$$y = c\left(\frac{\operatorname{cosec}x - \cot x}{\sin x}\right)$$

$$y = c\left(\frac{1}{\sin^2 x} - \frac{\cos x}{\sin^2 x}\right)$$

$$y = c\left(\frac{1-\cos x}{\sin^2 x}\right)$$

$$y = c\frac{(1-\cos x)}{(1+\cos^2 x)}$$

$$y = c\frac{(1-\cancel{\cos x})}{(1+\cos x)\cancel{(1-\cos x)}}$$

$$y(1+\cos x) = c$$

$$\frac{y}{c} = -(\sin x \operatorname{cosec} x - \sin x \cot x)$$

$$\frac{y}{c} = -\left(\sin x \frac{1}{\sin x} - \sin x \frac{\cos x}{\sin x}\right)$$

$$\frac{y}{c} = -(1-\cos x)$$

$$\frac{y}{c} = -1+\cos x$$

$$y - \cos x + 1 - c = 0$$

$$y(1-\cos x) - c = 0$$

General solution ~~(y cos x 1) c~~

$$(y(1+\cos x) - c)(y(1-\cos x) - c) = 0$$

4. solve $xyp^2 - (x^2+y^2)p + xy = 0$

$$a = xy \quad b = -(x^2+y^2) \quad c = xy$$

$$p = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$p = \frac{+(x^2+y^2) \pm \sqrt{(-(x^2+y^2))^2 - 4(xy)(xy)}}{2xy}$$

$$p = \frac{(x^2+y^2) \pm \sqrt{x^4+y^4+2x^2y^2-4x^2y^2}}{2xy}$$

$$p = \frac{(x^2+y^2) \pm \sqrt{x^4+y^4-2x^2y^2}}{2xy}$$

$$P = \frac{(x^2+y^2) \pm (\sqrt{x^2-y^2})^2}{2xy}$$

$$P = \frac{(x^2+y^2) \pm (x^2-y^2)}{2xy}$$

$$P = \frac{x^2+y^2+x^2-y^2}{2xy}$$

$$P = \frac{\cancel{2}x^{\cancel{2}}}{\cancel{2}\cancel{x}y} = \frac{x}{y}$$

$$P = \frac{x}{y}$$

$$\frac{dy}{dx} = \frac{x}{y}$$

$$dy\, y = dx\, x$$

$$\int y\, dy = \int x\, dx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + c$$

$$\boxed{\frac{x^2}{2} - \frac{y^2}{2} + c = 0}$$

$$P = \frac{x^2+y^2-x^2+y^2}{2xy}$$

$$P = \frac{\cancel{2}y^{\cancel{2}}}{\cancel{2}x\cancel{y}} = \frac{y}{x}$$

$$P = \frac{y}{x}$$

$$\frac{dy}{dx} = \frac{y}{x}$$

$$\int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\log y = \log x + \log c$$

$$\boxed{y = xc}$$

$$y - xc = 0$$

General solution is $\left(\frac{x^2}{2} - \frac{y^2}{2} + c\right)(y - xc) = 0$

5. $p^3 + 2xp^2 - y^2p^2 - 2xy^2p = 0$

consider

$$p^3 + 2xp^2 - y^2p^2 - 2xy^2p = 0$$

$$p(p^2 + 2xp - y^2p - 2xy^2) = 0$$

$\boxed{P = 0}$

$$p^2 + 2xp - y^2p - 2xy^2 = 0$$

$$p^2 + (2x - y^2)P - 2xy^2 = 0$$

$$a = 1 \quad b = 2x - y^2 \quad c = -2xy^2$$

$$P = \frac{-(2x-y^2) \pm \sqrt{(2x-y^2)^2 - 4(-2xy^2)}}{2(1)}$$

$$P = \frac{-(2x-y^2) \pm \sqrt{4x^2+y^4-4xy^2+8xy^2}}{2}$$

$$P = \frac{-(2x-y^2) \pm \sqrt{4x^2+y^4+4xy^2}}{2}$$

$$P = \frac{-2x+y^2 \pm \sqrt{(2x+y^2)^2}}{2}$$

$$P = \frac{-2x+y^2 \pm (2x+y^2)}{2}$$

$$P = \frac{-\cancel{2x}+y^2+\cancel{2x}+y^2}{2}$$

$$P = \frac{2y^2}{2} = y^2$$

$$\frac{dy}{dx} = y^2$$

$$\frac{dy}{y^2} = dx$$

$$\int \frac{1}{y^2}\,dy = \int dx$$

$$\int y^{-2}\,dy = \int dx$$

$$\frac{y^{-1}}{-1} = x$$

$$-\frac{1}{y} = x + c$$

$$\boxed{x + \frac{1}{y} + c = 0}$$

$$P = \frac{-2x+\cancel{y^2}-2x-\cancel{y^2}}{2}$$

$$P = -\frac{4x}{2} = -2x$$

$$\frac{dy}{dx} = -2x$$

$$dy = -2x\,dx$$

$$\int dy = -2\int x\,dx$$

$$y = -\cancel{2}\frac{x^2}{\cancel{2}} + c$$

$$\boxed{x^2 + y - c = 0}$$

$\therefore$ General solution is

$$\left(x + \frac{1}{y} + c\right)\left(x^2+y-c\right)(y-c) = 0$$

6. $p(p+y) = x(x+y)$

$p^2 + py - x(x+y) = 0$

$a = 1 \quad b = y \quad c = -x(x+y)$

$$p = \frac{-y \pm \sqrt{y^2 - 4(-x(x+y))}}{2(1)}$$

$$p = \frac{-y \pm \sqrt{y^2 - 4(-x^2 - y)}}{2}$$

$$p = \frac{-y \pm \sqrt{y^2 + 4x^2 + 4y}}{2}$$

$$p = \frac{-y \pm \sqrt{(2x+y)^{\cancel{2}}}}{2}$$

$$p = \frac{-y \pm (2x+y)}{2}$$

$$p = \frac{-y - (2x+y)}{2}$$

$$p = \frac{-y - 2x - y}{2}$$

$$p = \frac{-2y - 2x}{2}$$

$$p = \frac{-\cancel{2}(x+y)}{\cancel{2}}$$

$$p = -(x+y)$$

$$\frac{dy}{dx} = -(x+y)$$

$$p = \frac{-y + (2x+y)}{2}$$

$$p = \frac{-\cancel{y} + 2x + \cancel{y}}{2}$$

$$p = \frac{\cancel{2}x}{\cancel{2}}$$

$$p = x$$

$$\frac{dy}{dx} = x$$

$$dy = x\,dx$$

$$\int dy = \int x \int dx$$

$$y = \frac{x^2}{2} + c$$

$$\boxed{\frac{x^2}{2} - y + c = 0}$$

$$dy = -(x+y)\,dx$$

$$\frac{dy}{-(x+y)} = dx$$

$$\frac{dy}{dx} = -x-y$$

$$\frac{dy}{dx} + y = -x$$

$P(x) = 1 \quad Q(x) = -x$

$$I.F = e^{\int p(x)\,dx} = e^{x}$$

$$y(IF) = \int IF\ Q(x)\,dx + C$$

$$y(e^{x}) = \int e^{x}(-x)\,dx + C$$

$$ye^{x} = -\int x e^{x}\,dx + C$$

$$ye^{x} = -(xe^{x} - e^{x}) + C$$

$$ye^{x} = -e^{x}(x-1) + C$$

$$e^{x}y + e^{x}(x-1) - C = 0$$

$$e^{x}(y+x-1) - C = 0$$

General solution is $(2y-x^2-c)(e^{x}(y+x-1)-c) = 0$

7. Solve $\frac{dy}{dx} - \frac{dx}{dy} = \frac{x}{y} - \frac{y}{x}$

$$p - \frac{1}{p} = \frac{x}{y} - \frac{y}{x}$$

$$\frac{p^2-1}{p} = \frac{x}{y} - \frac{y}{x}$$

$$p^2 - 1 = p\left(\frac{x}{y} - \frac{y}{x}\right)$$

$$p^2 - p\left(\frac{x}{y} - \frac{y}{x}\right) - 1 = 0$$

$a = 1 \quad b = -\left(\frac{x}{y} - \frac{y}{x}\right) \quad c = -1$

$$P = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{\left(\frac{x}{y} - \frac{y}{x}\right) \pm \sqrt{\left(\frac{x}{y} - \frac{y}{x}\right)^2 - 4(-1)}}{2(1)}$$

$$P = \frac{\left(\frac{x}{y} - \frac{y}{x}\right) \pm \sqrt{\left(\frac{x}{y}\right)^2 + \left(\frac{y}{x}\right)^2 - 2\left(\cancel{\frac{x}{y}}\right)\left(\cancel{\frac{y}{x}}\right) + 4}}{2}$$

$$P = \frac{\left(\frac{x}{y} - \frac{y}{x}\right) \pm \sqrt{\left(\frac{x}{y} + \frac{y}{x}\right)^{\cancel{2}}}}{2}$$

$$P = \frac{1}{2}\left(\frac{x}{y} - \frac{y}{x} \pm \left(\frac{x}{y} + \frac{y}{x}\right)\right) \qquad P = \frac{1}{2}\left(\cancel{\frac{x}{y}} - \frac{y}{x} - \cancel{\frac{x}{y}} - \frac{y}{x}\right)$$

$$P = \frac{1}{2}\left(\frac{x}{y} - \cancel{\frac{y}{x}} + \frac{x}{y} + \cancel{\frac{y}{x}}\right) \qquad P = -\frac{\cancel{2}y}{\cancel{2}x} = -\frac{y}{x}$$

$$P = \frac{\cancel{2}x}{\cancel{2}y} = \frac{x}{y} \qquad P = -\frac{y}{x}$$

$$\frac{dy}{dx} = \frac{x}{y} \qquad \frac{dy}{dx} = -\frac{y}{x}$$

$$y\,dy = x\,dx \qquad -\frac{dy}{y} = \frac{dx}{x}$$

$$\int y\,dy = \int x\,dx \qquad \int -\frac{1}{y}\,dy = \int \frac{1}{x}\,dx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + c \qquad -\log y = \log x + \log c$$

$$\boxed{-\frac{x^2}{2} + \frac{y^2}{2} - c = 0} \qquad -\log y = \log(xc)$$

$$-y = xc$$

$$\boxed{xc + y = 0}$$

General solution is $\left(-\frac{x^2}{2} + \frac{y^2}{2} - c\right)(xc + y) = 0$

NOTE : The family of curve $G(x,y,c)=0$ posses an envelop then an equation of the envelop is the singular solution of differential equation.

ENVELOP : An envelop of a family of curves is a curve which touches each member of family

CLAIRAULTS EQUATION

It is in the equation of the form

$$y = p(x) + f(p)$$

Working procedure to find solution

i) Replace p by orbitary constant C to get the general solution of the given equation

ii) To obtain singular solution differentiate the given equation partially with respect to C between the given equation and equation obtained solve for C and replace C by value of C obtained.

The general solution of clairaults equation is of the form $y = c(x) + f(c)$

1. find general solution of equation $p = \sin(y - xp)$

$$p = \sin(y - xp)$$

$$\sin^{-1}p = y - xp$$

$$y = xp + \sin^{-1}p$$

Replace $p \to c$

$\boxed{y = xc + \sin^{-1}c}$ is General solution

2 find singular solution of equation

$$p = \log(px - y)$$

$$p = \log(px - y) \quad \text{--- (1)}$$

$$e^{p} = px - y$$

$y = px - e^p$

Replace $p \to c$

$\boxed{y = cx - e^c}$ is general solution $\to (2)$

Diff (2) p wrt c

$0 = x - e^c$

$x = e^c$

$\log x = \log e^c$

$\boxed{c = \log x}$

$(2) \Rightarrow \quad y = \log x(x) - e^{\log x}$

$\boxed{y = \log x(x) - x}$ is singular solution

3. find singular solution for $\sin px \cos y = \cos px \sin y + p$

$\sin px \cos y - \cos px \sin y = p \to (1)$

$\sin(px - y) = p$

$px - y = \sin^{-1} p$

$\sin^{-1} p = px - y$

$y = px - \sin^{-1} p$

$\boxed{y = xc - \sin^{-1} c}$ is general solution $\to (2)$

Dpw.r.t c.

$$0 = x - \frac{1}{\sqrt{1-c^2}}$$

$$+x = + \frac{1}{\sqrt{1-c^2}}$$

$$x^2 = \frac{1}{(1-c)^2}$$

$$1-c^2 = \frac{1}{x^2}$$

$$1-\frac{1}{x^2} = c^2$$

$$\frac{x^2-1}{x^2} = c^2$$

$$c = \pm\sqrt{\frac{x^2-1}{x^2}}$$

$$c = \pm\frac{\sqrt{x^2-1}}{x}$$ $\therefore$ ② becomes

$$y = \frac{\cancel{x}\left(\sqrt{x^2-1}\right)}{\cancel{x}} \sin^{-1}\left(\frac{\sqrt{x^2-1}}{x}\right)$$

4. Prove that $xp^2+px-py+1-y=0$ is clairaults equation and hence its singular solution

$$xp^2+px-py+1-y=0$$

$$px(p+1)-y(p+1) = -1$$

$$(p+1)(px-y) = -1$$

$$xp-y = -\frac{1}{p+1}$$

$$y = xp+\frac{1}{p+1} \rightarrow (1)$$

General solution is $y = cx+\frac{1}{c+1}$ $\rightarrow (2)$

to find singular solution,

Dp wrt c ②

$$0 = x+\left(-\frac{1}{(c+1)^2}\right)$$

$$x = \frac{1}{(c+1)^2}$$ $\qquad \sqrt{x} = \frac{1}{(c+1)}$

$$(c+1) = \pm\frac{1}{\sqrt{x}}$$

$$C = \frac{1}{\sqrt{x}} - 1$$

$$\boxed{C = \frac{1-\sqrt{x}}{\sqrt{x}}}$$

$\therefore$ (2) becomes $y = \left(\frac{1-\sqrt{x}}{\sqrt{x}}\right)x + \sqrt{x}$

$$y = \left(\frac{1-\sqrt{x}}{\cancel{\sqrt{x}}}\right)\cancel{\sqrt{x}}\sqrt{x} + \sqrt{x}$$

$$y = \sqrt{x} - x + \sqrt{x}$$

$$y = 2\sqrt{x} - x$$

$$\boxed{x + y = 2\sqrt{x}}$$

5. $(a^2 - x^2)p^2 + 2xyp + b^2 - y^2 = 0$

$$a^2p^2 - \underline{x^2p^2} + 2xyp + b^2 \underline{- y^2} = 0$$

$$a^2p^2 + b^2 - (x^2p^2 - 2xyp + y^2) = 0$$

$$a^2p^2 + b^2 - (xp - y)^2 = 0$$

$$a^2p^2 + b^2 = (xp - y)^2$$

$$\sqrt{a^2p^2 + b^2} = xp - y$$

$$y = xp - \sqrt{a^2p^2 + b^2}$$

Replace $p \to c$

$\boxed{y = xc - \sqrt{a^2c^2 + b^2}}$ is General solution

$$y = xc - \sqrt{a^2c^2 + b^2}$$

D.p w.r.t c

$$0 = x - \frac{1}{2\sqrt{a^2c^2 + b^2}}(a^2 2c) + 0$$

$$0 = x - \frac{\cancel{2}a^2c}{\cancel{2}\sqrt{a^2c^2+b^2}}$$

$$x = \frac{a^2c}{\sqrt{a^2c^2+b^2}}$$

$$x\left(\sqrt{a^2c^2+b^2}\right) = a^2c$$

SBS

$$x^2(a^2c^2+b^2) = a^4c^2$$

$$x^2a^2c^2 + x^2b^2 = a^4c^2$$

$$x^2a^2c^2 + x^2b^2 - a^4c^2 = 0$$

$$x^2a^2c^2 - a^4c^2 = -x^2b^2$$

$$c^2(x^2a^2 - a^4) = -x^2b^2$$

$$c^2 = \frac{-x^2b^2}{(x^2a^2-a^4)}$$

$$c^2 = \frac{-x^2b^2}{a^2(x^2-a^2)}$$

$$c = \frac{xb}{a\sqrt{a^2-x^2}}$$

$$\sqrt{a^2c^2+b^2} = \frac{a^2c}{x}$$

$$= \frac{a^{\cancel{2}}}{\cancel{x}}\left(\frac{\cancel{x}b}{\cancel{a}\sqrt{a^2-x^2}}\right) = \frac{ab}{\sqrt{a^2-x^2}}$$

$$\Rightarrow y = x \cdot \frac{xb}{a\sqrt{a^2-x^2}} - \frac{ab}{\sqrt{a^2-x^2}}$$

$$y = \frac{x^2b - a^2b}{a\sqrt{a^2-x^2}}$$

$$ay\sqrt{a^2-x^2} = b(x^2-a^2)$$

S.B.S

$$a^2y^2(\sqrt{a^2-x^2})^2 = b^2((x^2-a^2))^2$$
$$a^2y^2\cancel{(a^2-x^2)} = b^2(a^2-x^2)^{\cancel{2}}$$
$$a^2y^2 = b^2(a^2-x^2)$$
$$a^2y^2 = b^2a^2 - b^2x^2$$
$$a^2y^2+b^2x^2 = a^2b^2 \quad \div a^2b^2$$
$$\boxed{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1}$$

EQUATIONS REDUCIABLE TO CLAIRAULTS

1. find general solution of equation $(px-y)(py+x)=a^2p$ by reducing into clairaults form. taking substitution $X=x^2$ & $Y=y^2$

$$p=\frac{dy}{dx}=\frac{dy}{dY}\left(\frac{dY}{dX}\right)\frac{dX}{dx}$$

$$X=x^2 \qquad Y=y^2$$
$$\sqrt{X}=x \qquad \sqrt{Y}=y$$

D wrt X $\qquad$ D wrt Y

$$\frac{dx}{dX}=\frac{1}{2\sqrt{X}} \qquad \frac{dy}{dY}=\frac{1}{2\sqrt{Y}}$$

let $\frac{dY}{dX}=P$

$$p=\frac{1}{\cancel{2\sqrt{X}}}P\,\cancel{2}\sqrt{X} \Rightarrow p=\frac{\sqrt{X}}{\sqrt{Y}}P$$

$$\Rightarrow\left(\frac{\sqrt{X}}{\sqrt{Y}}P\sqrt{X}-\sqrt{Y}\right)\left(\frac{\sqrt{X}}{\cancel{\sqrt{Y}}}P\cancel{\sqrt{Y}}+\sqrt{X}\right)=a^2\frac{\sqrt{X}}{\sqrt{Y}}P$$

$$\left(\frac{XP-Y}{\sqrt{Y}}\right)(\sqrt{X}P+\sqrt{X})=a^2\frac{\sqrt{X}}{\sqrt{Y}}P$$

$$(xp-y)\sqrt{x}(p+1) = a^2\sqrt{x}p$$

$$(xp-y)(p+1) = a^2 p$$

$$xp - y = \frac{a^2 p}{p+1}$$

$$y = xp - \frac{a^2 p}{p+1}$$

Replace $p \to c$

$$y = xc - \frac{a^2 c}{c+1}$$

$$\boxed{y^2 = x^2 c - \frac{a^2 c}{c+1}}$$

2. Solve $e^{4x}(p-1) + e^{2y}p^2 = 0$ by using the substitution $u = e^{2x}$ $v = e^{2y}$

$u = e^{2x}$ $\qquad v = e^{2y}$

$$p = \frac{dy}{dx} = \frac{dy}{dv}\,\frac{dv}{du}\,\frac{du}{dx}$$

$u = e^{2x}$ $\qquad v = e^{2y}$

dwrt u $\qquad$ dwrt v

$1 = e^{2x}\frac{dx}{du}$ $\qquad 1 = e^{2y}(2)\frac{dy}{dv}$

$\boxed{\frac{du}{dx} = 2e^{2x}}$ $\qquad \boxed{\frac{dv}{dy} = \frac{1}{2e^{2y}}}$

$p = \frac{1}{\cancel{2}e^{2y}} \cdot \cancel{2}e^{2x} P$ $\qquad P = \frac{u}{v}P$

$e^{4x} = u^2$

$\therefore$ ① becomes

$$u^2\left(\frac{u}{v}p-1\right)+v\frac{u^2}{v^2}p^2=0$$

$$\frac{u^2(up-v)}{v}+\frac{u^2p^2}{v}=0$$

$$\frac{u^2(up-v)+u^2p^2}{v}=0$$

$$u^2(up-v)+u^2p^2=0$$

$$u^2[(up-v)+p^2]=0$$

$$up-v+p^2=0$$

$$\boxed{v=up+p^2}$$

$$v=cu+p^2$$

$e^{2y}=e^{2x}c+c^2$ is general solution

3. Solve $x^2(y-px)=p^2y$ by reducing to the clairaults form using substitution $X=x^2$ & $Y=y$

$$x^2(y-px)=p^2y \quad \text{—①}$$

$$X=x^2 \qquad Y=y^2$$

$$p=\frac{dy}{dx}=\frac{dy}{dY}\,\frac{dY}{dX}\,\frac{dX}{dx}$$

$$X=x^2 \qquad Y=y^2$$

$$\sqrt{X}=x \qquad \sqrt{Y}=y$$

d w r to X $\qquad$ d wrt Y

$$\frac{dx}{dX}=\frac{1}{2\sqrt{X}} \qquad \frac{dy}{dY}=\frac{1}{2\sqrt{Y}}$$

let $\frac{dX}{dY}=P$

$$p=\left(\frac{1}{\cancel{2}\sqrt{Y}}\right)P\,(\cancel{2}\sqrt{X})$$

$$P = \frac{\sqrt{X}}{\sqrt{Y}} p$$

(1) becomes $X(\sqrt{Y} - P\sqrt{X}) = P^2(\sqrt{Y})$

$$X\left(\sqrt{Y} - \frac{\sqrt{X}}{\sqrt{Y}} p\sqrt{X}\right) = \left(\frac{\sqrt{X}}{\cancel{\sqrt{Y}}}\right)^2 p^2 (\cancel{\sqrt{Y}})$$

$$\cancel{X}\left(Y - \frac{XP}{\cancel{\sqrt{Y}}}\right) = P^2 \frac{\cancel{X}}{\cancel{\sqrt{Y}}}$$

$$Y - XP = P^2$$

$Y = P^2 + XP$ Replace $P \to C$

$$\boxed{Y = C^2 + XC}$$

$y^2 = c^2 + x^2 c$ is general solution

d w r to C

$$0 = 2c + x^2$$

$$-2c = x^2$$

$$\boxed{c = -\frac{x^2}{2}}$$

$$y^2 = x^2\left(-\frac{x^2}{2}\right) + \cancel{2}\left(\frac{-x^2}{\cancel{2}}\right)^2$$

$$y^2 = -\frac{x^4}{2} + \frac{x^4}{4}$$

$$y^2 = \frac{-4x^4 + 2x^4}{8}$$

$$y^2 = \frac{-2x^4}{8}$$

$$y^2 = -\frac{x^4}{4}$$

$$\frac{x^4}{4} + y^2 = 0$$

$$\boxed{x^4 + 4y^2 = 0}$$

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