

MODULE-III

EIGENVALUES AND EIGENVECTORS

Upon Completion of this unit, students will be able to:

1. Study the Polynomials of Matrices.
2. Applications of Cayley-Hamilton Theorem and Eigen spaces of a linear transformation.
3. Identify Characteristic and Minimal Polynomials of Block Matrices.
4. Understanding Jordan Canonical form.

Introduction

Determinants give a transition from $Ax = b$ to $Ax = \lambda x$. In both cases the determinant leads to a “formal solution”: to Cramer’s rule for $x = A^{-1}b$, and to the polynomial $\det(A - \lambda I)$, whose roots will be the eigenvalues. (All matrices are now square; the eigenvalues of a rectangular matrix make no more sense than its determinant). The determinant can actually be used if $n = 2$ or 3 . For large n , computing $\det$ is more difficult than solving $Ax = b$.

The key equation was $Ax = \lambda x$. Most vectors x will not satisfy such an equation. They change direction when multiplied by A , so that Ax is not a multiple of x . This means that only certain special numbers are eigenvalues, and only certain special vectors x are eigenvectors which satisfy the key equation. The key equation involves two unknowns λ and x . The number λ (lambda) is an eigenvalue of the matrix A , and the vector x is the associated eigenvector.

Polynomials of Matrices

Consider a polynomial $f(t) = a_n t^n + \cdots + a_1 t + a_0$ over a field K . If A is any square matrix, then we define

$$f(A) = a_n A^n + \cdots + a_1 A + a_0 I$$

where I is the identity matrix. In particular, we say that A is a root of $f(t)$ if $f(A) = 0$, the zero matrix.

Problems:

1. Let $A = \begin{bmatrix} 1 & -2 \\ 4 & 5 \end{bmatrix}$. Find $f(A)$, where a) $f(t) = t^2 - 3t + 7$, b) $g(t) = t^2 - 6t + 13$

Solution: $f(A) = A^2 - 3A + 7I = \begin{bmatrix} 2 & -7 & -12 & -3 & 6 & 7 & 0 & -3 & -6 \\ 24 & 17 & -12 & -15 & 0 & 7 & 12 & 9 \end{bmatrix} + \begin{bmatrix} 24 & 17 & -12 & -15 & 0 & 7 & 12 & 9 \end{bmatrix} + \begin{bmatrix} 24 & 17 & -12 & -15 & 0 & 7 & 12 & 9 \end{bmatrix} = \begin{bmatrix} 24 & 17 & -12 & -15 & 0 & 7 & 12 & 9 \end{bmatrix}$

$g(A) = A^2 - 6A + 13I = \begin{bmatrix} 24 & 17 & -12 & -15 & 0 & 7 & 12 & 9 \end{bmatrix} + \begin{bmatrix} 24 & 17 & -12 & -15 & 0 & 7 & 12 & 9 \end{bmatrix} + \begin{bmatrix} 24 & 17 & -12 & -15 & 0 & 7 & 12 & 9 \end{bmatrix} = \begin{bmatrix} 24 & 17 & -12 & -15 & 0 & 7 & 12 & 9 \end{bmatrix}$

Thus, A is root of $g(t)$.

2. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Find $f(A)$, where a) $f(t) = t^2 - 5t - 2$, b) $g(t) = 2t^2 - 3t + 5$

Solution: $f(A) = A^2 - 5A - 2I = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} + \begin{bmatrix} -5 & -10 \\ -15 & -20 \end{bmatrix} + \begin{bmatrix} -20 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Thus, A is root of $f(t)$.

$g(A) = 2A^2 - 3A + 5I = \begin{bmatrix} 14 & 20 \\ 30 & 44 \end{bmatrix} + \begin{bmatrix} -3 & -12 \\ -12 & 0 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 16 & 8 \\ 18 & 37 \end{bmatrix}$

Applications of Cayley-Hamilton Theorem

The Cayley-Hamilton Theorem is a fundamental result in linear algebra that has many applications in various areas of mathematics, physics, and engineering. This theorem states that **every square matrix satisfies its own characteristic equation**. In other words, if A is a square matrix and λ is its eigenvalue, then the characteristic polynomial of A evaluated at A itself gives the zero matrix: $p(A)=0$, where $p(x)$ is the characteristic polynomial of A. This theorem has numerous applications, some of which are outlined below:

- Matrix Inversion:** The Cayley-Hamilton Theorem can be used to find the inverse of a matrix. If you have the characteristic polynomial of a matrix, you can find the inverse without explicitly computing cofactors and determinants.
- Diagonalization:** The theorem is often used to show that a matrix is diagonalizable. Specifically, if the characteristic polynomial of a matrix A splits into linear factors (i.e., it has all distinct eigenvalues), the matrix is diagonalizable.
- Jordan Canonical Form:** The theorem is related to the Jordan Canonical Form, which is a way to decompose a matrix into a block-diagonal form. It's useful for solving systems of linear differential equations and understanding the behavior of linear transformations.

4. **Linear Differential Equations:** The Cayley-Hamilton Theorem is used to solve linear homogeneous differential equations, especially those with constant coefficients. It helps in finding the general solution and understanding the behavior of the system.

5. **Matrix Exponentiation:** The Cayley-Hamilton Theorem can be used to simplify matrix exponentiation. For example, if you need to compute e^{At} , where A is a matrix, the theorem can help simplify the calculation.

1. Verify Cayley-Hamilton Theorem and find A^{-1} of the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$.

Solution: The characteristic equation of A is $\det(A - \lambda I) = 0$,

$$\text{i.e., } \begin{vmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(2-\lambda) - 6 = 0$$

$$\lambda^2 - 3\lambda - 4 = 0.$$

$$\text{Now } A^2 - 3A - 4I = \begin{bmatrix} 7 & 6 \\ 9 & 10 \end{bmatrix} - 3 \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Since A satisfies the characteristic equation, Cayley-Hamilton Theorem is verified.

$$\text{Now } A^{-1} \times (A^2 - 3A - 4I) = A - 3I - 4A^{-1}$$

$$A^{-1} = \frac{1}{4} (A - 3I) = \begin{bmatrix} -0.5 & 0.5 \\ 0.75 & -0.25 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}$$

2. Verify Cayley-Hamilton Theorem and find A^{-1} of the matrix $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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Solution: The characteristic equation is $\det(A - \lambda I) = 0$,

$$\begin{vmatrix} 2-\lambda & 0 & 1 \\ 1 & 2-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0,$$

$$(2-\lambda)[(2-\lambda)^2 - 0] - 0 + 1[0 - (2-\lambda)] = 0,$$

$$\text{i.e., } (2-\lambda)[\lambda^2 - 4\lambda + 4] - 2 + \lambda = 0,$$

$$2\lambda^2 - 8\lambda + 8 - \lambda^3 + 4\lambda^2 - 4\lambda + \lambda - 2 = 0,$$

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0.$$

$$\begin{aligned} \text{Now } A^3 - 6A^2 + 11A - 6I &= \begin{bmatrix} 14 & 0 & 13 \\ 0 & 8 & 0 \\ 13 & 0 & 14 \end{bmatrix} - 6 \begin{bmatrix} 5 & 0 & 4 \\ 0 & 4 & 0 \\ 4 & 0 & 5 \end{bmatrix} + 11 \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} - 6 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 14-30+22-6 & 0-0+0-0 & 13-24+11-0 \\ 0-0+0-0 & 8-24+22-6 & 0-0+0-0 \\ 13-24+22-6 & 0-0+0-0 & 14-30+22-6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

Since A satisfies the characteristic equation, Cayley-Hamilton Theorem is verified.

$$\text{Now } A^{-1} \times (A^3 - 6A^2 + 11A - 6I) = A^2 - 6A + 11I - 6A^{-1}$$

$$\begin{aligned} A^{-1} &= \frac{1}{6} (A^2 - 6A + 11I) = \begin{bmatrix} 0.6660 & 0.50 & -0.3330 \\ -0.333 & 0 & 0.666 \\ 0.666 & 0.50 & -0.333 \end{bmatrix} \\ \text{3. Using Cayley-Hamilton Theorem, find } A^{-1} \text{ and } A^4 \text{ of the matrix } A &= \begin{bmatrix} -6 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix} \end{aligned}$$

Solution: The characteristic equation is $\det(A - \lambda I) = 0$,

$$\text{i.e., } \begin{vmatrix} 7-\lambda & 2 & -2 \\ -6 & -1-\lambda & 2 \\ 6 & 2 & -1-\lambda \end{vmatrix} = 0,$$

$$\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0.$$

By Cayley-Hamilton Theorem, A satisfies the characteristic equation.

$$A^3 - 5A^2 + 7A - 3I = 0.$$

$$\text{Now } A^{-1} \times (A^3 - 5A^2 + 7A - 3I) = A^2 - 5A + 7I - 3A^{-1}$$

$$A^{-1} = \frac{1}{3} \begin{bmatrix} -12 & 0.666 & -0.666 \\ -0.666 & 1.666 & -2 \\ -0.666 & -0.666 & 1.666 \end{bmatrix}$$

$$\text{Now } A \times (A^3 - 5A^2 + 7A - 3I) = A^4 - 5A^3 + 7A^2 - 3A$$

$$A^4 = 5A^3 - 7A^2 + 3A = \begin{bmatrix} 241 & 80 & -80 \\ -240 & -79 & 80 \\ 240 & 80 & -79 \end{bmatrix}$$

Eigen spaces of a linear transformation

For a given linear operator $T : V \rightarrow V$, a nonzero vector x and a constant scalar λ are called an eigenvector and its eigenvalue, respectively, when $T(x) = \lambda x$. For a given eigenvalue λ , the set of all x such that $T(x) = \lambda x$ is called the λ -eigenspace. The set of all eigenvalues for a transformation is called its spectrum.

When the operator T is described by a matrix A , then we'll associate the eigenvectors, eigenvalues, eigenspaces, and spectrum to A as well. As A directly describes a linear operator on F^n , we'll take its eigenspaces to be subsets of F^n .

Theorem: Each λ -eigenspace is a subspace of V .

Proof. Suppose that x and y are λ -eigenvectors and c is a scalar. Then $T(x + cy) = T(x) + cT(y) = \lambda x + c\lambda y = \lambda(x + cy)$. Therefore $x + cy$ is also a λ -eigenvector. Thus, the set of λ -eigenvectors form a subspace of F^n .

Characteristic Polynomial

Let $A = [a_{ij}]$ be an n -square matrix. The matrix $M = A - tI_n$, where I_n is the n -square identity matrix and t is an indeterminate, may be obtained by subtracting t down the diagonal of A . The negative of M is the matrix $tI_n - A$, and its determinant

$$\Delta(t) = \det(tI_n - A) = (-1)^n \det(A - tI_n)$$

Which is a polynomial in t of degree n and is called the characteristic polynomial of A .

Problems:

- Find the characteristic polynomial of the matrix $A = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix}$.

Solution:

$$\Delta(t) = \det(tI_n - A) = \begin{vmatrix} t-1 & -3 \\ -4 & t-5 \end{vmatrix}$$

$$= (t-1)(t-5) - 12 = t^2 - 6t - 7$$

Characteristic Polynomial of Degrees 2 and 3

There are simple formulas for the characteristic polynomials of matrices of orders 2 and 3.

- (a) Suppose $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ then

$$\Delta(t) = \det(tI_n - A) = t^2 - (a_{11} + a_{22})t + \det(A) = t^2 - \text{trace}(A)t + \det(A)$$

- (b) Suppose $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then

$$\Delta(t) = \det(tI_n - A) = t^3 - \text{trace}(A)t^2 + (A_{11} + A_{22} + A_{33})t - \det(A)$$

Where A_{11}, A_{22}, A_{33} denote the cofactors of a_{11}, a_{22}, a_{33} .

- Find the characteristic polynomial of the following matrices

$$\begin{bmatrix} 5 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 7 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -2 \end{bmatrix}$$

$$i) A = \begin{bmatrix} & \\ 2 & 10 \end{bmatrix} \quad ii) B = \begin{bmatrix} & \\ 6 & 2 \end{bmatrix} \quad iii) C = \begin{bmatrix} & \\ 4 & -4 \end{bmatrix}$$

Solution:

$$a) \Delta(t) = t^2 - 15t + 44$$

$$b) \Delta(t) = t^2 - 9t + 20$$

$$c) \Delta(t) = t^2 - t - 12$$

$$3. \text{Find the characteristic polynomial of the matrix } A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 3 & 2 \\ 1 & 3 & 9 \end{bmatrix}$$

Solution:

$$\begin{aligned} \Delta(t) &= \det(tI_n - A) \\ &= t^3 - \text{trace}(A)t^2 + (A_{11} + A_{22} + A_{33})t - \det(A) = t^3 - 13t^2 + 31t - 17. \end{aligned}$$

Characteristic Polynomials and Block Triangular Matrices

Suppose M is a block triangular matrix, say $A = \begin{bmatrix} A_1 & B \\ 0 & A_2 \end{bmatrix}$, where A_1 and A_2 are square matrices.

Then $tI - A$ is also a block triangular matrix, with diagonal blocks $tI - A_1$ and $tI - A_2$. Thus,

$$|tI - M| = \begin{vmatrix} tI - A_1 & -B \\ 0 & tI - A_2 \end{vmatrix} = |tI - A_1| |tI - A_2|$$

That is, the characteristic polynomial of M is the product of the characteristic polynomials of the diagonal blocks A_1 and A_2 .

Note: Suppose M is a block triangular matrix with diagonal blocks $A_1, A_2, \dots, A_r$. Then the characteristic polynomial of M is the product of the characteristic polynomials of the diagonal blocks A_i . That is

$$\Delta_M(t) = \Delta_{A_1}(t) \Delta_{A_2}(t) \dots \Delta_{A_r}(t)$$

Problems:

1. Find the characteristic polynomial of the matrix $M = \begin{bmatrix} 9 & -1 & 5 & 7 \\ 8 & 3 & 2 & -4 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & -1 & 8 \end{bmatrix}$

Solution: Given $M = \begin{bmatrix} 9 & -1 & 5 \\ 8 & 3 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & -1 \end{bmatrix}$

$$A = \begin{bmatrix} 9 & -1 \\ 8 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 5 & 7 \\ -1 & 8 \end{bmatrix}$$

Here $\Delta_A(t) = t^2 - 12t + 35 = (t - 5)(t - 7)$

$\Delta_B(t) = t^2 - 11t + 30 = (t - 5)(t - 6)$

The characteristic polynomial of the matrix M is the product

$$\Delta_M(t) = \Delta_A(t) \Delta_B(t) = (t - 5)^2(t - 6)(t - 7).$$

2. Find the characteristic polynomial of the matrix $M = \begin{bmatrix} 2 & 5 & 1 \\ 1 & 4 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 6 \end{bmatrix}$

Then M is a block triangular matrix with diagonal blocks $\begin{bmatrix} 2 & 5 \\ 1 & 4 \end{bmatrix}$ and $\begin{bmatrix} 3 & 6 \\ 0 & 6 \end{bmatrix}$

Solution: Given $M = \begin{bmatrix} 2 & 5 & 1 & 1 \\ 1 & 4 & 2 & 2 \\ 0 & 0 & 6 & -5 \\ 0 & 0 & 2 & 3 \end{bmatrix}$

Then M is a block triangular matrix with diagonal blocks $A_1 = \begin{bmatrix} 2 & 5 \\ 1 & 4 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 6 & -5 \\ 2 & 3 \end{bmatrix}$

Here $\Delta_{A_1}(t) = t^2 - 6t + 3$

$\Delta_{A_2}(t) = t^2 - 9t + 28$

The characteristic polynomial of the matrix M is the product

$\Delta_M(t) = \Delta_{A_1}(t)\Delta_{A_2}(t) = (t^2 - 6t + 3)(t^2 - 9t + 28).$

3. Find the characteristic polynomial of the matrix $M = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 3 & 3 & 4 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & 0 & 6 \end{bmatrix}$

Solution: Given $M = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 3 & 3 & 4 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & 0 & 6 \end{bmatrix}$

The characteristic polynomial of the matrix M is the product

$$\Delta_M(t) = (t - 1)(t - 3)(t - 5)(t - 6).$$

Minimal Polynomials and Block Diagonal Matrices

Suppose M is a block diagonal matrix with diagonal blocks $A_1, A_2, \dots, A_r$. Then the minimal polynomial of M is equal to the least common multiple (LCM) of the minimal polynomials of the diagonal blocks A_i .

Problems:

1. Find the characteristic polynomial $\Delta(t)$ and the minimal polynomial $m(t)$ of the block diagonal

matrix $M = \begin{bmatrix} 2 & 5 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 \\ 0 & 0 & 3 & 5 & 0 \\ 0 & 0 & 0 & 0 & 7 \end{bmatrix}$

Solution: Given $M = \begin{bmatrix} 2 & 5 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 \\ 0 & 0 & 3 & 5 & 0 \\ 0 & 0 & 0 & 0 & 7 \end{bmatrix}$

Then M is a block triangular matrix with diagonal blocks $A_1 = \begin{bmatrix} 2 & 5 \\ 0 & 2 \end{bmatrix}$, $A_2 = \begin{bmatrix} 4 & 2 \\ 3 & 5 \end{bmatrix}$ and

$$A_3 = [7]$$

$$\text{Here } \Delta_{A_1}(t) = t^2 - 4t + 4 = (t - 2)^2$$

$$\Delta_{A_2}(t) = t^2 - 12t + 35 = (t - 2)(t - 7)$$

$$\Delta_{A_3}(t) = t - 7$$

The characteristic polynomial of the matrix M is the product

$$\Delta_M(t) = \Delta_{A_1}(t)\Delta_{A_2}(t)\Delta_{A_3}(t) = (t - 2)^3(t - 7)^2.$$

The minimal polynomial $m_1(t)$, $m_2(t)$, $m_3(t)$ of the diagonal blocks A_1 , A_2 , A_3 respectively, are equal to the characteristic polynomial $m_1(t) = (t - 2)^2$

$$m_2(t) = (t - 2)(t - 7)$$

$$m_3(t) = t - 7$$

The minimal polynomial $m(t)$ is to the least common multiple of $m_1(t)$, $m_2(t)$, $m_3(t)$. Thus $m(t) = (t - 2)^2(t - 7)$.

$$\begin{matrix} 4 & -2 & 2 & & 3 & -2 & 2 \\ & & & & & & \end{matrix}$$

2. Let $A = \begin{bmatrix} 6 & -3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -4 & 6 \end{bmatrix}$. The characteristic polynomial of both matrices is

$$\begin{matrix} 3 & -2 & 3 & & 2 & -3 & 5 \\ & & & & & & \end{matrix}$$

$\Delta(t) = (t - 2)(t - 1)^2$. Find the minimal polynomial $m(t)$ of each matrix.

Solution: The minimal polynomial $m(t)$ must divide $\Delta(t)$. Thus $m(t)$ must be exactly one of the following

$$f(t) = (t - 2)(t - 1) \text{ or } g(t) = (t - 2)(t - 1)^2$$

(a) By the Cayley Hamilton theorem, $g(A) = \Delta(A) = 0$, so we need only test $f(t)$.

$$\begin{matrix} & & 2 & -2 & 2 & 3 & -2 & 2 & 0 & 0 & 0 \\ f(A) = (A - 2I)(A - I) = & \begin{bmatrix} 6 & -5 & 4 \\ 3 & -2 & 1 \end{bmatrix} & \begin{bmatrix} 6 & -4 & 4 \\ 3 & -2 & 2 \end{bmatrix} = & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{matrix}$$

Thus $m(t) = f(t) = (t - 2)(t - 1)$ is minimal polynomial of A.

(b) Again $g(B) = \Delta(B) = 0$, so we need only test $f(t)$.

$$\begin{matrix} & & 1 & -2 & 2 & 2 & -2 & 2 \\ f(B) = (B - 2I)(B - I) = & \begin{bmatrix} 4 & -6 & 6 \\ 2 & -3 & 3 \end{bmatrix} & \begin{bmatrix} 4 & -5 & 6 \\ 2 & -3 & 4 \end{bmatrix} \neq 0 \end{matrix}$$

Thus $m(t) \neq f(t)$ so $m(t) = (t - 2)(t - 1)^2$ is minimal polynomial of B.

3. Find the characteristic polynomial $\Delta(t)$ and the minimal polynomial $m(t)$ of the matrix $M =$

$$\begin{bmatrix} 4 & 1 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 1 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

Solution: Given $M =$

$$\begin{bmatrix} 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

Then M is a block triangular matrix with diagonal blocks $A_1 = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{bmatrix}$, $A_2 = \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$

$$\text{Here } \Delta_{A_1}(t) = (t - 4)^3$$

$$\Delta_{A_2}(t) = (t - 4)^2$$

The characteristic polynomial of the matrix M is the product

$$\Delta_M(t) = \Delta_{A_1}(t)\Delta_{A_2}(t) = (t - 4)^5.$$

The minimal polynomial $m(t)$ is to the least common multiple of $\Delta_{A_1}(t)$, $\Delta_{A_2}(t)$. Thus $m(t) = (t - 4)^3$.

4. Find the characteristic polynomial $\Delta(t)$ and the minimal polynomial $m(t)$ of the matrix $M =$

$$\begin{bmatrix} 2 & 7 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -2 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 7 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -2 & 4 \end{bmatrix}$$

Solution: Given $M =$

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & -2 & 4 \end{bmatrix}$$

Then M is a block triangular matrix with diagonal blocks $A_1 = \begin{bmatrix} 2 & 7 \\ 1 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$

$$\text{Here } \Delta_{A_1}(t) = (t - 2)^2$$

$$\Delta_{A_2}(t) = (t - 2)(t - 3)$$

The characteristic polynomial of the matrix M is the product

$$\Delta_M(t) = \Delta_{A_1}(t)\Delta_{A_2}(t) = (t - 2)^3(t - 3).$$

The minimal polynomial $m(t)$ is to the least common multiple of $\Delta_{A_1}(t)$, $\Delta_{A_2}(t)$. Thus $m(t) = (t - 2)^2(t - 3)$.

Jordan Canonical form

The matrix of the form

$$\begin{bmatrix} \lambda & 1 & 0 & \dots & 0 & 0 \\ 0 & \lambda & 1 & & 0 & 0 \\ \vdots & & & \ddots & & \\ 0 & 0 & 0 & \dots & \lambda & 1 \\ & & & & 0 & \lambda \end{bmatrix}_J$$

is called Jordan block matrix belonging to λ . In this matrix A, λ 's are on the diagonal and 1's are on the superdiagonal and other elements are zero.

Problems:

1. Determine all possible Jordan Canonical forms for a linear operator $T: V \rightarrow V$ whose characteristic polynomial is $\Delta(x) = (x - 2)^3(x - 5)^2$

Solution: Since $x - 2$ has exponent 3 in $\Delta(x)$ and $x - 5$ has exponent 2 in $\Delta(x)$, therefore Jordan Canonical form will be the matrix of order 5×5 .

$$\Delta(x) = (x - \lambda_1)^3(x - \lambda_2)^2 \text{ where } \lambda_1 = 2 \text{ and } \lambda_2 = 5.$$

Now, $\lambda_1 = 2$ must appear three times on the main diagonal and $\lambda_2 = 5$ must appear two times.

Hence the possible Jordan Canonical forms are

canonical forms are

$$(i) \left[\begin{array}{ccc|cc} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ \hline 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{array} \right]$$

$$(ii) \left[\begin{array}{ccc|cc} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ \hline 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{array} \right]$$

$$(iii) \left[\begin{array}{ccc|cc} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ \hline 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{array} \right]$$

$$(iv) \left[\begin{array}{ccc|cc} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ \hline 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{array} \right]$$

$$(v) \left[\begin{array}{ccc|cc} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ \hline 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{array} \right]$$

$$(vi) \left[\begin{array}{ccc|cc} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ \hline 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{array} \right]$$

characteristic polynomial

2. If A is a 5×5 matrix with characteristic polynomial $\Delta(x) = (x - 2)^3(x + 7)^2$ and minimal polynomial $m(x) = (x - 2)^2(x + 7)$. What is the Jordan form of A ?

Solution:

Since $x - 2$ has exponent 3 in $\Delta(x)$ and $x + 7$ has exponent 2 in $\Delta(x)$, therefore Jordan Canonical form will be the matrix of order 5×5 . Also, in $m(x)$, the exponent of $(x - 2)$ is 2 and $(x + 7)$ is 1. Therefore, Jordan form will have one block of order 2 and the other two blocks must be of order 2 or 1. Hence the required Jordan form of A is

$$\left[\begin{array}{cc|ccc} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ \hline 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ \hline 0 & 0 & 0 & 0 & -7 \end{array} \right]$$

3. Determine all possible Jordan canonical forms for a matrix of order 5 whose minimum polynomial is $m(x) = (x - 2)^2$ **Solution:**

Since minimum polynomial of the matrix of order 5×5 is $(x - 2)^2$, therefore its characteristic polynomial will be $(x - 2)^5$. Thus, Jordan form will have one block of order 2 and the other two blocks must be of order 2 or 1. Hence the possible Jordan form of A are

$$(i) \left[\begin{array}{cc|ccc} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ \hline 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ \hline 0 & 0 & 0 & 0 & 2 \end{array} \right] \quad (ii) \left[\begin{array}{cc|ccc} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ \hline 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ \hline 0 & 0 & 0 & 0 & 2 \end{array} \right]$$

Video Links:

<https://www.youtube.com/watch?v=wSxKJUJloJw>

<https://www.youtube.com/watch?v=lWSrdaqhUyE>

<https://www.youtube.com/watch?v=j0QA95Ox2oI>

<https://www.youtube.com/watch?v=GVixvieNnyc>

https://www.youtube.com/watch?v=UKj_ufj1dK4