

MODULE-3

Eigen Values and Eigen Vectors

Finding Eigen Values and Eigen Vectors of the matrix.

1. $A = \begin{bmatrix} -19 & 7 \\ -42 & 16 \end{bmatrix}$

Solution:

The characteristic equation of a matrix $= |A - \lambda I| = 0$

A is a given matrix. I = Identity matrix according to the given matrix A. λ is a constant.

$$|A - \lambda I| = \begin{vmatrix} -19 - \lambda & 7 \\ -42 & 16 - \lambda \end{vmatrix} = 0$$

On expanding the determinant we get

$$(-19 - \lambda)(16 - \lambda) - (-42) \cdot 7 = 0$$

$$-304 + 19\lambda - 16\lambda + \lambda^2 + 294 = 0$$

$$\lambda^2 + 3\lambda - 10 = 0$$

Solve the quadratic equation, we get $\lambda = 2, -5$. is called Eigen values.

To find the Eigen vector corresponding to $\lambda = 2$.

Substitute $\lambda = 2$ in $[A - \lambda I][X] = 0$

$$\begin{bmatrix} -19 - 2 & 7 \\ -42 & 16 - 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$\begin{bmatrix} -21 & 7 \\ -42 & 14 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

Cofactor of -21 is 14.

Cofactor of 7 is $-(-42) = 42$.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 14 \\ 42 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

It is divided by 14, we get $x = 1, y = 3$.

To find the Eigen vector corresponding to $\lambda = -5$.

Substitute $\lambda = -5$ in $[A - \lambda I][X] = 0$

$$\begin{bmatrix} -19 + 5 & 7 \\ -42 & 16 + 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$\begin{bmatrix} -14 & 7 \\ -42 & 21 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

Cofactor of -14 is 21.

Cofactor of 7 is $-(-42)=42$.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 21 \\ 42 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

It is divided by 21, we get $x=1, y=2$.

$$\text{Eigen vector} = \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}.$$

$$2. \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

Solution:

The characteristic equation of a matrix $= |A - \lambda I| = 0$

A is a given matrix. I = Identity matrix according to the given matrix A. λ is a constant.

$$|A - \lambda I| = \begin{vmatrix} 8 - \lambda & -6 & 2 \\ -6 & 7 - \lambda & -4 \\ 2 & -4 & 3 - \lambda \end{vmatrix} = 0$$

On expanding we get ,

$$(8-\lambda)[(7-\lambda)(3-\lambda) - (-4)(-4)] + 6[-6(3-\lambda) + 8] + 2[24 - 2(7-\lambda)] = 0$$

$$(8-\lambda)[21 - 7\lambda - 3\lambda + \lambda^2 - 16] + 6[-18 + 6\lambda + 8] + 2[24 - 14 + 2\lambda] = 0$$

$$(8-\lambda)[5 - 10\lambda + \lambda^2] + 6[-10 + 6\lambda] + 2[10 + 2\lambda] = 0$$

$$40 - 80\lambda + 8\lambda^2 - 5\lambda + 10\lambda^2 - \lambda^3 - 60 + 36\lambda + 20 + 4\lambda = 0$$

$$-45\lambda + 18\lambda^2 - \lambda^3 = 0$$

Multiply the equation with -,

$$\lambda^3 - 18\lambda^2 + 45\lambda = 0$$

Using calculator to find the roots of the cubic equation,

We get $\lambda=0, 3, 15$.

The system of equation is

$$(8-\lambda)x - 6y + 2z = 0$$

$$-6x + (7-\lambda)y - 4z = 0$$

$$2x - 4y + (3-\lambda)z = 0$$

Case(i) **Eigen vector corresponding to $\lambda=0$ in the above equation**

$$8x - 6y + 2z = 0$$

$$-6x + 7y - 4z = 0$$

$$2x - 4y + 3z = 0$$

Applying the rule of cross multiplication using any two equation out of three, (any combination)

$$\frac{x}{\begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -6 & -4 \\ 2 & 3 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -6 & 7 \\ 2 & -4 \end{vmatrix}}$$

$$\frac{x}{21-16} = \frac{-y}{-18+8} = \frac{z}{24-14}$$

$$\frac{x}{5} = \frac{-y}{-10} = \frac{z}{10}$$

Divided by 5

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{2}$$

Case(ii) **Eigen vector corresponding to $\lambda=3$ in the above equation**

$$5x-6y+2z=0$$

$$-6x+4y-4z=0$$

$$2x-4y+0z=0$$

Applying the rule of cross multiplication using second and third equation

$$\frac{x}{\begin{vmatrix} 4 & -4 \\ -4 & 0 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -6 & -4 \\ 2 & 0 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -6 & 4 \\ 2 & -4 \end{vmatrix}}$$

$$\frac{x}{0-16} = \frac{-y}{0+8} = \frac{z}{24-8}$$

$$\frac{x}{-16} = \frac{-y}{8} = \frac{z}{16}$$

Divided by -8

$$\frac{x}{2} = \frac{y}{1} = \frac{z}{-2}$$

Case(iii) **Eigen vector corresponding to $\lambda=15$ in the above equation**

$$-7x-6y+2z=0$$

$$-6x-8y-4z=0$$

$$2x-4y-12z=0$$

Applying the rule of cross multiplication using second and third equation

$$\frac{x}{\begin{vmatrix} -8 & -4 \\ -4 & -12 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -6 & -4 \\ 2 & -12 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -6 & -8 \\ 2 & -4 \end{vmatrix}}$$

$$\frac{x}{96-16} = \frac{-y}{72+8} = \frac{z}{24+16}$$

$$\frac{x}{80} = \frac{-y}{80} = \frac{z}{40}$$

Divided by 40

$$\frac{x}{2} = \frac{y}{-2} = \frac{z}{1}$$

$$\text{Eigen vector of the matrix} = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}.$$

$$3. \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

Solution:

The characteristic equation of a matrix = $|A - \lambda I| = 0$

A is a given matrix. I = Identity matrix according to the given matrix A. λ is a constant.

$$|A - \lambda I| = \begin{vmatrix} 1 - \lambda & 1 & 3 \\ 1 & 5 - \lambda & 1 \\ 3 & 1 & 1 - \lambda \end{vmatrix} = 0$$

On expanding we get ,

$$(1-\lambda)[(5-\lambda)(1-\lambda) - (1)(1)] - 1[1(1-\lambda) - 3] + 3[1 - 3(5-\lambda)] = 0$$

$$(1-\lambda)[5 - 5\lambda - \lambda + \lambda^2 - 1] - 1[-2 - \lambda] + [-42 + 9\lambda] = 0$$

$$(1-\lambda)[4 - 6\lambda + \lambda^2] + 2 + \lambda - 40 + 4\lambda = 0$$

$$-\lambda^3 + 7\lambda^2 + 0\lambda + 36 = 0$$

$$x^3 - (\text{trace})x^2 + (A_{11} + A_{22} + A_{33})x - \text{Det}(A) = 0$$

$$x^3 - (1 + 5 + 1)x^2 + (4 - 8 + 4)x + 36 = 0$$

$$x^3 - 7x^2 + 36 = 0$$

$$X = -2, 6, 3$$

The system of equation is

$$(1-\lambda)x - 1y + 3z = 0$$

$$1x + (5-\lambda)y + 1z = 0$$

$$3x + y + (1-\lambda)z = 0$$

Case(i) **Eigen vector corresponding to $\lambda = -2$ in the above equation**

$$3x - 1y + 3z = 0$$

$$X + 7y + z = 0$$

$$3x + y + 3z = 0$$

Applying the rule of cross multiplication using any two equation out of three, (any combination) (2 and 3rd equation is considered)

$$\frac{x}{\begin{vmatrix} 7 & 1 \\ 1 & 3 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 1 & 1 \\ 3 & 3 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & 7 \\ 3 & 1 \end{vmatrix}}$$

$$\frac{x}{21 - 1} = \frac{-y}{3 - 3} = \frac{z}{1 - 21}$$

$$\frac{x}{20} = \frac{-y}{0} = \frac{z}{-20}$$

Divided by 20

$$\frac{x}{1} = \frac{y}{0} = \frac{z}{-1}$$

Case(ii) **Eigen vector corresponding to $\lambda = 3$ in the above equation**

$$-2x - 1y + 3z = 0$$

$$1x + 2y + 1z = 0$$

$$3x + y - 2z = 0$$

Applying the rule of cross multiplication using second and third equation

$$\frac{x}{\begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}}$$

$$\frac{x}{-5} = \frac{-y}{-5} = \frac{z}{1-6}$$

$$\frac{x}{-5} = \frac{y}{5} = \frac{z}{-5}$$

Divided by

$$\frac{x}{-1} = \frac{y}{1} = \frac{z}{-1}$$

Case(iii) **Eigen vector corresponding to $\lambda=6$ in the above equation**

$$-5x - 1y + 3z = 0$$

$$1x + y + 1z = 0$$

$$3x + y - 5z = 0$$

Applying the rule of cross multiplication using second and third equation

$$\frac{x}{\begin{vmatrix} -1 & 1 \\ 1 & -5 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 1 & 1 \\ 3 & -5 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix}}$$

$$\frac{x}{5-1} = \frac{-y}{-5-3} = \frac{z}{1+3}$$

$$\frac{x}{4} = \frac{-y}{-8} = \frac{z}{4}$$

Divided by 4

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{1}$$

$$\text{Eigen vector of the matrix} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ -1 & -1 & 1 \end{bmatrix}.$$

Find the Characteristic and minimal polynomial of the block diagonal Matrix $\begin{bmatrix} 2 & 5 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 \\ 0 & 0 & 3 & 5 & 0 \\ 0 & 0 & 0 & 0 & 7 \end{bmatrix}$

Solution:

Mark the block diagram to understand

$$\left[\begin{array}{cc|ccc} 2 & 5 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 \\ 0 & 0 & 3 & 5 & 0 \\ 0 & 0 & 0 & 0 & 7 \end{array} \right]$$

Let us consider the block $A_1 = \begin{bmatrix} 2 & 5 \\ 0 & 2 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 4 & 2 \\ 3 & 5 \end{bmatrix}$ $A_3 = 7$

Let us find the Characteristic equation of A_1 and A_2 , A_3 using

For A_1 $x^2 - (\text{trace } A)x + \text{Det}(A) = 0$

$$x^2 - 4x + 4 = 0, (x - 2)^2$$

For A_2 $x^2 - 9x + 14 = 0$

$$(x - 2)(x - 7) = 0$$

For A_3 $(x - 7)$

$$\text{Char. polynomial} = \Delta_1(t) \cdot \Delta_2(t) \cdot \Delta_3(t) = (x - 2)^2(x - 2)(x - 7) \cdot (x - 7)$$

$$\text{Minimal polynomial} = \text{LCM}(\Delta_1(t), \Delta_2(t), \Delta_3(t))$$

$$= \text{LCM}((x - 2)^2, (x - 2)(x - 7), (x - 7)(x - 2)^2)$$

$$\text{Minimal polynomial} = (x - 2)^2(x - 7)$$

2) Find the Characteristic and minimal polynomial of the block diagonal Matrix $\begin{bmatrix} 4 & -1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$

Solution:

Mark the block diagram to understand

$$\begin{bmatrix} 4 & -1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

Let us consider the block $A_1 = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

Let us find the Characteristic equation of A_1 and A_2 using

For A_1 $x^2 - (\text{trace } A)x + \text{Det}(A) = 0$

$$x^2 - 6x + 9 = 0, (x - 3)^2$$

A_{22} is Minor of second row and second column

For A_2 $x^3 - (\text{trace } A)x^2 + (A_{11} + A_{22} + A_{33})x - \text{Det}(A) = 0$

$$x^3 - (3 + 3 + 3)x^2 + \left(\begin{vmatrix} 3 & 1 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 0 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 0 & 3 \end{vmatrix} \right)x - 27 = 0$$

$$x^3 - 9x^2 + (9 + 9 + 9)x - 27 = 0$$

$$x^3 - 9x^2 + 27x - 27 = 0$$

$$X = 3, 3, 3$$

$$\text{Char. polynomial} = \Delta_1(t) \cdot \Delta_2(t) = (x - 3)^2(x - 3)^3 = (x - 3)^5$$

$$\text{Minimal polynomial} = \text{LCM}(\Delta_1(t), \Delta_2(t)) = \text{LCM}((x - 3)^2, (x - 3)^3)$$

$$\text{Minimal polynomial} = (x - 3)^3$$

3) Find the Characteristic and minimal polynomial of the block diagonal Matrix $\begin{bmatrix} 2 & 7 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -2 & 4 \end{bmatrix}$

Solution:

Mark the block diagram to understand

$$\left[\begin{array}{cc|cc} 2 & 7 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -2 & 4 \end{array} \right]$$

Let us consider the block $A_1 = \begin{bmatrix} 2 & 7 \\ 0 & 2 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$

Let us find the Characteristic equation of A_1 and A_2 using

For A_1 $x^2 - (\text{trace } A)x + \text{Det}(A) = 0$

$$x^2 - 4x + 4 = 0, (x - 2)^2$$

Let us consider the block $A_2 = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$

Let us find the Characteristic equation of A_2

For A_2 $x^2 - (\text{trace } A)x + \text{Det}(A) = 0$

$$x^2 - 5x + 6 = 0, (x - 2)(x - 3)$$

$$\text{Char. polynomial} = \Delta_1(t) \cdot \Delta_2(t) = (x - 2)^2 \cdot (x - 2)(x - 3) = (x - 2)^3 \cdot (x - 3)$$

$$\text{Minimal polynomial} = \text{LCM}(\Delta_1(t), \Delta_2(t)) = \text{LCM}((x - 2)^2, (x - 2)(x - 3))$$

$$\text{Minimal polynomial} = (x - 2)^2(x - 3)$$

3) Find the Characteristic and minimal polynomial of the block diagonal Matrix $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 4 & 1 \end{bmatrix}$

Solution:

Mark the block diagram to understand

$$\left[\begin{array}{cc|cc} 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 4 & 1 \end{array} \right]$$

Let us consider the block $A_1 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$

Let us find the Characteristic equation of A_1

For A_1 $x^2 - (\text{trace } A)x + \text{Det}(A) = 0$

$$x^2 - 2x - 3 = 0, (x - 3)(x + 1)$$

Let us consider the block $A_2 = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$

Let us find the Characteristic equation of A_2

For A_2 $x^2 - (\text{trace } A)x + \text{Det}(A) = 0$

$$x^2 - 4x - 5 = 0, (x - 5)(x + 1)$$

$$\text{Char. polynomial} = \Delta_1(t) \cdot \Delta_2(t) = (x - 3)(x + 1) \cdot (x - 5)(x + 1)$$

$$= (x+1)^2 \cdot (x-3) \cdot (x-5)$$

$$\text{Minimal polynomial} = \text{LCM}(\Delta_1(t), \Delta_2(t)) = \text{LCM}((x-3)(x+1), (x-5)(x+1))$$

$$\text{Minimal polynomial} = (x-3)(x+1) \cdot (x-5)$$

$$5. \begin{bmatrix} 4 & -2 & 2 \\ 6 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix} \text{ Find the Characteristic and minimal polynomial}$$

Solution:

$$x^3 - (\text{trace})x^2 + (A_{11} + A_{22} + A_{33})x - \text{Det}(A) = 0$$

$$x^3 - (4 - 3 + 3)x^2 + (-1 + 6 + 0)x - 2 = 0$$

$$x^3 - 4x^2 + 5x - 2 = 0$$

$$\text{Char. polynomial} = \Delta(t) = (x-2)(x-1)^2$$

$$M(t) = (x-2)(x-1) \quad \text{or} \quad (x-2)(x-1)^2$$

$$f(A) = (A - 2I)(A - I) = \begin{bmatrix} 2 & -2 & 2 \\ 6 & -5 & 4 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 & 2 \\ 6 & -4 & 4 \\ 3 & -2 & 2 \end{bmatrix} = [0]$$

$$\text{Hence } M(t) = (x-2)(x-1)$$

$$5. \begin{bmatrix} 3 & -2 & 2 \\ 4 & -4 & 6 \\ 2 & -3 & 5 \end{bmatrix} \text{ Find the Characteristic and minimal polynomial}$$

Solution:

$$x^3 - (\text{trace})x^2 + (A_{11} + A_{22} + A_{33})x - \text{Det}(A) = 0$$

$$x^3 - (3 - 4 + 5)x^2 + (-2 + 11 - 4)x - 2 = 0$$

$$x^3 - 4x^2 + 5x - 2 = 0$$

$$\text{Char. polynomial} = \Delta(t) = (x-2)(x-1)^2$$

$$M(t) = (x-2)(x-1) \quad \text{or} \quad (x-2)(x-1)^2$$

$$f(A) = (A - 2I)(A - I) = \begin{bmatrix} 1 & -2 & 2 \\ 4 & -6 & 6 \\ 2 & -3 & 3 \end{bmatrix} \begin{bmatrix} 2 & -2 & 2 \\ 4 & -5 & 6 \\ 2 & -3 & 4 \end{bmatrix} = \begin{bmatrix} -2 & 2 & -2 \\ -4 & 4 & -4 \\ -2 & 2 & -2 \end{bmatrix} \neq 0$$

$$\text{Hence } M(t) = (x-2)(x-1)^2$$

$$6) \begin{bmatrix} 3 & -2 & 2 \\ 4 & -4 & 6 \\ 2 & -3 & 5 \end{bmatrix} \text{ Verify Cayley Hamilton Theorem and Hence Find } A^{-1}$$

Solution:

$$x^3 - (\text{trace})x^2 + (A_{11} + A_{22} + A_{33})x - \text{Det}(A) = 0$$

$$x^3 - (3 - 4 + 5)x^2 + (-2 + 11 - 4)x - 2 = 0$$

$$x^3 - 4x^2 + 5x - 2 = 0$$

Replace $x=A$,

$$A^3 - 4A^2 + 5A - 2I = 0$$

$$\begin{aligned} & A^3 - 4A^2 + 5A - 2I \\ &= \begin{bmatrix} 3 & -2 & 2 \\ 4 & -4 & 6 \\ 2 & -3 & 5 \end{bmatrix} \begin{bmatrix} 3 & -2 & 2 \\ 4 & -4 & 6 \\ 2 & -3 & 5 \end{bmatrix} \begin{bmatrix} 3 & -2 & 2 \\ 4 & -4 & 6 \\ 2 & -3 & 5 \end{bmatrix} - 4 \begin{bmatrix} 3 & -2 & 2 \\ 4 & -4 & 6 \\ 2 & -3 & 5 \end{bmatrix} \begin{bmatrix} 3 & -2 & 2 \\ 4 & -4 & 6 \\ 2 & -3 & 5 \end{bmatrix} + 5 \begin{bmatrix} 3 & -2 & 2 \\ 4 & -4 & 6 \\ 2 & -3 & 5 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 7 & -6 & 6 \\ 12 & -18 & 26 \\ 6 & -13 & 21 \end{bmatrix} - 4 \begin{bmatrix} 5 & -4 & 4 \\ 8 & -10 & 14 \\ 4 & -7 & 11 \end{bmatrix} + \begin{bmatrix} 15 & -10 & 10 \\ 20 & -20 & 30 \\ 10 & -15 & 25 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 7-20+15-2 & -6+16-10-0 & 6-16+10-0 \\ 12-32+20+0 & -18+40-20-2 & 26-56+30+0 \\ 6-16+10-0 & -13+28-15+0 & 21-44+25-2 \end{bmatrix} \\ &= \begin{bmatrix} 22-22 & -16+16 & 16-16 \\ 32-32 & -40+40 & 56-56 \\ 16-16 & -28+28 & 46-46 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

Cayley Hamilton Theorem is verified.

To find A^{-1}

Multiply A^{-1}

$$A^3 \cdot A^{-1} - 4A^2 \cdot A^{-1} + 5A \cdot A^{-1} - 2I \cdot A^{-1} = 0$$

$$A^2 - 4A + 5I - 2A^{-1} = 0$$

$$-2A^{-1} = A^2 - 4A + 5I$$

$$A^{-1} = -\frac{1}{2}(A^2 - 4A + 5I)$$

$$\begin{aligned} A^{-1} &= -\frac{1}{2} \left(\begin{bmatrix} 3 & -2 & 2 \\ 4 & -4 & 6 \\ 2 & -3 & 5 \end{bmatrix} \cdot \begin{bmatrix} 3 & -2 & 2 \\ 4 & -4 & 6 \\ 2 & -3 & 5 \end{bmatrix} - 4 \begin{bmatrix} 3 & -2 & 2 \\ 4 & -4 & 6 \\ 2 & -3 & 5 \end{bmatrix} + 5 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \\ &= -\frac{1}{2} \left(\begin{bmatrix} 3 & -2 & 2 \\ 4 & -4 & 6 \\ 2 & -3 & 5 \end{bmatrix} \cdot \begin{bmatrix} 3 & -2 & 2 \\ 4 & -4 & 6 \\ 2 & -3 & 5 \end{bmatrix} - \begin{bmatrix} 12 & -8 & 8 \\ 16 & -16 & 24 \\ 8 & -12 & 20 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \right) \\ &= -\frac{1}{2} \left(\begin{bmatrix} 5 & -4 & 4 \\ 8 & -10 & 14 \\ 4 & -7 & 11 \end{bmatrix} - \begin{bmatrix} 12 & -8 & 8 \\ 16 & -16 & 24 \\ 8 & -12 & 20 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \right) \\ &= -\frac{1}{2} \begin{bmatrix} 5-12+5 & -4+8+0 & 4-8+0 \\ 8-16+0 & -10+16+5 & 14-24+0 \\ 4-8+0 & -7+12+0 & 11-20+5 \end{bmatrix} \end{aligned}$$

$$= -\frac{1}{2} \left(\begin{bmatrix} -1 & 4 & -4 \\ -8 & -11 & 10 \\ -4 & 5 & -4 \end{bmatrix} \right)$$

6) $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -2 & 4 \end{bmatrix}$ Verify Cayley Hamilton Theorem and hence find A^{-1} .

Solution:

Let us consider the block $A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ and $A_2 = \begin{bmatrix} -1 & 1 \\ -2 & 4 \end{bmatrix}$

Let us find the Characteristic equation of A_1

For A_1 $x^2 - (\text{trace } A)x + \text{Det}(A) = 0$

$$x^2 - 3x + 2 = 0$$

Let us consider the block $A_2 = \begin{bmatrix} -1 & 1 \\ -2 & 4 \end{bmatrix}$

Let us find the Characteristic equation of A_2

For A_2 $x^2 - (\text{trace } A)x + \text{Det}(A) = 0$

$$x^2 - 3x - 2 = 0$$

$$(x^2 - 3x - 2)(x^2 - 3x + 2) = 0$$

$$x^4 - 6x^3 + 9x^2 + 0x - 4 = 0$$

Replace $x=A$,

$$A^4 - 6A^3 + 9A^2 - 4I = 0$$

$$A^4 - 6A^3 + 9A^2 - 4I =$$

$$A^4 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 15 & 0 & 0 \\ 0 & 16 & 0 & 0 \\ 0 & 0 & -17 & 39 \\ 0 & 0 & -78 & 178 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 7 & 0 & 0 \\ 0 & 8 & 0 & 0 \\ 0 & 0 & -5 & 11 \\ 0 & 0 & -22 & 50 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & -6 & 14 \end{bmatrix}$$

$$A^4 - 6A^3 + 9A^2 - 4I$$

$$= \begin{bmatrix} 1 & 15 & 0 & 0 \\ 0 & 16 & 0 & 0 \\ 0 & 0 & -17 & 39 \\ 0 & 0 & -78 & 178 \end{bmatrix} - 6 \begin{bmatrix} 1 & 7 & 0 & 0 \\ 0 & 8 & 0 & 0 \\ 0 & 0 & -5 & 11 \\ 0 & 0 & -22 & 50 \end{bmatrix} + 9 \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & -6 & 14 \end{bmatrix} - 4 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-6+9-4 & 15-42+27-0 & 0 & 0 \\ 0 & 16-48+36-4 & 0 & 0 \\ 0 & 0 & -17+30-9-4 & 39-66+27+0 \\ 0 & 0 & -78+132-54+0 & 178-300+126-4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

To find A^{-1}

Multiply A^{-1}

$$A^4 \cdot A^{-1} - 6A^3 \cdot A^{-1} + 9A^2 \cdot A^{-1} - 4I \cdot A^{-1} = 0$$

$$A^3 - 6A^2 + 9A - 4A^{-1} = 0$$

$$-4A^{-1} = A^3 - 6A^2 + 9A$$

$$A^{-1} = \frac{-1}{4} [A^3 - 6A^2 + 9A]$$

$$A^{-1} = \frac{-1}{4} \left[\begin{bmatrix} 1 & 7 & 0 & 0 \\ 0 & 8 & 0 & 0 \\ 0 & 0 & -5 & 11 \\ 0 & 0 & -22 & 50 \end{bmatrix} - 6 \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & -6 & 14 \end{bmatrix} + 9 \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -2 & 4 \end{bmatrix} \right]$$

$$A^{-1} = \frac{-1}{4} \left[\begin{bmatrix} 1 & 7 & 0 & 0 \\ 0 & 8 & 0 & 0 \\ 0 & 0 & -5 & 11 \\ 0 & 0 & -22 & 50 \end{bmatrix} - \begin{bmatrix} 6 & 18 & 0 & 0 \\ 0 & 24 & 0 & 0 \\ 0 & 0 & -6 & 18 \\ 0 & 0 & -36 & 84 \end{bmatrix} + \begin{bmatrix} 9 & 9 & 0 & 0 \\ 0 & 18 & 0 & 0 \\ 0 & 0 & -9 & 9 \\ 0 & 0 & -18 & 36 \end{bmatrix} \right]$$

$$A^{-1} = \frac{-1}{4} \left[\begin{bmatrix} 1-6+0+0 & 7-18+0+0 & 0 & 0 \\ 0 & 8-24+0+0 & 0 & 0 \\ 0 & 0 & -5+6-9 & 11-18+9 \\ 0 & 0 & -22+36-18 & 50-84+36 \end{bmatrix} \right]$$

$$A^{-1} = \frac{-1}{4} \left[\begin{bmatrix} -5 & -11 & 0 & 0 \\ 0 & -16 & 0 & 0 \\ 0 & 0 & -8 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix} \right]$$

Jordan Canonical Form

Let $T:V$ to V be a Linear operator whose characteristic and minimal polynomials are respectively

$$\Delta(t) = (t - \lambda_1)^{m_1} \dots (t - \lambda_r)^{m_r} \quad m(t) = (t - \lambda_1)^{m_1} \dots (t - \lambda_r)^{m_r}$$

where

Where λ_1 are distinct scalars. Then T has a block diagonal matrix representation J which each diagonal entry is a Jordan block $J_{ij} = J(\lambda_1)$.

- i) There is at least one J_{ij} of order m_i ; all other J_{ij} are of order $\leq m_i$
- ii) The sum of the orders of the J_{ij} is n_i
- iii) The number of J_{ij} equals the geometric multiplicity of λ_i
- iv) The number of J_{ij} of each possible order is uniquely determined by T.

Problems:

Suppose the characteristic and minimal polynomials of an operator T are respectively

$$\Delta(t) = (t - 2)^4(t - 5)^3 \quad m(t) = (t - 2)^2(t - 5)^3$$

Solution:

The Jordan canonical form of T is one of the following block diagonal matrices,

$$\text{diag} \left(\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 5 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{bmatrix} \right), \text{diag} \left(\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, [2], [2], \begin{bmatrix} 5 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{bmatrix} \right)$$

The first matrix occurs if T has Two independent Eigen values belonging to the Eigen value 2. The second matrix occurs if T has three independent Eigen vectors belonging to 2.

2)

Determine all possible Jordan canonical forms for a linear operator T: $v \rightarrow Tv$ whose characteristic polynomial $\Delta(t) = (t - 2)^3(t - 5)^2$. In each case find the minimal polynomial $m(t)$.

Solution:

Since $t-2$ has power 3 in $\Delta(t)$, 2 must appear three times on the diagonal, similarly 5 must appear twice.

Thus there are six possibilities:

$$\text{a) } \text{diag} \left[\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix} \right] \quad m(t) = (t - 2)^3(t - 5)^2$$

$$\text{b) } \text{diag} \left[\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}, [5], [5] \right] \quad m(t) = (t-2)^3(t-5)$$

$$\text{c) } \text{diag} \left[\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, [2], \begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix} \right] \quad m(t) = (t-2)^2(t-5)^2$$

$$\text{d) } \text{diag} \left[\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, [2], [5], [5] \right] \quad m(t) = (t-2)^2(t-5)$$

$$\text{e) } \text{diag} \left[[2], [2], [2], \begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix} \right] \quad m(t) = (t-5)^2(t-2)$$

$$\text{f) } \text{diag} [[2], [2], [2], [5], [5]] \quad m(t) = (t-2)(t-5)$$

Rational canonical Form.

Problem:1

Let V be a vector space of dimension 8 over the rational field $\mathbb{Q}$ and let T be a linear operator on V whose minimal polynomial is $m(t) = f_1(t)f_2(t)^2 = (t^4 - 4t^3 + 6t^2 - 4t - 7)(t-3)^2$.

Since $\dim V = 8$.

The characteristics polynomial $\Delta(t) = f_1(t)f_2(t)^4$

The rational canonical form M of T must have one block the companion matrix of $f_1(t)$ and one block the companion matrix of $f_2(t)^2$

There are two possibilities

$$\text{diag}((t^4 - 4t^3 + 6t^2 - 4t - 7)(t-3)^2(t-3)^2).$$

$$\text{diag}((t^4 - 4t^3 + 6t^2 - 4t - 7)(t-3)^2(t-3)(t-3)).$$

$$\text{diag} \left[\begin{bmatrix} 0 & 0 & 0 & 7 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & -6 \\ 0 & 0 & 1 & 4 \end{bmatrix}, \begin{bmatrix} 0 & -9 \\ 1 & 6 \end{bmatrix}, \begin{bmatrix} 0 & -9 \\ 1 & 6 \end{bmatrix} \right]$$

$$\text{diag} \left[\begin{bmatrix} 0 & 0 & 0 & 7 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & -6 \\ 0 & 0 & 1 & 4 \end{bmatrix}, \begin{bmatrix} 0 & -9 \\ 1 & 6 \end{bmatrix}, [3], [3], \right]$$

Problem:2

Let V be a vector space of dimension 7 over $\mathbb{R}$ and let T be a linear operator on V whose minimal polynomial is $m(t) = f_1(t)f_2(t)^3 = (t^2 - 2t + 5)(t - 3)^3$.

Since $\dim V = 7$.

The characteristics polynomial $\Delta(t) = f_1(t)^2 f_2(t)^3$ (or) $f_1(t)^1 f_2(t)^5$

The sum of the orders of the companion matrices must be add upto 7.

Companion matrix must be $(t^2 - 2t + 5)$ & one must be $(t - 3)^3 = (t^3 - 9t^2 + 27t - 27)$

M must be one of the following block

$$\text{diag}(((t^2 - 2t + 5)^2)(t^3 - 9t^2 + 27t - 27)).$$

$$\text{diag}(((t^2 - 2t + 5))(t^3 - 9t^2 + 27t - 27)(t - 3)^2 =$$

$$\text{diag}(((t^2 - 2t + 5))(t^3 - 9t^2 + 27t - 27)(t^2 - 6t + 9)$$

$$\text{diag}(((t^2 - 2t + 5))(t^3 - 9t^2 + 27t - 27)(t - 3)(t - 3)$$

$$\text{diag} \left[\begin{bmatrix} 0 & -5 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & -5 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 27 \\ 1 & 0 & -27 \\ 0 & 1 & 9 \end{bmatrix} \right]$$

$$\text{diag} \left[\begin{bmatrix} 0 & -5 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 27 \\ 1 & 0 & -27 \\ 0 & 1 & 9 \end{bmatrix}, \begin{bmatrix} 0 & -9 \\ 1 & 6 \end{bmatrix}, \right]$$

$$\text{diag} \left[\begin{bmatrix} 0 & -5 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 27 \\ 1 & 0 & -27 \\ 0 & 1 & 9 \end{bmatrix}, [5], [5] \right]$$

