

CENTROID OF PLANE FIGURES

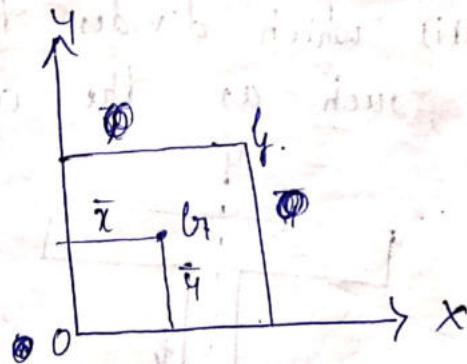
Centre of gravity:

It is the point where the whole weight of the body is assumed to be concentrated.

Centroid:

Centroid is the point where the whole area of the plane figure is assumed to be concentrated.

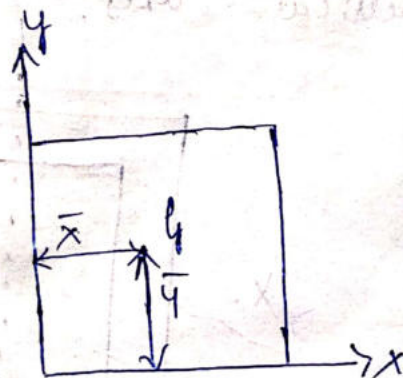
Centroid means the determination of $\bar{x}$ & $\bar{y}$ as shown in fig.



The difference between centre of gravity & centroid is the term centre of gravity applies to bodies with mass and weight, and centroid applies to plane areas.

Axes of reference:

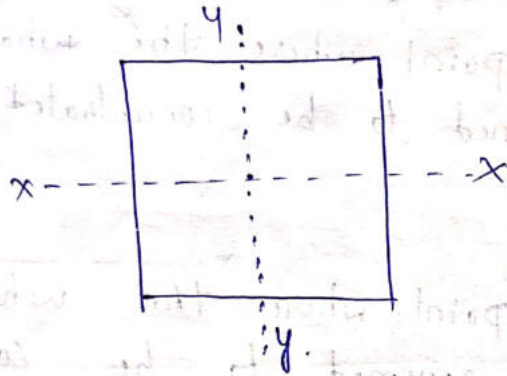
These



These are the axes with respect to which the centroid of a given fig. is determined.

Centroidal Axis :-

The axis which passes through the centroid of the given fig. is known as centroidal axis, such as the axis $x-x$ and the axis $y-y$ shown in fig.



Symmetrical Axis :-

It is the axis which divides the whole figure into equal parts, such as the axis $x-x$ and the axis $y-y$.

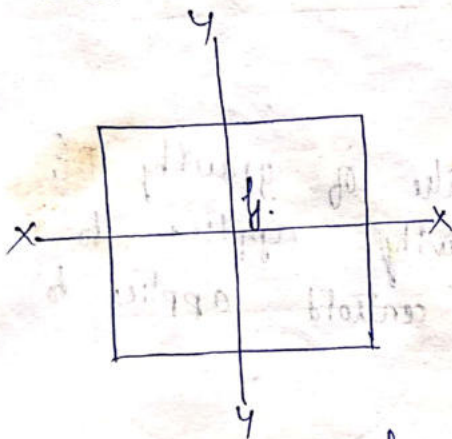


Fig (a)

Symmetrical axis.

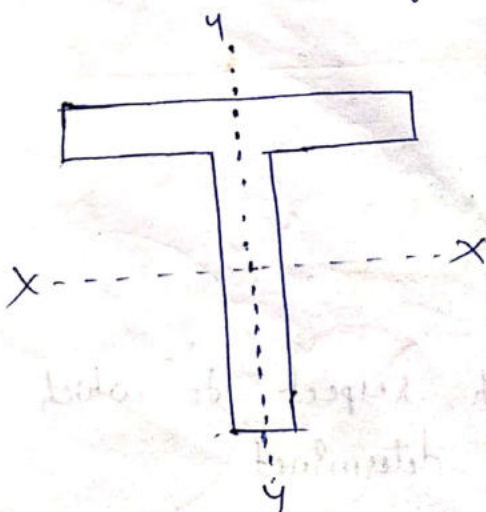
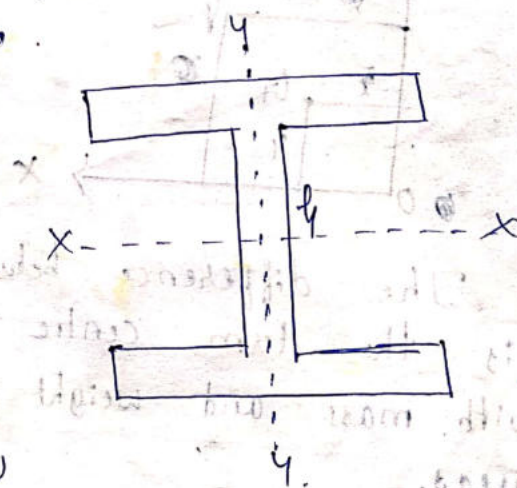
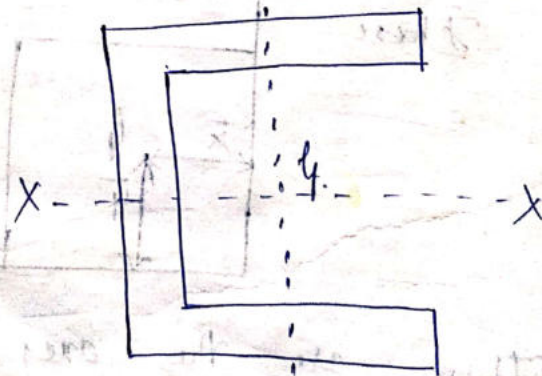


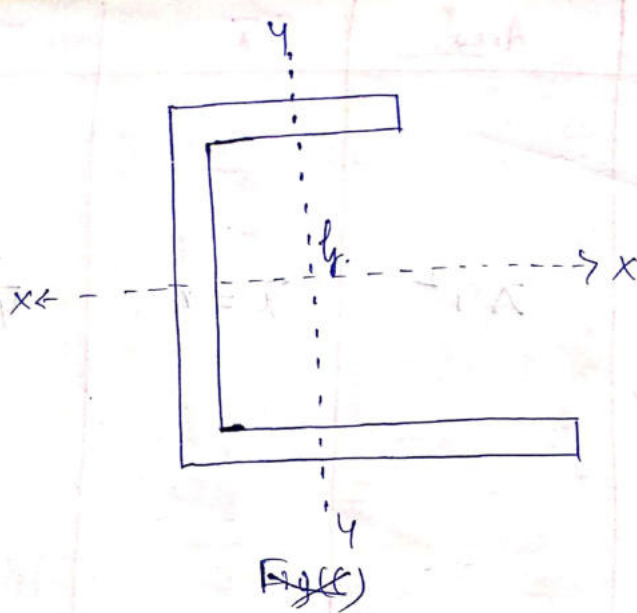
Fig (b)

Symmetry about the $y-y$ axis



Symmetry about the $x-x$ axis

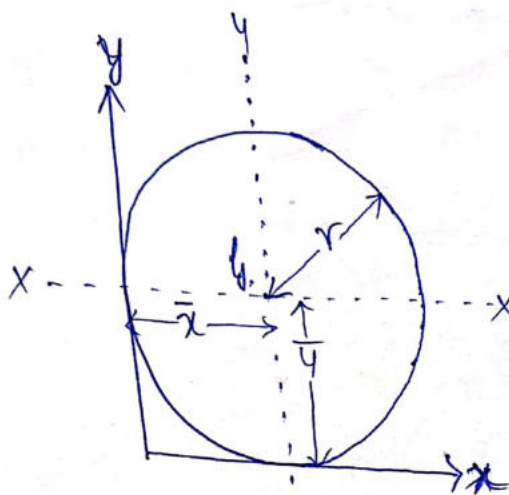
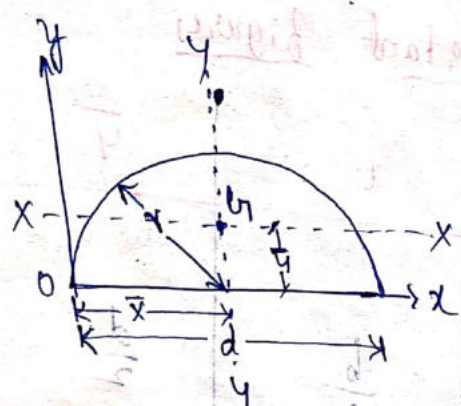
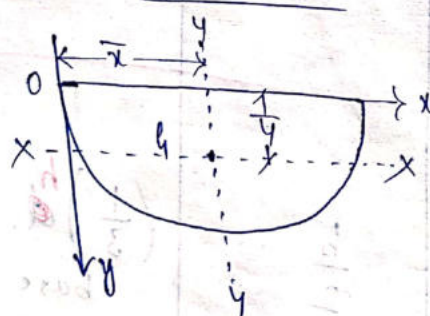
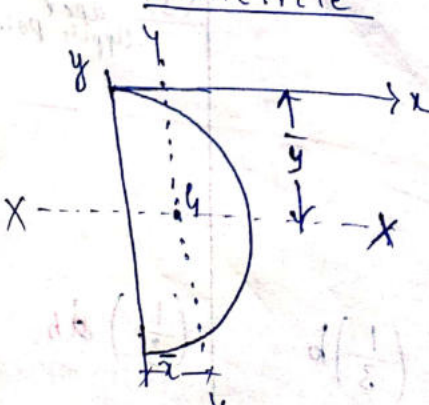
Q.



Neither the $x-x$ axis nor the $y-y$ axis is the axis of symmetry

Centroids of some Important figures

Shape.	Area	$\bar{x}$	$\bar{y}$
Rectangle.	bd	$\frac{b}{2}$	$\frac{d}{2}$
Triangle.	$\frac{1}{2} b h$	$\frac{b}{2}$	$\left(\frac{1}{3}\right) h$ from base $\left(\frac{2}{3}\right) h$ from apex upper point
Right-angled triangle.	$\frac{1}{2} b h$	$\left(\frac{1}{3}\right) b$	$\left(\frac{1}{3}\right) h$

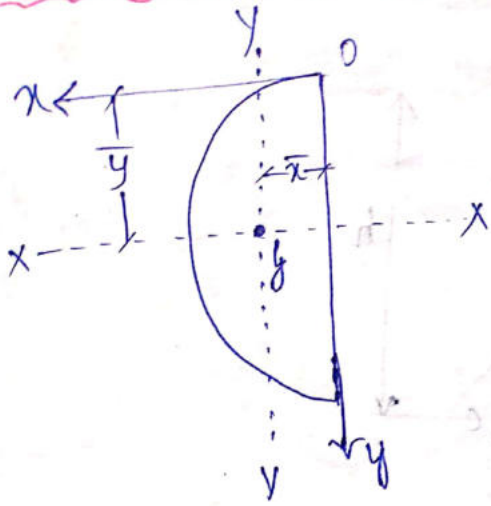
Shape	Area	$\bar{x}$	$\bar{y}$
Circle. 	πr^2	$\bar{x} = r$	$\bar{y} = r$
Semicircle. 	$\frac{4r}{3\pi}$ $\frac{\pi r^2}{2}$	$\frac{d}{2}$	$\frac{4r}{3\pi}$
Semicircle 	$\frac{\pi r^2}{2}$	$\frac{d}{2}$	$-\frac{4r}{3\pi}$
Semicircle 	$\frac{\pi r^2}{2}$	$\frac{4r}{3\pi}$	$-\frac{d}{2}$

Shape

Area

$\bar{x}$

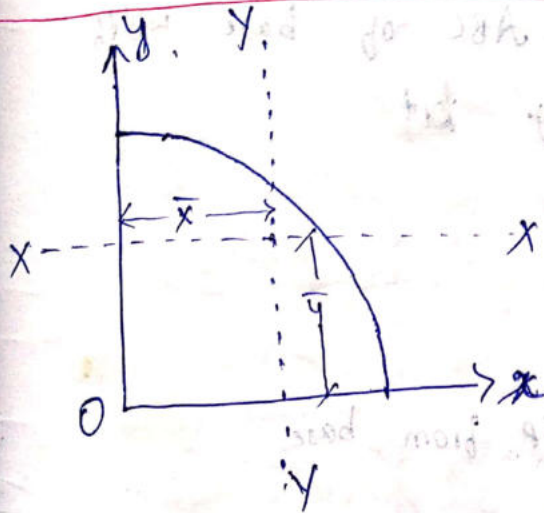
$\bar{y}$



$$\frac{\pi r^2}{2}$$

$$-\frac{4r}{3\pi}$$

$$-\frac{d}{2}$$



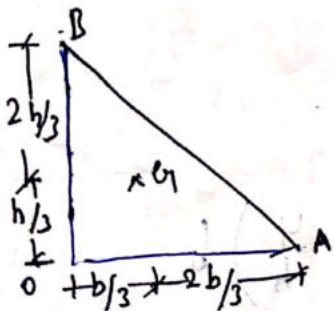
$$\frac{\pi r^2}{4}$$

$$\frac{4r}{3\pi}$$

$$\frac{4r}{3\pi}$$

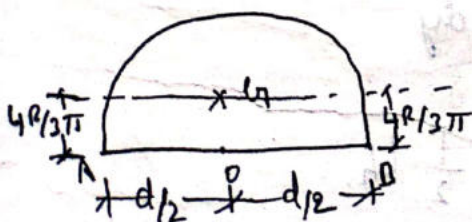
NOTE:-

(i)



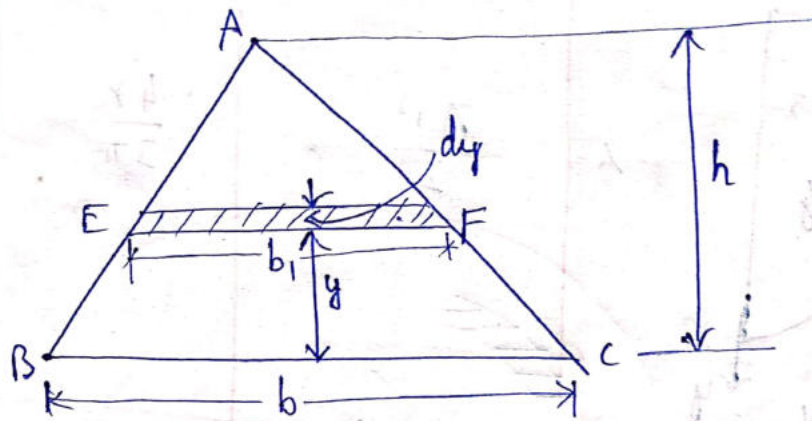
For the right angled Δ centroid always lies at a distance of $b/3$ or $h/3$ from the point where two $\perp$ lines intersect (Point O), and at a distance of $2h/3$ or $2b/3$ from other two points (Point A & B).

(ii)



For the semicircle centroid always lies at a distance of $\frac{4r}{3\pi}$ from the diametric axis, & at a distance of $d/2$ from either points (A & B).

Centroid of triangle :-



Consider the triangle ABC of base width 'b' and height 'h' as shown in fig.

Let

b_1 = Width of elemental strip

dy = thickness of elemental strip

y = distance of elemental strip from base

Since $\triangle AEF$ & $\triangle ABC$ are the similar triangles,

$$\frac{b_1}{b} = \frac{h-y}{h}$$

$$b_1 = \left(\frac{h-y}{h} \right) b = \left(1 - \frac{y}{h} \right) b$$

Area of the element = $dA = b_1 dy$

$$dA = \left(1 - \frac{y}{h} \right) b dy$$

Area of the triangle $A = \frac{1}{2} bh$

Moment of area of strip about BC = $\text{Area of strip} \times y$

$$= \left(1 - \frac{y}{h} \right) b dy \times y$$

$$\bar{y} = \frac{\text{Moment of area}}{\text{Total area}} = \frac{\int y dA}{A}$$

Moment of elemental strip about base = $dA \cdot y$

∴ Moment of whole area = $\int_0^h y dA = \int_0^h y \left(1 - \frac{y}{h}\right) b dy$

$$= \int_0^h \left(y - \frac{y^2}{h}\right) b dy$$

$$= b \left[\frac{y^2}{2} - \frac{y^3}{3h} \right]_0^h$$

$$= b \left[\left(\frac{h^2}{2} - \frac{h^3}{3h} \right) - 0 \right]$$

$$= b \left[\frac{h^2}{2} - \frac{h^2}{3} \right]$$

$$= b \left[\frac{3h^2 - 2h^2}{6} \right] = b \left(\frac{h^2}{6} \right)$$

$$= b \left[\frac{h^3}{6h} \right] \quad \int y dA = \frac{bh^3}{6}$$

$$\bar{y} = \frac{\int y dA}{A}$$

$$= \frac{\frac{bh^3}{6}}{\frac{bh}{2}}$$

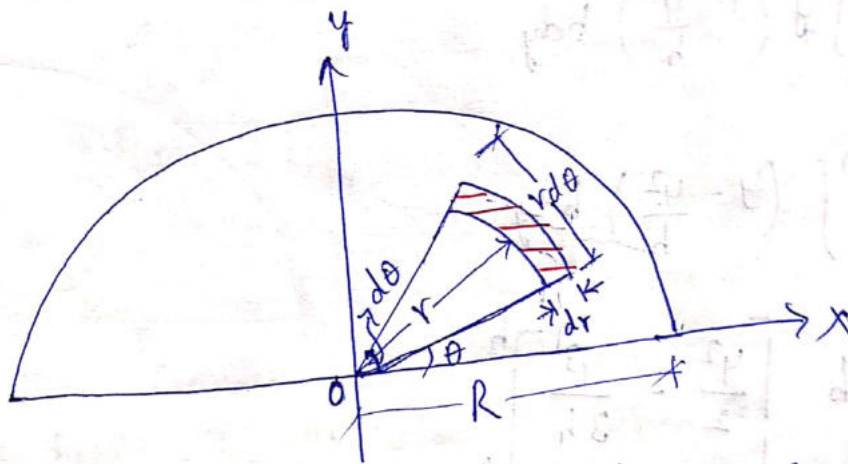
$$= \frac{bh^3}{6} \times \frac{2}{bh}$$

$$\boxed{\bar{y} = \frac{h}{3}}$$

Thus distance of centroid is at a distance of $\frac{h}{3}$ from the base (or) $\frac{2h}{3}$ from the apex.

Centroid of semicircle

Consider the semicircle of radius



Consider the semicircle of radius 'R' as shown in fig. Due to symmetry centroid must lie on y-axis. Let its distance from diametral axis be $\bar{y}$. To find $\bar{y}$ consider an elemental strip at a distance 'r' from the centre 'O' of the semicircle, radial width being dr and bound by radii θ & $\theta + d\theta$.

The elemental area may be treated as a rectangle of sides $r d\theta$ & dr .

$$\therefore \text{Area of element} = r d\theta \cdot dr$$

Moment of area of element about x-axis is

$$= r d\theta \cdot dr \times r \sin \theta$$

$$= r^2 \sin \theta dr d\theta.$$

$\therefore$ Total moment of area about diametral axis,

$$= \int_0^\pi \int_0^R r^2 \sin \theta dr d\theta.$$

$$= \int_0^\pi \left[\frac{r^3}{3} \right]_0^R \sin \theta d\theta$$

$$\left[\frac{r^3}{3} \right]_0^R = \left[\frac{r^3}{3} \right]_0^R$$

$$= \int_0^{\pi} \frac{R^3}{3} \sin \theta d\theta$$

$$= \frac{R^3}{3} \int_0^{\pi} \sin \theta d\theta$$

$$= \frac{R^3}{3} [-\cos \theta]_0^{\pi} = -\frac{R^3}{3} [\cos \theta]_0^{\pi}$$

$$= -\frac{R^3}{3} [(\cos \pi) - (\cos 0)]$$

$$= -\frac{R^3}{3} [-1 - 1]$$

$$= \frac{2R^3}{3}$$

Area of semicircle $A = \frac{1}{2} \pi R^2$

$$\bar{y} = \frac{\text{Moment of area}}{\text{Total area.}}$$

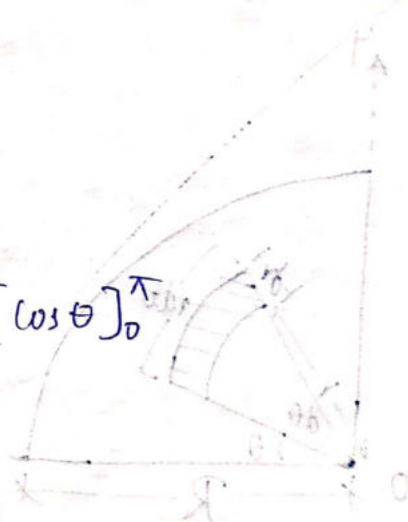
$$\bar{y} = \frac{\frac{2R^3}{3}}{\frac{\pi R^2}{2}}$$

$$\bar{y} = \frac{2R^3}{3} \times \frac{2}{\pi R^2}$$

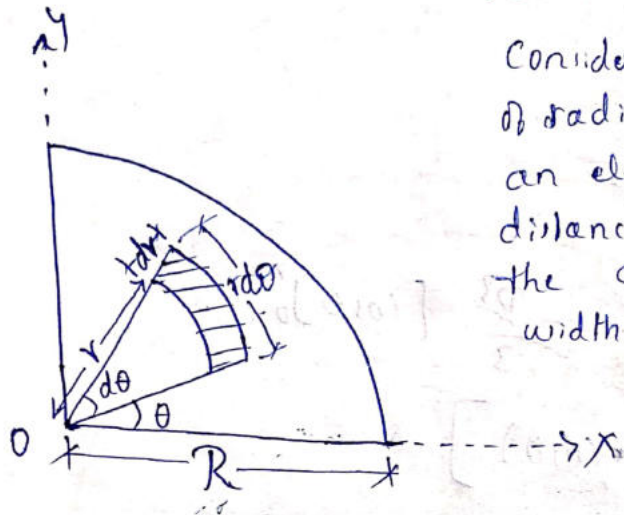
$$\boxed{\bar{y} = \frac{4R}{3\pi}}$$

Thus, the centroid of the ^{semi}circle is at a distance

$\frac{4R}{3\pi}$ from the diametrical axis.



Centroid of Quadrant of a circle



Consider the quadrant of a circle of radius R as shown in fig. Consider an elemental strip at a radial distance r from the centre O of the quadrant of the circle, radial width being dr & bound by θ & $\theta + d\theta$.

Moment about diametral x -axis is given by $\int_0^{\pi/2} \int_0^R r^2 \sin \theta dr d\theta$

[Note! Change from that of a semicircle is only in the limit of integration for θ].

$$\int_0^{\pi/2} \int_0^R r^2 \sin \theta dr d\theta$$

$$= \int_0^{\pi/2} \left[\frac{r^3}{3} \right]_0^R \sin \theta d\theta$$

$$= \frac{R^3}{3} \int_0^{\pi/2} \sin \theta d\theta$$

$$= \frac{R^3}{3} [-\cos \theta]_0^{\pi/2}$$

$$= -\frac{R^3}{3} \left[\cos \frac{\pi}{2} - \cos 0 \right]$$

$$= -\frac{R^3}{3} [0 - 1]$$

$$= \frac{R^3}{3}$$

Area of quadrant of a circle = $\frac{1}{4} \pi R^2$

$$\bar{y} = \frac{\text{Moment of area}}{\text{Total area.}}$$

$$\bar{y} = \frac{\frac{R^3}{3}}{\frac{\pi R^2}{4}}$$

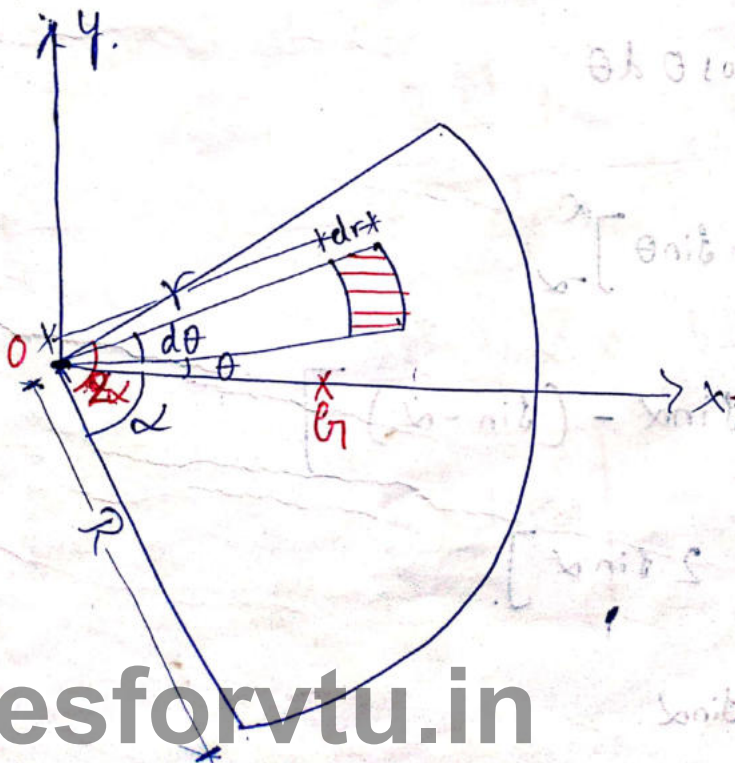
$$\bar{y} = \frac{R^3}{3} \times \frac{4}{\pi R^2}$$

$$\boxed{\bar{y} = \frac{4R}{3\pi}}$$

Similarly by taking moment of area about y-axis

$$\boxed{\bar{x} = \frac{4R}{3\pi}}$$

Centroid of Sector of a Circle



Consider the sector of a circle of angle 2α as shown in fig. Due to symmetry centroid lies on x-axis. To find its distance from the centre 'O', consider the elemental area shown.

$$\text{Area of the elemental strip} = r d\theta \cdot dr$$

Moment of area of the elemental strip about y-axis.

$$= r d\theta \cdot dr \cdot r \cos\theta$$

$$= r^2 \cos\theta \cdot dr d\theta$$

$\therefore$ Total moment of area about y-axis

$$= \int_{-\alpha}^{\alpha} \int_0^R r^2 \cos\theta dr d\theta$$

$$= \int_{-\alpha}^{\alpha} \left[\frac{r^3}{3} \right]_0^R \cos\theta d\theta$$

$$= \left[\frac{R^3}{3} \right]_{-\alpha}^{\alpha} \cos\theta d\theta$$

$$= \frac{R^3}{3} \left[+\sin\theta \right]_{-\alpha}^{\alpha}$$

$$= +\frac{R^3}{3} \left[\sin\alpha - (\sin - \alpha) \right]$$

$$= +\frac{R^3}{3} \left[2 \sin\alpha \right]$$

Total area of the sector

$$= \int_{-\alpha}^{\alpha} \int_0^R r d\theta \cdot dr$$

$$= \int_{-\alpha}^{\alpha} \left[\frac{r^2}{2} \right]_0^R d\theta$$

$$= \frac{R^2}{2} \int_{-\alpha}^{\alpha} d\theta$$

$$= \frac{R^2}{2} [\theta]_{-\alpha}^{\alpha}$$

$$= \frac{R^2}{2} [\alpha - (-\alpha)]$$

$$= \frac{2R^2\alpha}{2}$$

$$= \underline{\underline{R^2\alpha}}$$

The distance of centroid from centre 'O'

$$= \frac{\text{Moment of area about y-axis}}{\text{Area of the fig.}}$$

$$= \frac{\frac{2R^3}{3} \sin\alpha}{\frac{R^2\alpha}{1}}$$

$$= \frac{2R^3}{3} \sin\alpha \times \frac{1}{R^2\alpha}$$

$$= \frac{2R}{3} \sin\alpha$$

Centroid from 'O' = $\frac{2R \sin\alpha}{3\alpha}$

Centre of gravity of a plane fig.

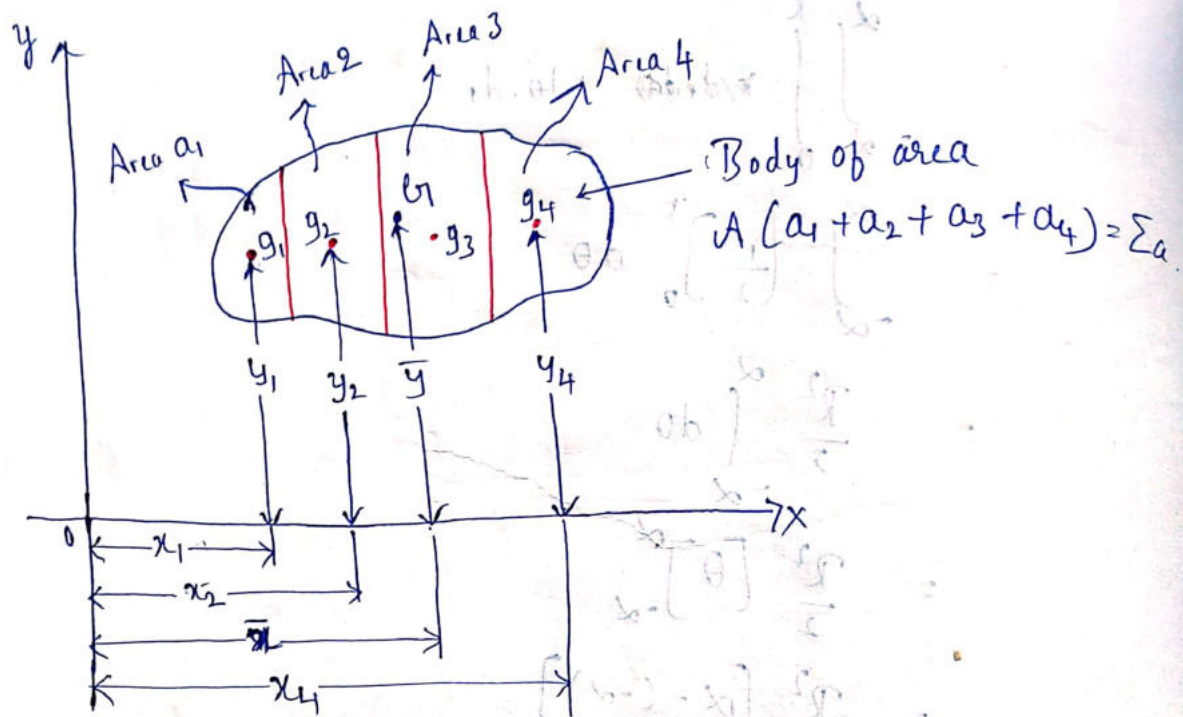


Fig. shows a plane figure of total area A and unit thickness. Let 'y' be the C.G. of whole area with coordinates $(\bar{x}, \bar{y})$. Divide this area 'A' into a number of small areas a_1, a_2, a_3 & a_4 with centroids g_1, g_2, g_3 & g_4 respectively.

$$A = \Sigma a = a_1 + a_2 + a_3 + a_4 \longrightarrow (1)$$

Let $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ & (x_4, y_4) be their co-ordinates respectively.

Moment of the total area 'A' about 'y' axis.

$$A \bar{x} = \Sigma a \bar{x} \longrightarrow (2)$$

From varignon's theorem.

Sum of moments of all areas about

$$A \bar{x} = a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4$$

$$\Sigma a \cdot \bar{x} = \Sigma a x \longrightarrow (3)$$

$$\Sigma a \cdot \bar{x} = \Sigma a x$$

$$\boxed{\bar{x} = \frac{\Sigma a x}{\Sigma a}}$$

∴ y axis is equal to the moment of whole area about the same axis.

Steps in solving the problem:-

- * Identify whether the given fig. is symmetric about x -axis & y -axis.
- * If the given fig. is symmetric about y -axis then $\bar{x}$ is known. & if the fig. is symmetric about x -axis then $\bar{y}$ is known.
- * Draw the reference axis.
- * Divide the given fig. into a number of plane areas (Δ^k , $\square^k$, semicircle, quadrant of a circle) respectively.
- * Enter the values in the tabular column given below.

a.) If the given fig. is symmetric about only y -axis & not symmetric about x -axis then

Component	Area (a)	Centroidal dist. from x -axis (y)	Centroidal dist. from y -axis (x)
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$$\bar{y} = \frac{\sum ay}{\sum a}$$

b.) If the fig. is symmetric about only x -axis & not symmetric about y -axis then

Component	Area (a)	Centroidal dist. from y -axis (x)	Centroidal dist. from x -axis (y)
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$$\bar{x} = \frac{\sum ax}{\sum a}$$

c.) If the fig. is not symmetric about both x & y axis.

Component	Area (a)	Centroidal dist. from y -axis (x)	Centroidal dist. from x -axis (y)	$\bar{x}$	$\bar{y}$
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$$\bar{x} = \frac{\sum ax}{\sum a}, \quad \bar{y} = \frac{\sum ay}{\sum a}$$

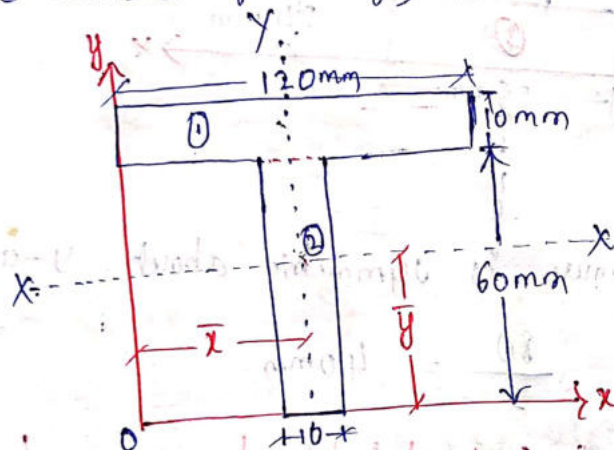
Similarly,

$$\sum a\bar{y} = \sum ay$$

$$\bar{y} = \frac{\sum ay}{\sum a}$$

Problems:-

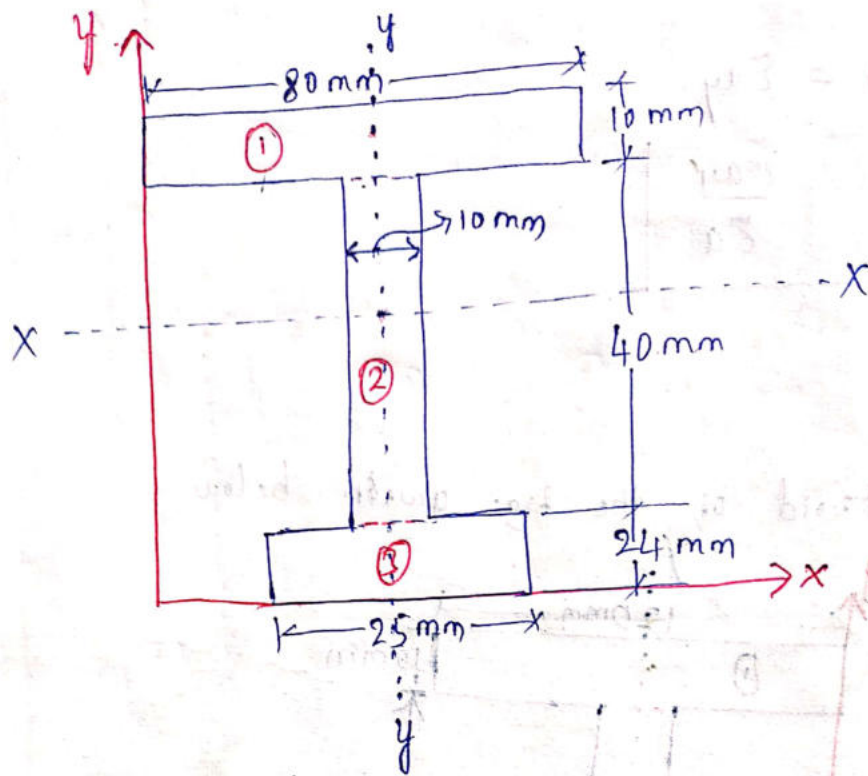
1. Find the centroid of the fig. given below



Component	Area, a (mm^2)	Centroidal distance from y-axis ($\bar{x}$)	Centroidal distance from x-axis ($\bar{y}$)	$a\bar{x}$	$a\bar{y}$
Rectangle ①	$A = b \times d$ $120 \times 10 = 1200$	$\bar{x} = b/2$ $\frac{120}{2} = 60$	$y = d/2$ $60 + \frac{10}{2} = 65$	72000	78000
Rectangle ②	$10 \times 60 = 600$	$\frac{120}{2} = 60$	$\frac{60}{2} = 30$	36000	18000
Sum	$\sum a = 1800$			108000	96000
	$\bar{x} = \frac{\sum a\bar{x}}{\sum a} = \frac{108000}{1800} = 60 \text{ mm}$ $\bar{y} = \frac{\sum a\bar{y}}{\sum a} = \frac{96000}{1800} = 53.33 \text{ mm}$				

NOTE:- Given figure is symmetric about y-axis, therefore we can write directly $\bar{x} = \frac{120}{2} = 60 \text{ mm}$

2-5 Find the centroid for the fig. given below.



Soln:- The given figure is symmetric about $\bar{Y}$ -axis,

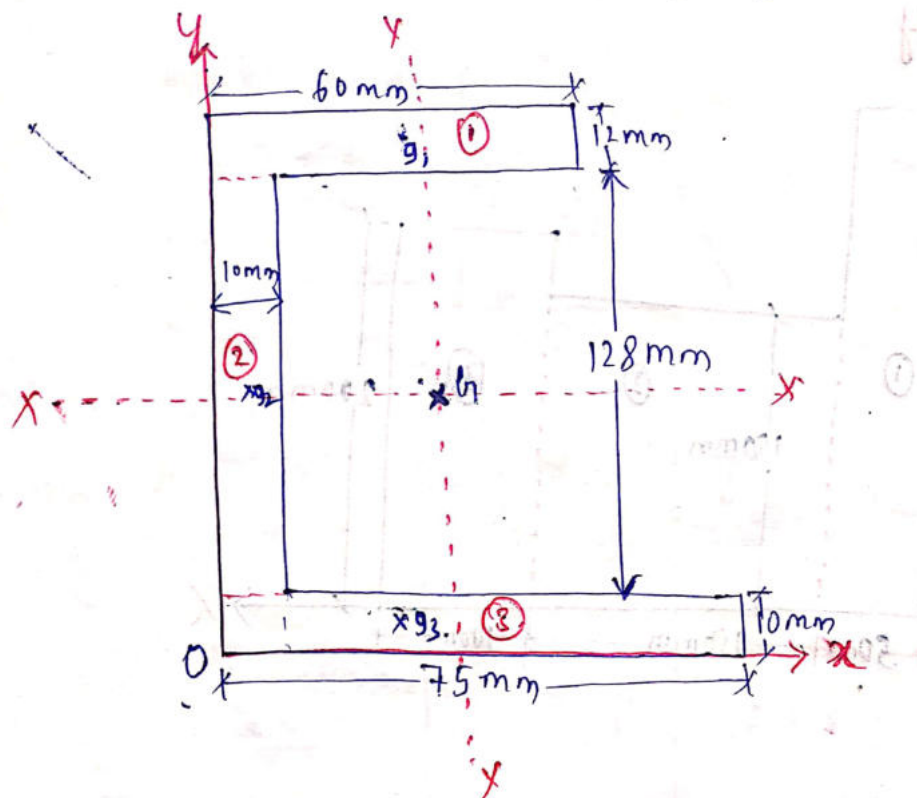
$$\text{Hence } \bar{x} = \frac{80}{2} = 40 \text{ mm.}$$

Component	Area, a (mm ²)	Centroidal distance from $\bar{x}$ -axis (y)	ay
Rectangle ①	$10 \times 80 = 800$	$24 + 40 + \frac{10}{2} = 69$	55,200
Rectangle ②	$10 \times 40 = 400$	$24 + \frac{40}{2} = 44$	17,600
Rectangle ③	$25 \times 24 = 600$	$\frac{24}{2} = 12$	7,200
$\Sigma a = 1800$			$\Sigma ay = 80,000$

$$\bar{y} = \frac{80,000}{1800} = 44.44 \text{ mm}$$

$$\bar{x} = 40 \text{ mm}$$

3.} Determine the centroid for the fig. given below.

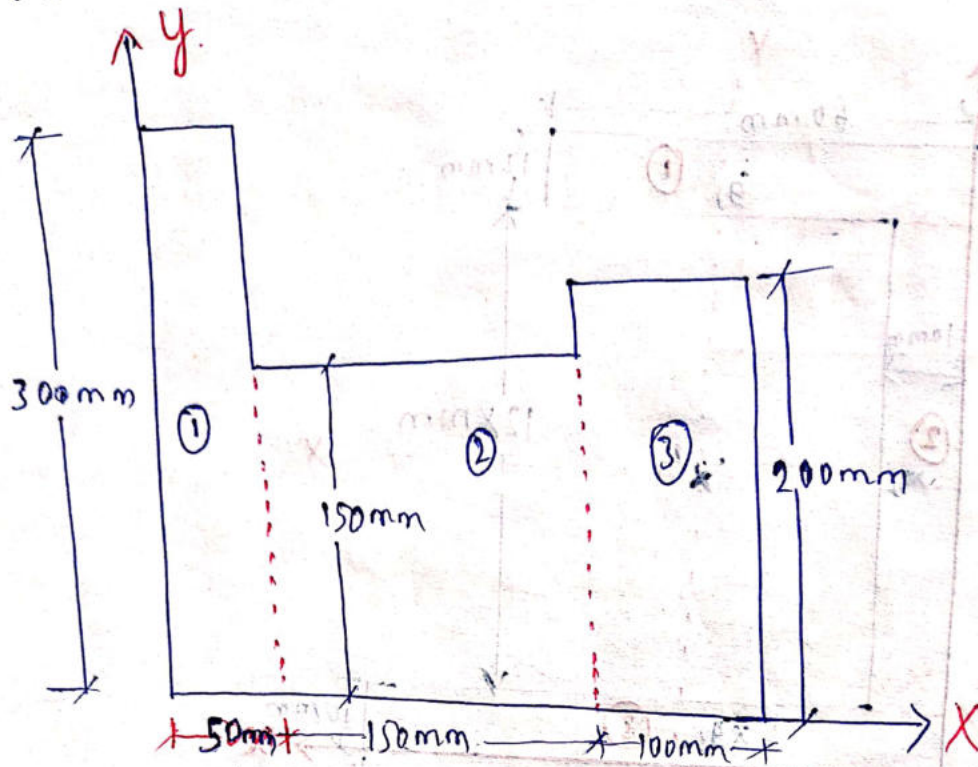


Component	Area, a mm^2	x	y	ax	ay
$\square^k ①$	$60 \times 12 = 720$	$\frac{60}{2} = 30$	$10 + \frac{128}{2} + \frac{12}{2} = 144$	21600	103680
$\square^k ②$	$10 \times 128 = 1280$	$\frac{10}{2} = 5$	$10 + \frac{128}{2} = 74$	6400	94720
$\square^k ③$	$75 \times 10 = 750$	$\frac{75}{2} = 37.5$	$\frac{10}{2} = 5$	28125	3750
Σ	$\Sigma a = 2750$			$\Sigma ax = 56125$	$\Sigma ay = 202150$

$$\bar{x} = \frac{\Sigma ax}{\Sigma a} = \frac{56125}{2750} = 20.4 \text{ mm}$$

$$\bar{y} = \frac{\Sigma ay}{\Sigma a} = \frac{202150}{2750} = 73.5 \text{ mm}$$

4.} Determine the centroid of for the fig. given below.



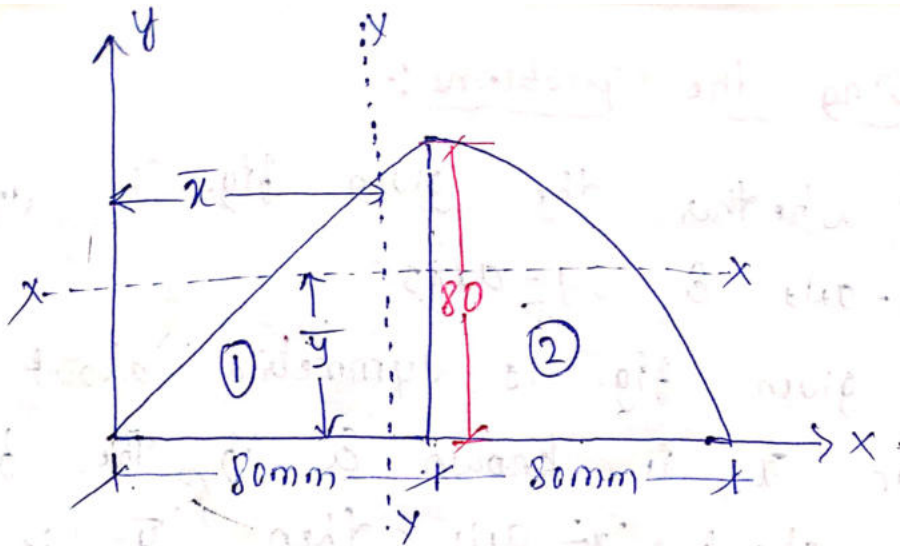
Soln:

Component No.	Area (a) mm ²	x	y	ax	ay
1	$50 \times 300 = 15000$	$\frac{50}{2} = 25$	$\frac{300}{2} = 150$	375000	2250000
2	$150 \times 150 = 22500$	$50 + \frac{150}{2} = 125$	$\frac{150}{2} = 75$	2812500	1687500
3	$100 \times 200 = 20000$	$50 + 150 + \frac{100}{2} = 250$	$\frac{200}{2} = 100$	2000000	2000000
Sum	$\Sigma a = 57500$			$\Sigma ax = 8187500$	$\Sigma ay = 5937500$

$$\bar{x} = \frac{\Sigma ax}{\Sigma a} = \frac{8187500}{57500} = 142.4 \text{ mm}$$

$$\bar{y} = \frac{\Sigma ay}{\Sigma a} = \frac{5937500}{57500} = 103.26 \text{ mm}$$

5.8



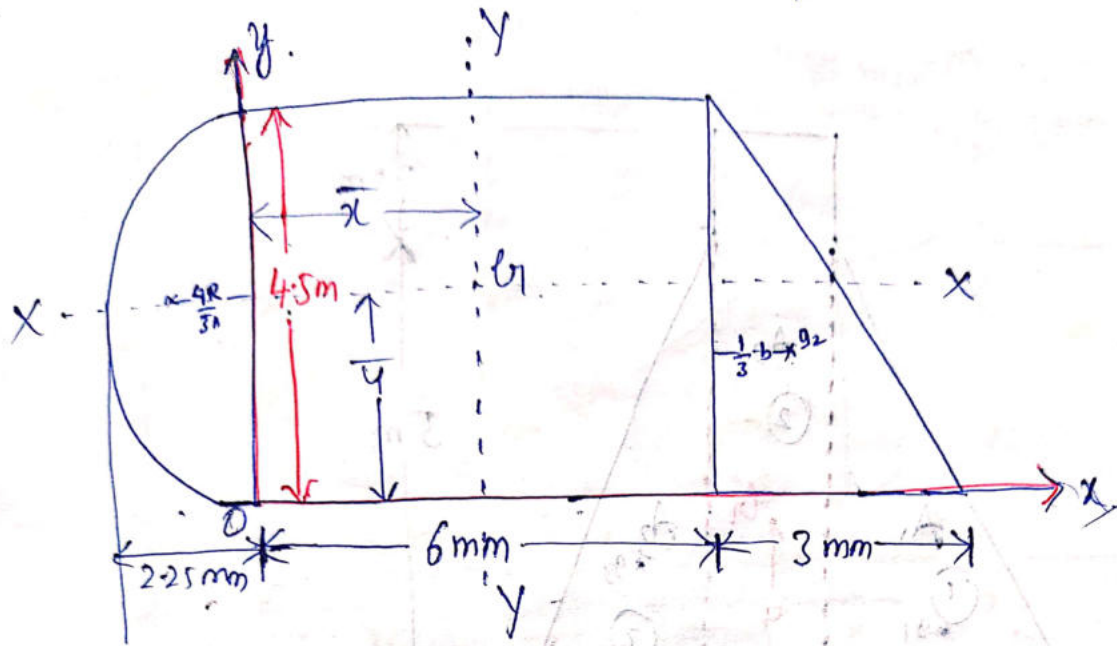
Component	Area, (a) mm ²	$\bar{x}$	$\bar{y}$	$a\bar{x}$	$a\bar{y}$
Δ ^{1e} ①	$\frac{1}{2} \times 80 \times 80$ = 3200	$\frac{2}{3} \times 80$ = 53.33	$\frac{1}{3} \times 80$ = 26.67	170656.6	85333.33
Quarter circle ②	$\frac{\pi r^2}{4} = \frac{\pi \times 80^2}{4}$ = 5026.55	$80 + \frac{4 \times 80}{3 \times \pi}$ = 113.953	$\frac{4 \times 80}{3 \times \pi}$ = 33.953	572790.45 170666.45	170666.72
	$\Sigma a = 8226.55$		$\Sigma \bar{x}$	743447.05 = 341329.05	$\Sigma a\bar{y} =$ 256000.78

$$\bar{x} = \frac{\Sigma a\bar{x}}{\Sigma a} = \frac{743447.05}{8226.55}$$

$$\bar{x} = 90.373 \text{ mm}$$

$$\bar{y} = \frac{\Sigma a\bar{y}}{\Sigma a} = \frac{256000.78}{8226.55} = 31.119 \text{ mm}$$

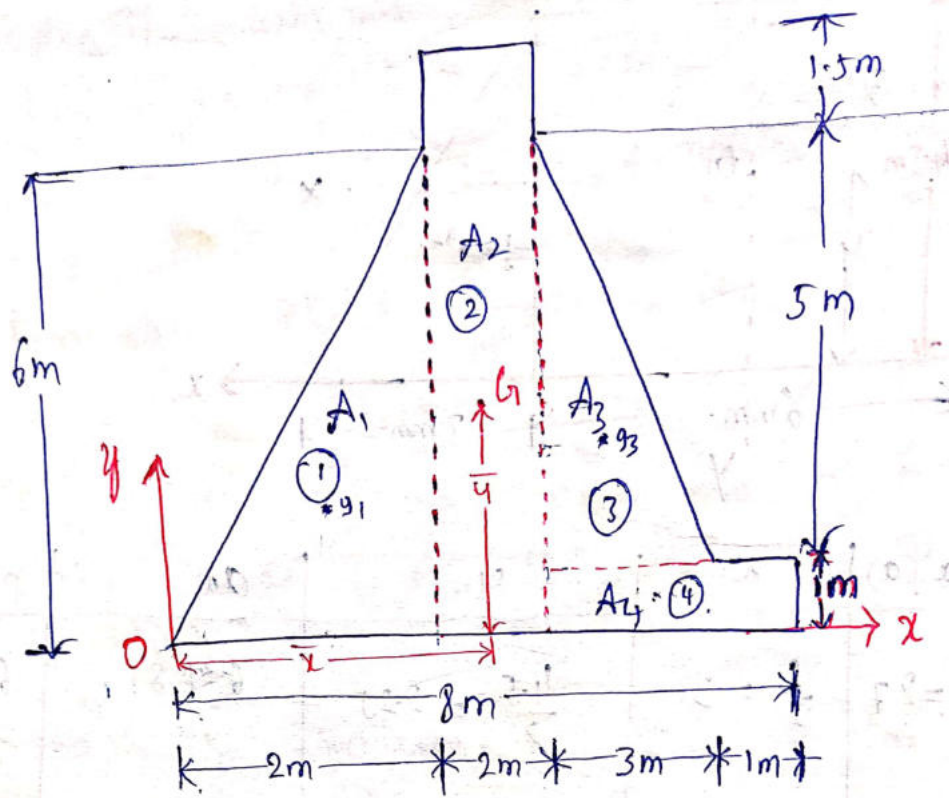
6) Calculate the centroid of fig. given below with to 'O'.



Component	Area (a)	x	y	ax	ay
Rectangle	$6 \times 4.5 = 27$	$\frac{6}{2} = 3$	$\frac{4.5}{2} = 2.25$	60.75	60.75
△	$\frac{1}{2} \times 3 \times 4.5 = 6.75$	$6 + \frac{1}{3} \times 3 = 7$	$\frac{1}{3} \times 4.5 = 1.5$	47.25	10.125
Semicircle	$\frac{\pi \times (2.25)^2}{2} = 7.95$	$\frac{4 \times 2.25}{3\pi} = 0.955$	$\frac{4.5}{2} = 2.25$	-7.59	17.88
$\Sigma a = 41.7$				$\Sigma ax = 120.66$	$\Sigma ay = 88.755$

$$\bar{x} = \frac{\Sigma ax}{\Sigma a} = \frac{120.66}{41.7} = 2.893 \text{ mm}$$

$$\bar{y} = \frac{88.755}{41.7} = 2.129 \text{ mm}$$

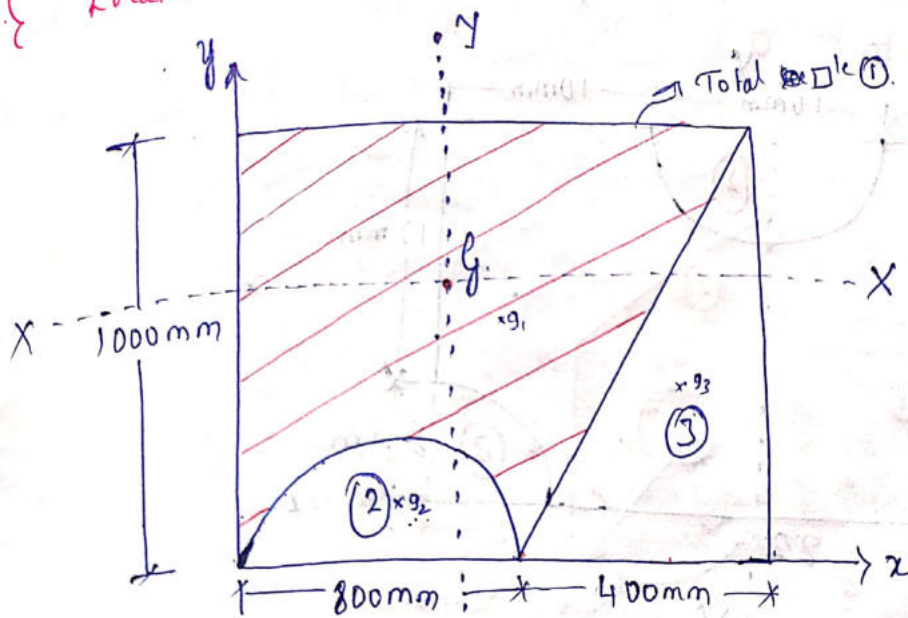


Component	Area (a)	x	y	ax	ay
Δ^c ①	$\frac{1}{2} \times 2 \times 6 = 6$	$\frac{2}{3} \times 6 = 4$ $= \frac{2}{3} \times 2 = 1.33m$	$\frac{1}{3} \times 6 = 2m$	7.98	12
$\square^c$ ②	$2 \times (1.5 + 5 + 1) = 15$	$2 + \frac{2}{2} = 3m$	$\frac{1.5 + 5 + 1}{2} = 3.75$	45	56.25
Δ^c ③	$\frac{1}{2} \times 3 \times 5 = 7.5$	$2 + 2 + \frac{1}{3} \times 3 = 5m$	$1 + \frac{1}{3} \times 5 = 2.67$	37.5	20.025
$\square^c$ ④	$4 \times 1 = 4$	$2 + 2 + \frac{4}{2} = 6$	$\frac{1}{2} = 0.5$	24	2
$\Sigma a = 32.5$				$\Sigma ax = 114.5$	$\Sigma ay = 90.275$

$$\bar{x} = \frac{\Sigma ax}{\Sigma a} = \frac{114.5}{32.5} = 3.52m$$

$$\bar{y} = \frac{\Sigma ay}{\Sigma a} = \frac{90.275}{32.5} = 2.77m$$

7.1 Locate the centroid of the shaded area shown in fig.



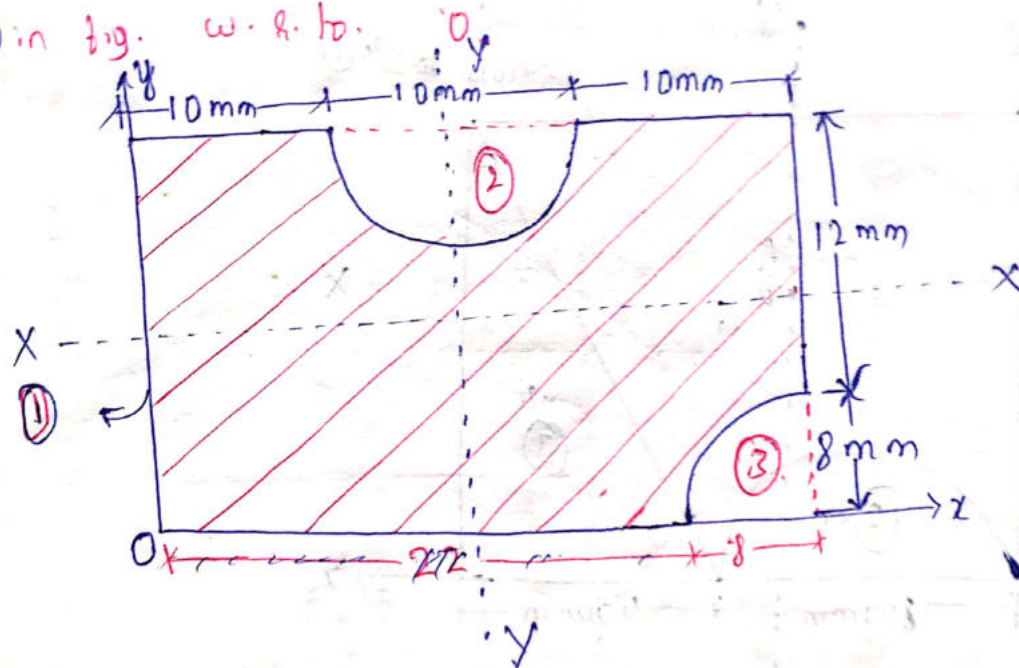
Soln:

Component	Area (a) mm ²	$x = \frac{a_1 x_1 + a_2 x_2 + a_3 x_3}{\sum a}$	$y = \frac{a_1 y_1 + a_2 y_2 + a_3 y_3}{\sum a}$	$a x$	$a y$
Rect ①	$1000 \times (800 + 400)$ $= 1200000$	$\frac{1200}{2} = 600$	$\frac{1000}{2} = 500$	720000000	600000000
Semicircle ②	$\frac{\pi R^2}{2} = \frac{\pi \times 400^2}{2}$ $= 251327.41$	$R = \frac{800}{2} = 400$	$\frac{4R}{3\pi}$ $= \frac{4 \times 400}{3 \times \pi}$ $= 169.76$	-100530964	-42665341.12
Alt ③	$\frac{1}{2} \times 400 \times 1000$ $= 200000$	$800 + \frac{2b}{3}$ $= 800 + 2 \times \frac{400}{3}$ $= 1066.67$	$\frac{h}{3} = \frac{1000}{3}$ $= 333.33$	-213334000	-66666666.6
$\sum a = 748672.6$		$\sum a x = 40613503.6$		$\sum a y = 490667992.2$	

$$\bar{x} = \frac{\sum a x}{\sum a} = \frac{40613503.6}{748672.6} = 542.473 \text{ mm}$$

$$y = \frac{\sum a y}{\sum a} = \frac{490667992.2}{748672.6} = 655.38 \text{ mm}$$

8) Determine the centre of gravity of the lamina shown in fig. w.r.to.

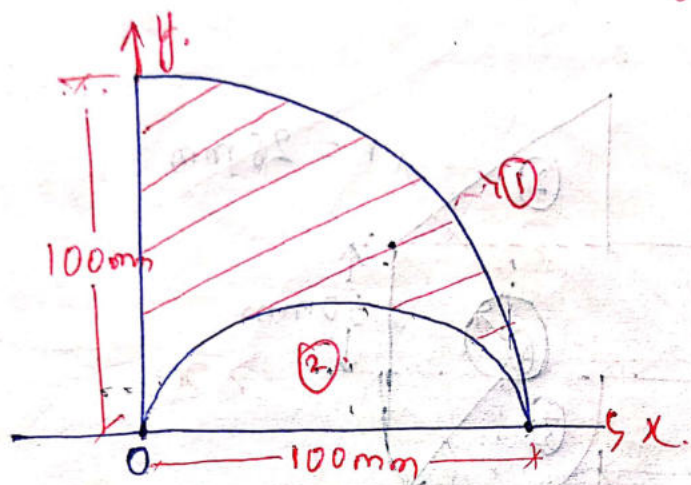


Component	Area(a) mm ²	x	y	ax	ay
Total or ①	$(10+10+10) \times (12+8)$ $= 600$	$\frac{10+10+10}{2}$ $= 15$	$\frac{12+8}{2} = 10$	9000	6000
Semicircle ②	$\frac{\pi R^2}{2} = \frac{\pi \times 5^2}{2}$ $= -39.27$	$10 + R = 10 + 5$ $= 15$	$(12+8) - \frac{4R}{3\pi}$ $= 20 - \frac{4 \times 5}{3\pi}$ $= 17.87$	-589.05	-701.75
Quadrant of the circle ③	$\frac{\pi R^2}{4} = \frac{\pi \times 8^2}{4}$ $= -50.26$	$30 - \frac{4R}{3\pi}$ $30 - \frac{4 \times 8}{3\pi}$ $= 26.6$	$\frac{4R}{3\pi} = \frac{4 \times 8}{3\pi}$ $= 3.395$	-1337.15	+170.63
Σa	510.47			$\Sigma ax = 7073.85$	$\Sigma ay = 5127.62$

$$\bar{x} = \frac{\Sigma ax}{\Sigma a} = \frac{7073.85}{510.47} = 13.85 \text{ mm}$$

$$\bar{y} = \frac{\Sigma ay}{\Sigma a} = \frac{5127.62}{510.47} = 10.04 \text{ mm}$$

Q. Find the centroid of the shaded area shown in fig. obtained by cutting a semicircle of diameter 100mm from the quadrant of a circle of radius 100mm.



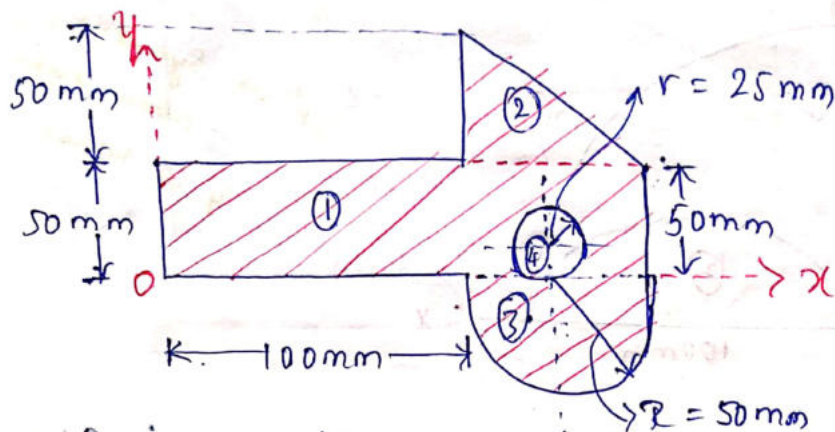
Component	Area (mm ²)	x	y	a _x	a _y
Total quadrant of the circle	$\frac{\pi R^2}{4} = \frac{\pi \times 100^2}{4}$ $= 7854$	$\frac{4R}{3\pi} = \frac{4 \times 100}{3\pi}$ $= 42.44$	$\frac{4R}{3\pi} = \frac{4 \times 100}{3\pi}$ $= 42.44$	333323.76	333323.76
Semicircle	$\frac{\pi R^2}{2} = \frac{\pi \times 50^2}{2}$ $= -3926.99$	$R = 50$	$\frac{4R}{3\pi} = \frac{4 \times 50}{3\pi}$ $= 21.22$	-196349.5	-83330.72
	$\Sigma a = 3927.01$			$\Sigma a_x = 136974.26$	$\Sigma a_y = 249993.04$

$$\bar{x} = \frac{\Sigma a_x}{\Sigma a} = \frac{136974.26}{3927.01}$$

$$= 34.88 \text{ mm}$$

$$\bar{y} = \frac{\Sigma a_y}{\Sigma a} = \frac{249993.04}{3927.01} = 63.66 \text{ mm}$$

10. Determine the location of the centroid of the shaded portion of the lamina, shown in fig. w.r.to. the origin 'O'

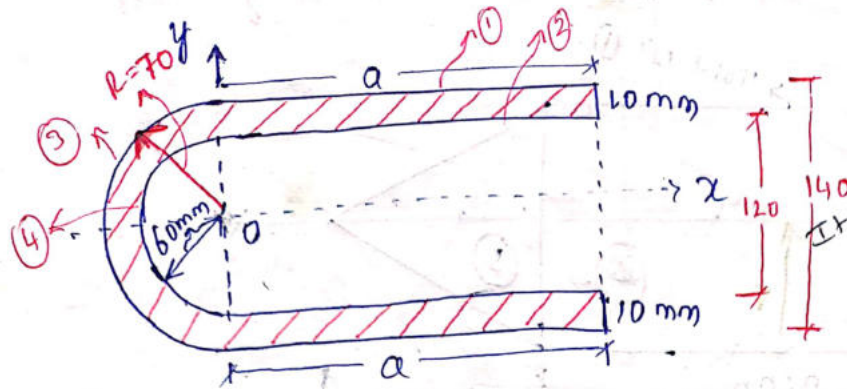


Component	Area (a) mm ²	$\bar{x}$ mm	$\bar{y}$ mm	$a\bar{x}$	$a\bar{y}$
Rect ①	$(100+50+50) \times 50$ $= 10000$	$\frac{100+100}{2} = 100$	$\frac{50}{2} = 25$	1000000	250000
Δ ②	$\frac{1}{2} \times 100 \times 50$ $= 2500$	$100 + \frac{b}{3}$ $= 100 + \frac{100}{3}$ $= 133.33$	$50 + \frac{h}{3}$ $= 50 + \frac{50}{3} = 66.67$	333325	166675
Semi-circle ③	$\frac{\pi R^2}{2} = \frac{\pi \times 50^2}{2}$ $100 + R = 150$ $= 3927.5$	$100 + R = 150$	$-\frac{4R}{3\pi}$ $= -\frac{4 \times 50}{3\pi}$ $= -21.22$	589125	-83341.55
Circle ④	$\pi R^2 = \pi \times 25^2$ $= 1963.5$	$100 + 50 = 150$	$R = 25$	-294525	-49087.5
	$\Sigma a = 14463.5$			$\Sigma a\bar{x} = 1627925$	$\Sigma a\bar{y} = 284245.95$

$$\bar{x} = \frac{\Sigma a\bar{x}}{\Sigma a} = \frac{1627925}{14463.5} = 112.55 \text{ mm}$$

$$\bar{y} = \frac{\Sigma a\bar{y}}{\Sigma a} = \frac{284245.95}{14463.5} = 19.65 \text{ mm}$$

1st For the shaded area shown in fig, find 'a' so that centroid is at 'O'.



is symmetric about x-axis
y is known

Given data:-

Since centroid is lying at 'O' so $\bar{x} = 0$,
 1) For outer $\Delta^e \Rightarrow R = 60 + 10 = 70$. For inner $\Delta^e \Rightarrow b = a$
 2) For inner $\Delta^e \Rightarrow a = 60$. For outer $\Delta^e \Rightarrow b = a, d = 60 + 10 + 10 = 80$.
 $\Rightarrow b = a$
 $d = 60 \text{ mm}$

Component	Area (a) mm ²	$\bar{x}$	$a\bar{x}$
Rect ① [Outer]	$a \times [10 + 60 + 60 + 10]$ $= 140a$	$\frac{a}{2}$	$70a^2$
Rect ② [Inner]	$-a \times [60 + 60]$ $= -120a$	$\frac{a}{2}$	$-60a^2$
Semicircle ③ [Outer]	$\frac{\pi R^2}{2} = \frac{\pi (60+10)^2}{2}$ 7696.9	$-\frac{4R}{3\pi} = -\frac{4 \times 70}{3\pi}$ $= -29.7$	-228597.93
Semicircle ④ [Inner]	$-\frac{\pi R^2}{2} = -\frac{\pi \times 60^2}{2}$ $= -5654.86$	$-\frac{4R}{3\pi} = -\frac{4 \times 60}{3\pi}$ $= -25.465$	14400.1
Σa	$140a - 120a + 7696.9 - 5654.86$		

$$\Sigma a\bar{x} = 70a^2 - 60a^2 - 228597.93 + 14400.1$$

$$\Sigma a\bar{x} = 10a^2 - 84596.93$$

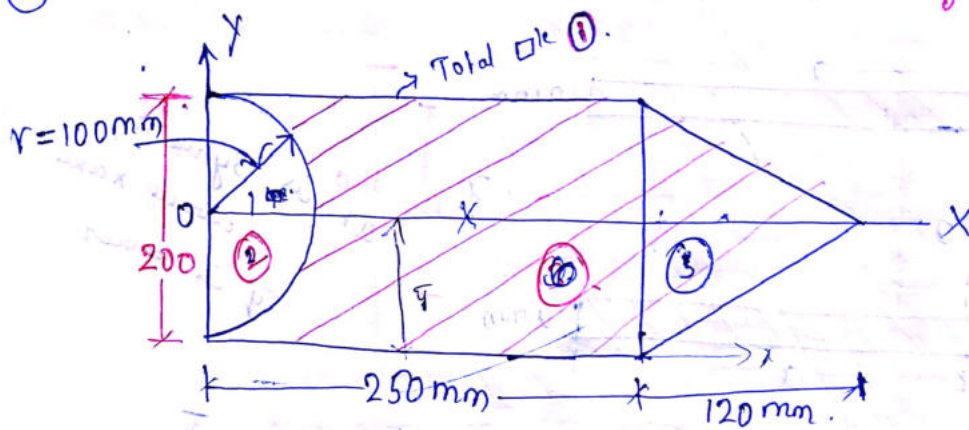
$$\bar{x} = \frac{\Sigma a\bar{x}}{\Sigma a} = \frac{10a^2 - 84596.93}{140a - 120a + 7696.9 - 5654.86} = 0$$

$$\Rightarrow 10a^2 - 84596.93 = 0$$

$$a^2 = 8459.693$$

$$a = 91.97 \text{ mm}$$

125 Find C.G. of the shaded area for the given fig. with respect to given x & y axes.



The given fig. is symmetric about x-axis.

$\therefore \bar{y}$ is known.

$$\bar{y} = 0$$

Component	Area (a) mm ²	$\bar{x}$	ax
Total Rect (1)	$b \times d = 250 \times 200$ $= 50000$	$\frac{b}{2} = \frac{250}{2} = 125$	6250000
Semicircle (2)	$-\frac{\pi R^2}{2} = -\frac{\pi \times 100^2}{2}$ $= -15708$	$\frac{4R}{3\pi} = \frac{4 \times 100}{3\pi}$ $= 42.44$	-666647.52
Alt (3)	$\frac{1}{2}bh = \frac{1}{2} \times 200 \times 120$ $= 12000$	$250 + \frac{h}{3}$ $= 250 + \frac{200}{3}$ $= 290$	3480000
$\Sigma a = 46292$		$\Sigma ax =$	9063352.48

$$\bar{x} = \frac{\Sigma ax}{\Sigma a}$$

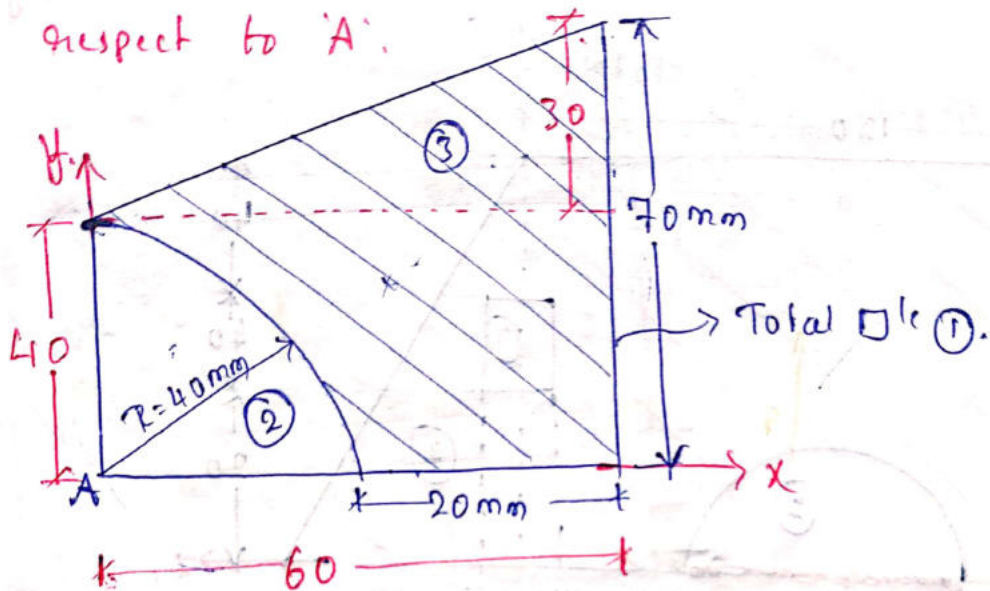
$$= \frac{9063352.48}{46292}$$

$$\bar{x} = 195.78 \text{ mm}$$

$$\bar{y} = 0$$

13. Locate the centroid of lamina shown in Fig. with respect to 'A'.

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Component	Area (a) mm ²	$\bar{x}$ (mm)	$\bar{y}$ (mm)	$a\bar{x}$	$a\bar{y}$
Total $\square$ ①	$(40+20) \times 40$ $= 2400$	$\frac{60}{2} = 30$	$\frac{40}{2} = 20$	72000	48000
Quadrant of the $\square$ ②	$-\frac{\pi R^2}{4} = -\frac{\pi \times 40^2}{4}$ $= -1256.63$	$\frac{4R}{3\pi} = \frac{4 \times 40}{3\pi}$ $= 16.97$	$\frac{4R}{3\pi} = \frac{4 \times 40}{3\pi}$ $= 16.97$	-21325	-21325
Δ ③	$\frac{1}{2} \times 60 \times (70-40)$ $= 900$	$\frac{2b}{3} = \frac{2 \times 60}{3}$ $= 40$	$40 + \frac{2h}{3} = 40 + \frac{2 \times 30}{3} = 60$	36000	45000
$\Sigma a =$	2043.37		$\Sigma a\bar{y} = 86675$	$\Sigma a\bar{x} = 71675$	71675 71675

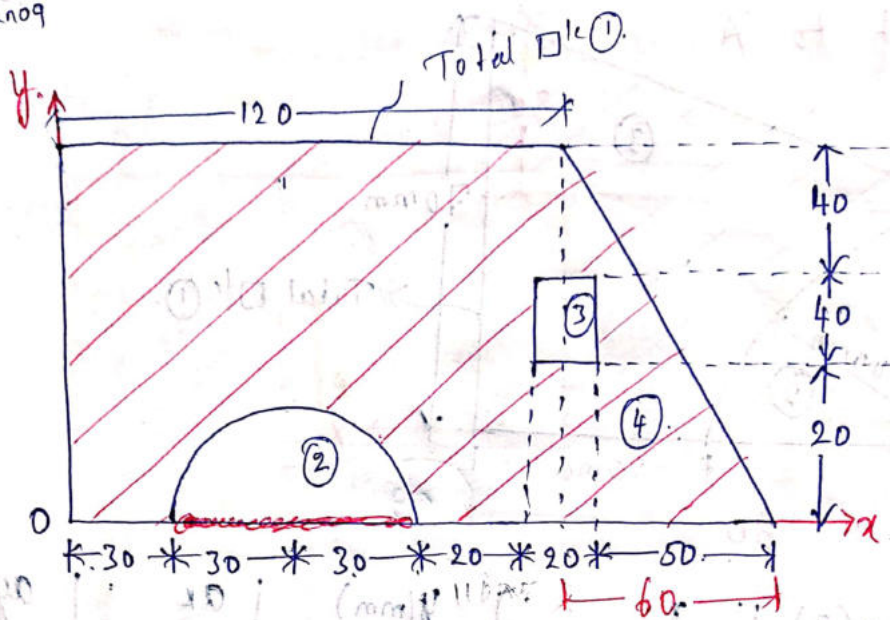
$$\bar{x} = \frac{\Sigma a\bar{x}}{\Sigma a} = \frac{86675}{2043.37} = 42.417 \text{ mm}$$

$$\bar{y} = \frac{\Sigma a\bar{y}}{\Sigma a} = \frac{71675}{2043.37} = 35.07 \text{ mm}$$

14. Locate the centroid of an area shown in fig.

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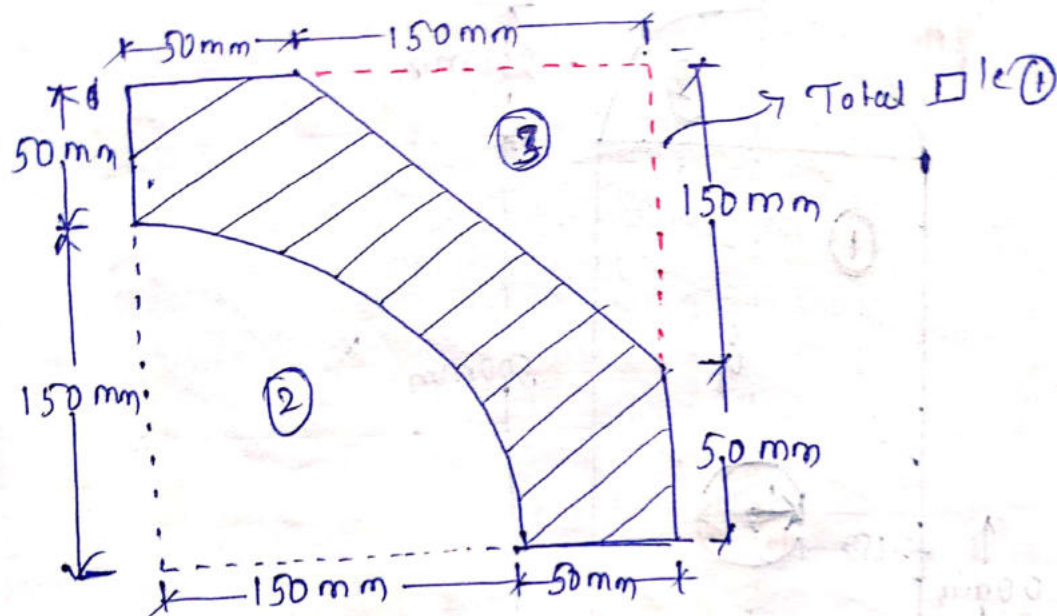


Component	Area (a) mm ²	x (mm)	y (mm)	ax (mm)	ay (mm)
Total rect ①	$120 \times (40 + 40 + 20)$ $= 12000$	$\frac{b}{2} = \frac{120}{2} = 60$	$\frac{d}{2} = \frac{40 + 40 + 20}{2}$ $= 50$	720000	600000
Jemicircle ②	$\frac{\pi R^2}{2} = \frac{\pi \times 30^2}{2}$ $= -1413.7$	$30 + R =$ $30 + 30 = 60$	$\frac{4R}{3\pi} = \frac{4 \times 30}{3\pi}$ $= 12.73$	-84822	-17996.4
Rect ③	$20 \times 40 = 800$	$30 + 30 + 30 +$ $\frac{20 + 20}{2}$ $= 120$	$20 + \frac{d}{2} =$ $20 + \frac{40}{2} = 40$	-96000	-32000
Δ ④	$\frac{1}{2} \times (50 + \frac{20}{2}) \times$ $(20 + 40 + 40)$ $= 3000$	$30 + 30 + 30 + 20$ $\frac{20 + 50}{2}$ $= 140$	$\frac{h}{3} = \frac{40 + 40 + 20}{3}$ $= 33.33$	420000	99990
Σa	12786.3			$\Sigma ax = 959178$	$\Sigma ay = 649993.6$

$$\bar{x} = \frac{\Sigma ax}{\Sigma a} = \frac{959178}{12786.3} = 75.026 \text{ mm}$$

$$\bar{y} = \frac{\Sigma ay}{\Sigma a} = \frac{649993.6}{12786.3} = 50.84 \text{ mm}$$

15. Determine the centroid of the shaded area shown in fig.



Component	Area (a) mm ²	x	y	a _x	a _y
Total Rect	$(50+150) \times (150+50) = 40,000$	$\frac{b}{2} = \frac{150+50}{2} = 100$	$\frac{d}{2} = \frac{150+50}{2} = 100$	4000000	4000000
Quadrant of the circle (2)	$\frac{\pi R^2}{4} = \frac{\pi \times 150^2}{4} = 17671.45$	$\frac{4R}{3\pi} = \frac{4 \times 150}{3\pi} = 63.66$	$\frac{4R}{3\pi} = \frac{4 \times 150}{3\pi} = 63.66$	-1124964.5	-1124964.5
Rect (3)	$\frac{1}{2}bh = \frac{1}{2} \times 150 \times 150 = 11250$	$50 + \frac{2}{3}b = 50 + \frac{2}{3} \times 150 = 150$	$50 + \frac{2}{3}h = 50 + \frac{2}{3} \times 150 = 150$	-1687500	-1687500
Σa	11078.55				

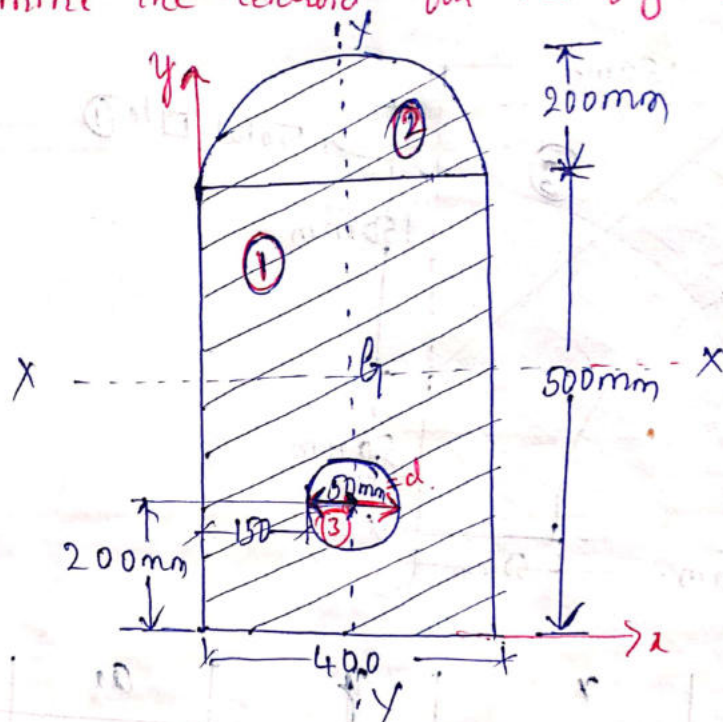
$$\Sigma a_x = 1187535.5$$

$$\Sigma a_y = 1187535.5$$

$$\bar{x} = \frac{\Sigma a_x}{\Sigma a} = \frac{1187535.5}{11078.55} = 107.2 \text{ mm}$$

$$\bar{y} = \frac{\Sigma a_y}{\Sigma a} = 107.2 \text{ mm}$$

16. Determine the centroid for the fig. given below



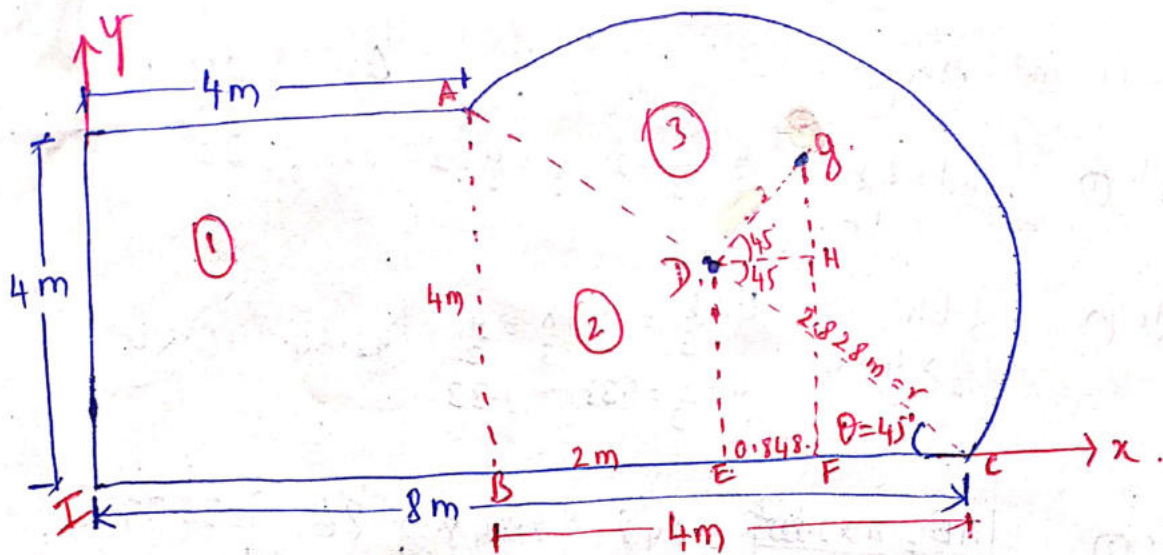
The given fig. is symmetric about y-axis,

$$\therefore \bar{x} = \frac{400}{2} = 200 \text{ mm.}$$

Component	Area (a) mm ²	y (mm)	ay (mm ³)
Rect ①	$400 \times 500 = 200000$	$\frac{d}{2} = \frac{500}{2} = 250$	50000000
Semicircle ②	$\frac{\pi R^2}{2} = \frac{\pi \times 200^2}{2}$ $= 62831.853$	$500 + \frac{4R}{3\pi}$ $500 + \frac{4 \times 200}{3\pi}$ $= 584.88$	36749094.18
Hole ③	$-\pi R^2 = -\pi \times 25^2$ $= -1963.75$	200	-392600
$\Sigma a = 260868.103$			$\Sigma ay = 86356494.18$

$$\bar{y} = \frac{\Sigma ay}{\Sigma a} = \frac{86356494.18}{260868.103}$$

$$\bar{y} = 331.035 \text{ mm}$$



From fig. $AC^2 = AB^2 + BC^2$

$$AC^2 = 4^2 + 4^2$$

$$AC = 5.656 \text{ m. (d)}$$

$$CD = r = \frac{5.656}{2} = 2.828 \text{ m.}$$

$$\theta = \tan^{-1}\left(\frac{4}{4}\right) = 45^\circ$$

From Δ^c GHG

$$DG = \frac{4r}{3\pi} = \frac{4 \times 2.828}{3\pi} = 1.2 \text{ m.}$$

$$\cos 45 = \frac{DH}{DG} \Rightarrow DH = DG \cos 45 = 1.2 \cos 45 = 0.848 \text{ m.} = EF$$

$$GH = DG \sin 45 = 0.848 \text{ m.}$$

$$EC = 2.828 \cos 45 = 2 \text{ m.}$$

$$BE = 4 - 2 = 2 \text{ m.}$$

$$DE = 2.828 \sin 45 = 2 \text{ m.}$$

$$x \text{ for } \Delta^c = 4 + 2 + 0.848 = 6.848 \text{ m.}$$

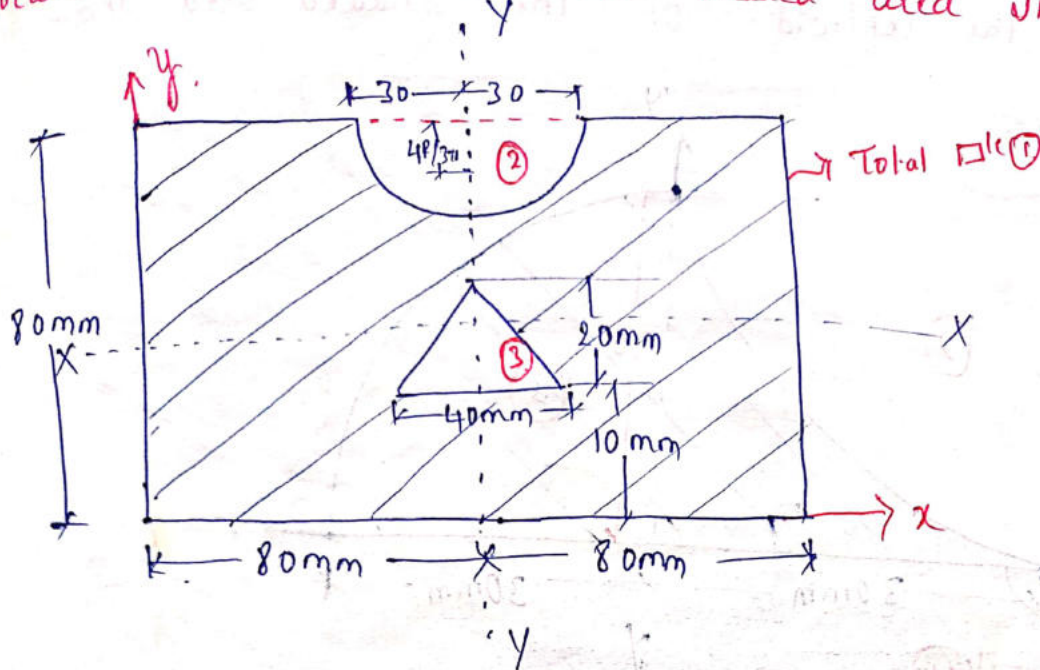
$$y \text{ for } \Delta^c = DE + GH = 2 + 0.848 = 2.848 \text{ m.}$$

Component	Area (a)	x	y	ax	ay
□ ①	$bd = 4 \times 4 = 16$	$\frac{b}{2} = \frac{4}{2} = 2$	$\frac{d}{2} = \frac{4}{2} = 2$	32	32
△ ②	$\frac{1}{2}bh = \frac{1}{2} \times 4 \times 4 = 8$	$4 + \frac{b}{3} = 4 + \frac{4}{3} = 5.33$	$\frac{h}{3} = \frac{4}{3} = 1.33$	42.64	10.64
◐ ③	$\frac{\pi R^2}{2} = \frac{\pi \times 2.828^2}{2} = 12.56$	6.848	2.848	86	35.77
	$\Sigma a = 36.56$			$\Sigma ax = 160.44$	$\Sigma ay = 78.41$

$$\bar{x} = \frac{\Sigma ax}{\Sigma a} = \frac{160.44}{36.56} = 4.389 \text{ m}$$

$$\bar{y} = \frac{\Sigma ay}{\Sigma a} = \frac{78.41}{36.56} = 2.145 \text{ m}$$

Locate the centroid of the shaded area shown in fig



The given fig. is symmetric about Y-axis
 $\therefore \bar{x} = 80\text{mm}$

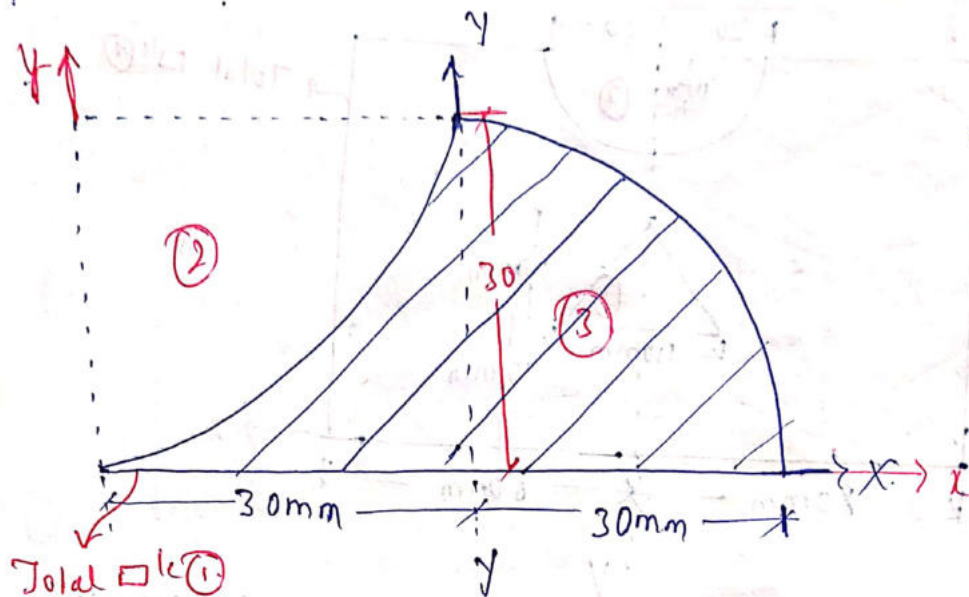
Component	Area, a (mm^2)	y	ay
Rectangle ①	$bd = (80+80) \times 80 = 12800$	$\frac{d}{2} = \frac{80}{2} = 40$	512000
Semicircle ②	$\frac{\pi R^2}{2} = \frac{\pi \times 30^2}{2} = -1413.7$	$80 - \frac{4R}{3\pi} = 80 - \frac{4 \times 30}{3\pi} = 67.26$	-28552.242 -95085.46
△ ③	$-\frac{1}{2}bh = -\frac{1}{2} \times 40 \times 20 = -400$	$10 + \frac{h}{3} = 10 + \frac{20}{3} = 16.66$	-6664
	$\Sigma a = 10987$		$\Sigma ay = 410250.54$

$$\bar{y} = \frac{\Sigma ay}{\Sigma a}$$

$$= \frac{410250.54}{10987}$$

$$\bar{y} = 37.34 \text{ mm.}$$

18. Locate the centroid of the shaded area shown in fig.



Component	Area (a) mm ²	$\bar{x}$	$\bar{y}$	$a\bar{x}$	$a\bar{y}$
Total Rect (1)	$b \times d = 30 \times 30$ $= 900$	$\frac{b}{2} = \frac{30}{2} = 15$	$\frac{d}{2} = \frac{30}{2} = 15$	13500	13500
Quadrant of the circle (2)	$\frac{\pi R^2}{4} = \frac{\pi \times 30^2}{4}$ $= -706.8$	$\frac{4R}{3\pi} = \frac{4 \times 30}{3\pi}$ $= 12.73$	$30 - \frac{4R}{3\pi} = 30 - \frac{4 \times 30}{3\pi}$ $= 17.27$	-8997.56	-12206.4
Quadrant of the circle (3)	$\frac{\pi R^2}{4} = \frac{\pi \times 30^2}{4}$ $= 706.8$	$30 + \frac{4R}{3\pi} = 30 + \frac{4 \times 30}{3\pi}$ $= 42.73 \text{ mm}$	$\frac{4R}{3\pi} = 12.73$	30201.56	8997.56
	$\Sigma a = 900$			$\Sigma a\bar{x} = 34704$	

$$\bar{x} = \frac{\Sigma a\bar{x}}{\Sigma a} = \frac{34704}{900}$$

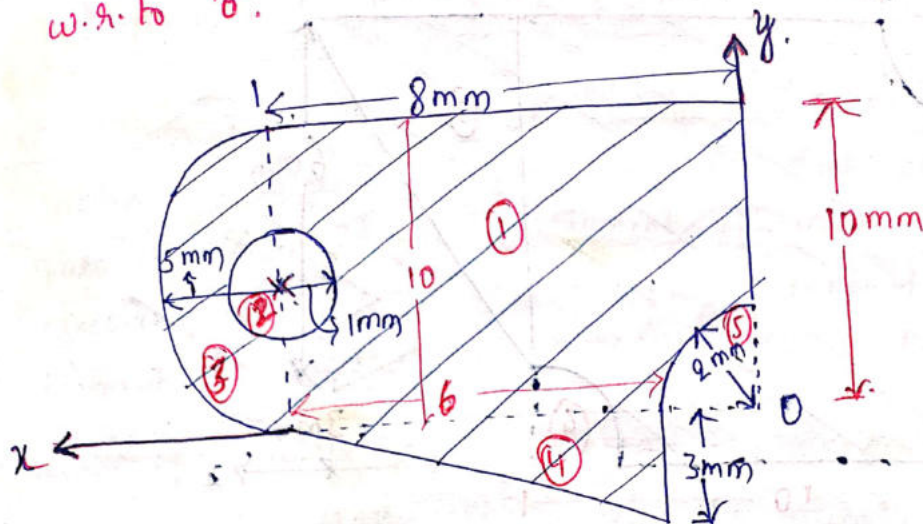
$$\bar{x} = 38.56$$

$$\Sigma a\bar{y} = 10291.16$$

$$\bar{y} = \frac{\Sigma a\bar{y}}{\Sigma a} = \frac{10291.16}{900}$$

$$\bar{y} = 11.43 \text{ mm}$$

19. Locate the centroid of the plane shown in Fig. w.r. to 'O'.

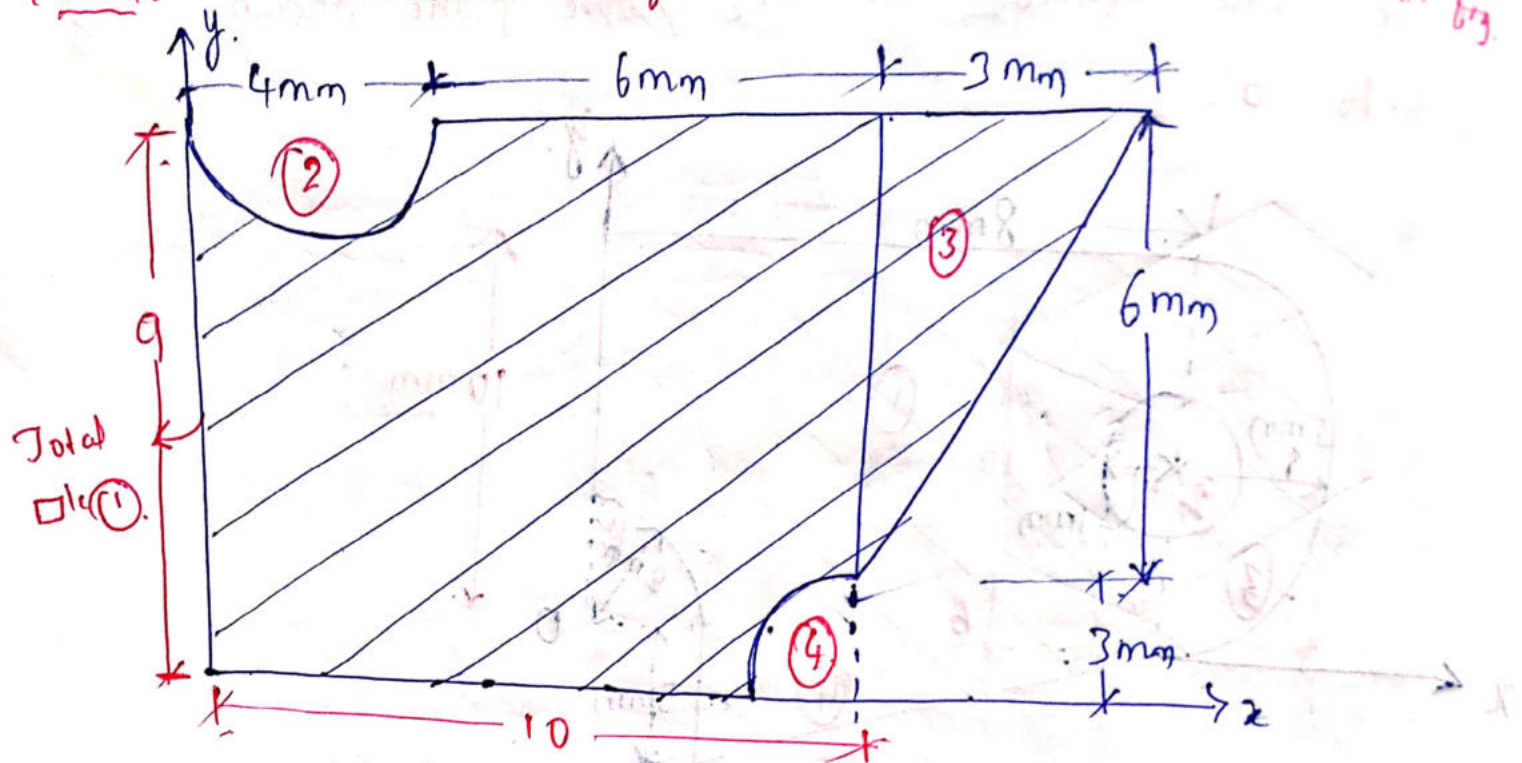


Component	Area 'a' (mm ²)	x	y	ax	ay
Total $\square$ ①	$bd = 8 \times 10 = 80$	$\frac{b}{2} = \frac{8}{2} = 4$	$\frac{d}{2} = \frac{10}{2} = 5$	-320	400
Semicircle ②	$\frac{\pi r^2}{2} = \frac{\pi \times 12}{2} = -3.142$	$\frac{4r}{3\pi} = \frac{4 \times 5}{3\pi} = 2.08$	$\frac{4r}{3\pi} = \frac{4 \times 5}{3\pi} = 2.08$	25.136	-15.71
Semicircle ③	$\frac{\pi r^2}{2} = \frac{\pi \times 5^2}{2} = 39.26$	$-(8 + \frac{4r}{3\pi}) = -(8 + \frac{4 \times 5}{3\pi}) = -10.12$	$r = 5$	-397.31	+196.3
Δ ④	$\frac{1}{2}bh = \frac{1}{2} \times 6 \times 3 = 9$	$-(2 + \frac{b}{3}) = -(2 + \frac{6}{3}) = -4$	$-\frac{h}{3} = -\frac{3}{3} = -1$	-36	-9
Quadrant of the circle ⑤	$-\frac{\pi r^2}{4} = -\frac{\pi \times 2^2}{4} = -3.142$	$-\frac{4r}{3\pi} = -\frac{4 \times 2}{3\pi} = -0.848$	$\frac{4r}{3\pi} = \frac{4 \times 2}{3\pi} = 0.848$	2.66	-2.66
Σa	121.98			-725.5	$\Sigma ay = 176.33$ 568.93

$$\bar{x} = \frac{\Sigma ax}{\Sigma a} = \frac{-725.5}{121.98} = -5.94 \text{ mm}$$

$$\bar{y} = \frac{\Sigma ay}{\Sigma a} = \frac{568.93}{121.98} = 4.664 \text{ mm}$$

2) Locate the centroid of the shaded area shown in fig.



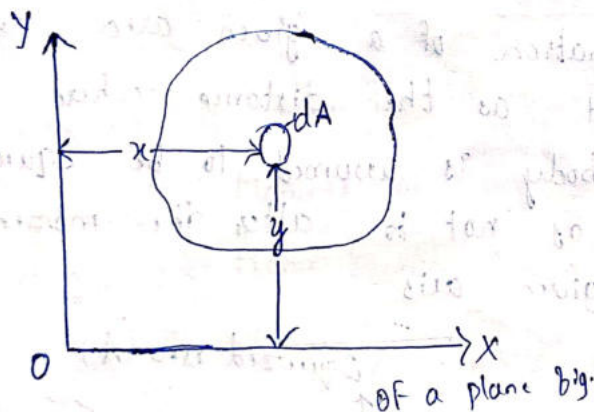
Component	Area	x	y	ax	ay
Total $\square$ (1)	$10 \times 9 = 90$	5	4.5	450	405
Δ (2)	$-\frac{\pi R^2}{2} = -\frac{\pi \times 2^2}{2}$ $= -6.28$	2	$9 - \frac{4 \times 2}{3\pi} = 8.151$	-12.56	-51.188
Δ (3)	$\frac{1}{2} \times 3 \times 6 = 9$	$10 + \frac{b}{3} = 10 + \frac{3}{3} = 11$	$3 + \frac{2h}{3} = 3 + \frac{2 \times 6}{3} = 7$	99	63
Δ (4)	$-\frac{\pi R^2}{4} = -\frac{\pi \times 3^2}{4}$ $= -7.06$	$10 - \frac{4 \times 3}{3\pi} = 8.72$	$\frac{4R}{3\pi} = 1.273$	-61.56	-8.98
	$\Sigma a = 85.66$			$\Sigma ax = 474.88$	$\Sigma ay = 407.83$
	$\bar{x} = \frac{\Sigma ax}{\Sigma a} = \frac{474.88}{85.66} = 5.543 \text{ mm}$				
	$\bar{y} = \frac{\Sigma ay}{\Sigma a} = \frac{407.83}{85.66} = 4.76 \text{ mm}$				

Moment of Inertia.

Syllabus:- Moment of inertia of an area, polar moment of inertia, Radius of gyration, I_r axis theorem and parallel axis theorem, Moment of inertia of $\square$ lar, circular, and Δ lar areas from method of integration, Moment of inertia of composite areas, Numerical problems.

[Moment of inertia of an area is another important property of cross-sections of the members like beam column and many mechanical components. Determination of moment of inertia of the plane areas is of great importance in the study of subjects like strength of materials, structural designs, steel structures, and machine designs.]

Moment of inertia of an area (or) plane figure:-



Moment of inertia is the resistance offered to change in rotational motion of an area about an axis.

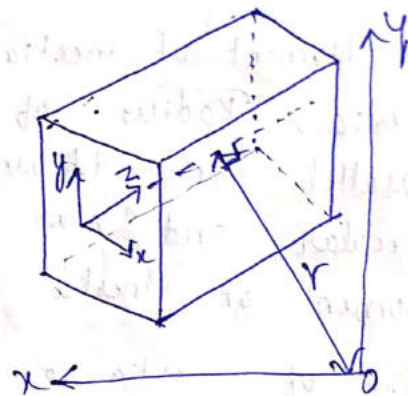
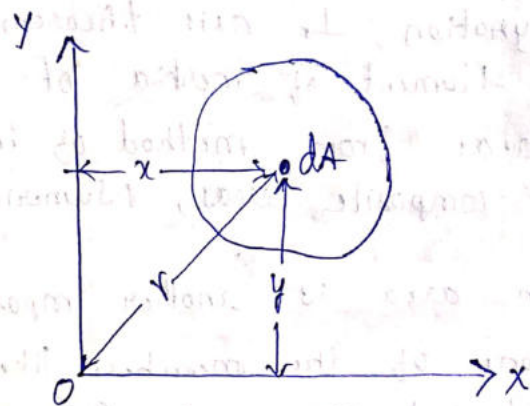
Consider a plane area as shown in fig. dA is an elemental area with co-ordinates x & y .

Moment of elemental area about y -axis = First moment of inertia. If first moment of inertia is again multiplied by I_r distance y , which gives dAy^2 is known as second moment of area (or) moment of inertia (or) moment of area.

Moment of inertia about x -axis = $I_{xx} = \int y^2 dA$
 Moment of inertia about y -axis = $I_{yy} = \int x^2 dA$

Unit of moment of inertia is denoted by mm^4 or cm^4 or m^4 .

Polar moment of inertia

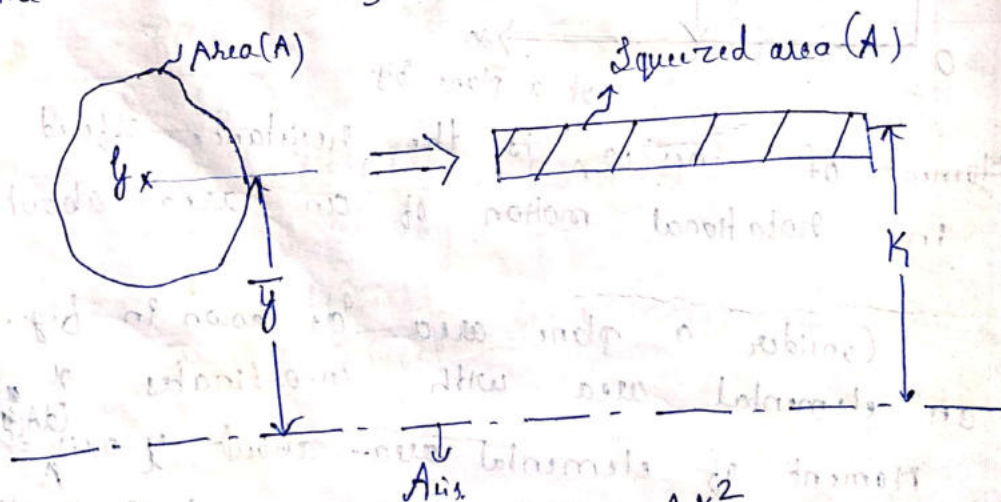


Moment of inertia about an axis I to the plane of an area is known as polar moment of inertia. It is noted by I_{zz} .

$$I_{zz} = \sum r^2 dA.$$

Radius of gyration

Radius of gyration of a given area, about an axis is defined as the distance where the whole area of the body is assumed to be squeezed and concentrated so as not to alter the moment of inertia about the given axis.



Mathematically,

Radius of gyration

$$I = A k^2$$

$$k = \sqrt{\frac{I}{A}}$$

where

I - Moment of Inertia

A - Area of body

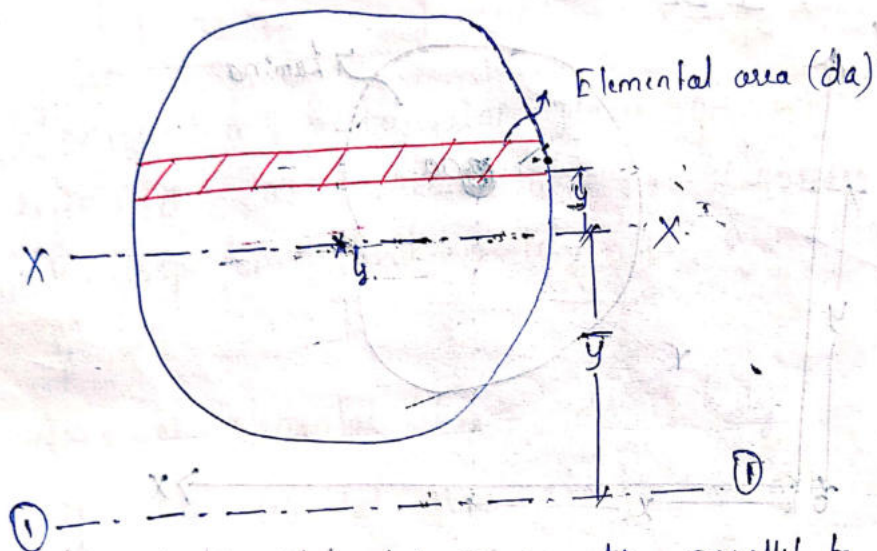
$$I = A k^2$$

Theorems of moment of inertia

→ Parallel axis theorem:

Statement: If the moment of inertia of a plane area about an axis through its C.G. is denoted by I_{xx} , then M.I. moment of the area about any reference axis ①-① parallel to x-axis and at a distance $\bar{y}$ from C.G. is given by.

$$I_{1-1} = I_{xx} + A\bar{y}^2$$



Consider an elemental area da parallel to xx axis

- Let
- I_{xx} - Moment of inertia about $x-x$ axis.
 - I_{1-1} - Moment of inertia about reference axis ①-①.
 - A - Area of body.
 - $\bar{y}$ - Distance of C.G. of body from ①-①.

[In fig. I_{xx} , I_{1-1} and elemental area are parallel to each other. Hence it is called parallel axis theorem]

$$\begin{aligned} \text{Moment of } \overset{\text{inertia}}{\Delta} \text{ elemental area about ①-①} &= \sum da (y + \bar{y})^2 \\ &= \sum da (y^2 + \bar{y}^2 + 2y\bar{y}) \\ &= \sum da y^2 + \sum da \bar{y}^2 + 2 \sum da y \bar{y} \\ &= \sum da \bar{y}^2 + \sum da y^2 + 2\bar{y} \sum da y \\ &= \bar{y}^2 \sum da + \sum da y^2 + 2\bar{y} \sum da y \end{aligned}$$

Now,

$$\sum da = A$$

$$\sum da y^2 = \text{M.I. about } xx = I_{xx}$$

$$\sum da y = 0 \quad (\text{i.e. Moment of areas about centroid} = 0)$$

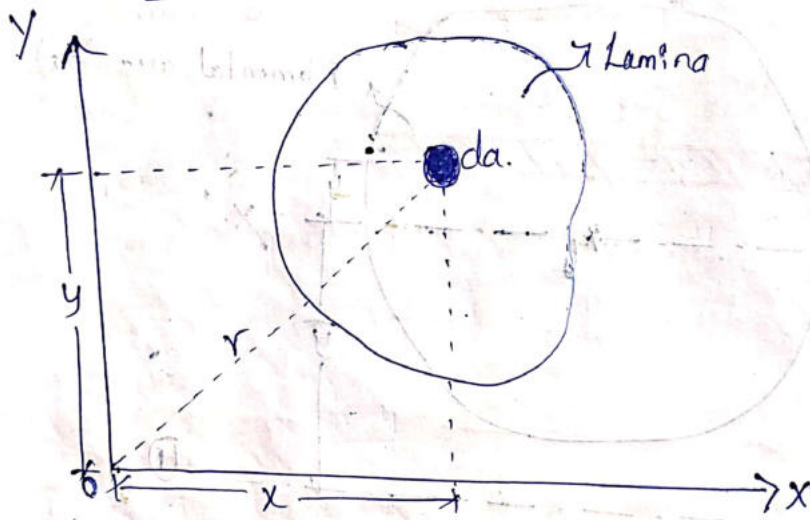
$$\therefore I_{1-1} = A\bar{y}^2 + I_{xx} + 0$$

$$\textcircled{0A} \quad I_{1-1} = I_{xx} + A\bar{y}^2$$

2.1 Perpendicular axis theorem :-

Statement :- If I_{xx} and I_{yy} be the moment of inertia of the area about xx and yy axes respectively then moment of inertia about zz axis is given by

$$I_{zz} = I_{xx} + I_{yy}$$



Consider an elemental area 'da' at a distance 'r' from 'O'.

Moment of inertia of element about z-z axis = $da r^2$

M.I. of whole area $I_{zz} = \int da r^2$

From fig. $r^2 = x^2 + y^2$

$$= \int da (x^2 + y^2)$$

$$I_{zz} = \int da x^2 + \int da y^2 \rightarrow \textcircled{1}$$

$$\left. \begin{array}{l} \text{W.K.T, } \int da x^2 = I_{yy} \\ \int da y^2 = I_{xx} \end{array} \right\} \rightarrow \textcircled{2}$$

From $\textcircled{1}$ & $\textcircled{2}$

$$I_{zz} = I_{xx} + I_{yy}$$