MODULE-2

Taylor's series expansion:-

If $y = f(x)$ is a continous function then expansion of $f(x)$ at point $x = a$ is given by

$$y = f(x) = y_0 + (x-a)y'(a) + \frac{(x-a)^2}{2!}y''(a) + \frac{(x-a)^3}{3!}y'''(a) + \cdots\cdots$$

Maclaurin's series:-

If $y = f(x)$ is continous function then expansion of function at point $x = 0$ is given by $y = f(x)$, $x = 0$

$$y = f(x) = y_0 + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \frac{x^4}{4!}y^{iv}(0) + \cdots\cdots$$

Problems:-

1. Expand $y = e^x$ at $x = 0$ or expand e^x as maclaurin's series.

Sol:- Consider, $y = e^x$

$$y = f(x) = y_0 + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \cdots\cdots$$

Since given to find M.S $\Rightarrow x = 0$

function	Value
$y = e^x$	$y_0 = y(0) = e^0 = 1$
$y' = e^x$	$y'(0) = e^0 = 1$
$y'' = e^x$	$y''(0) = e^0 = 1$
$y''' = e^x$	$y'''(0) = e^0 = 1$
$y^{IV} = e^x$	$y^{IV}(0) = e^0 = 1$

$$y = f(x) = e^x = 1 + x(1) + \frac{x^2}{2!}(1) + \frac{x^3}{3!}(1) + \frac{x^4}{4!}(1)$$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$$

2. Expand $\sqrt{1+\sin 2x}$ as M.S

Sol:- Consider, $\sqrt{1+\sin 2x}$

$$y = f(x) = y_0 + x\cdot y'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \frac{x^4}{4!}y^{iv}(0) + \cdots$$

Since given to find MS $\Rightarrow x=0$

$$y = \sqrt{1+\sin 2x} = \sqrt{\sin^2 x + \cos^2 x + 2\sin x.\cos x}$$

$$= \sqrt{(\sin x + \cos x)^{\cancel{2}}}$$

$$y = \sin x + \cos x$$

function	Value
$y = \sin x + \cos x$	$y(0) = \sin(0) + \cos(0) = 1$
$y' = \cos x - \sin x$	$y'(0) = \cos(0) - \sin(0) = 1$
$y'' = -\sin x - \cos x$	$y''(0) = -\sin(0) - \cos(0) = -1$
$y''' = -\cos x + \sin x$	$y'''(0) = -\cos(0) - \sin(0) = -1$
$y^{IV} = \sin x + \cos x$	$y^{IV}(0) = \sin(0) + \cos(0) = 1$

$$y = \sqrt{1+\sin 2x} = 1 + x(1) + \frac{x^2}{2!}(1) + \frac{x^3}{3!}(-1) + \frac{x^4}{4!}(1)$$

$$y = \sqrt{1+\sin 2x} = 1 + x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24}$$

3. Expand $\log(\sec x)$ as M.S

Sol:- WKT, MS is given by

$$y = f(x) = y^0 + x.y'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \cdots$$

$$y = \log(\sec x)$$

d.w.r.t x

$$y_1 = \frac{1}{\cancel{\sec x}}.\cancel{\sec x}.\tan x \Rightarrow y_1 \tan x$$

Function	Value at $x=0$
Let $y=\log(\sec x)$ d.w.r.t x	$y(0)=\log(\sec(0))=\log 1=0$
$y_1=\tan x$ d.w.r.t x	$y_1(0)=\tan(0)=0$
$y_2=\sec^2 x$ d.w.r.t x	$y_2(0)=\sec^2(0)=1$
$y_2=1+\tan^2 x$ $=1+y_1^2$ $y_3=0+2y_1y_2$ d.w.r.t x	$y_3(0)=2y_1(0).\,y_2(0)=0$ $y_3(0)=0$
$y_4=2[y_1y_3+y_2y_2]$ d.w.r.t x	$y_4(0)=2[y_1(0)y_3(0)+y_2^2(0)]$ $y_4(0)=2$
$y_5=2[y_1y_4+y_3y_2+2y_2y_3]$	$y_5(0)=2[y_1(0)y_4(0)+y_3(0)y_2(0)+2y_2(0)y_3(0)]$
$y_5=2[y_1y_4+3y_2y_3]$ d.w.r.t x	$y_5(0)=0$
$y_6=2[y_1y_5+y_4y_2+3(y_2y_4+y_3y_3)]$	$y_6(0)=2[y_1(0)y_5(0)+y_4(0)y_2(0)+3y_2(0)y_4(0)+3y_3^2(0)]$ $y_6(0)=2(2\times1+3\times1\times2)$ $y_6(0)=16$

$$\log(\sec x)=0+x_1(0)+\frac{x^2}{2!}(1)+\frac{x^3}{3!}(0)+\frac{x^4}{4!}(2)+\frac{x^5}{5!}(0)+\frac{x^6}{6!}(16)$$

$$\log(\sec x)=\frac{x^2}{2}+\frac{x^4}{12}+\frac{x^6}{45}$$

4. Expand $\tan x$ as MS

Sol:-

Function	Value at $x=0$
$y = \tan x$ d.w.r.t x	$y = \tan(0) = 0$
$y_1 = \sec^2 x$ $= 1+\tan^2 x$ $y_1 = 1+y^2$ d.w.r.t x	$y_1(0) = \sec^2(0) = 1$
$y_2 = 0 + 2yy_1$ d.w.r.t x	$y_2(0) = 2[y(0)\cdot y_1(0)]$ $y_2(0) = 0$
$y_3 = 2[y\cdot y_2 + y_1 y_1]$ d.w.r.t x	$y_3(0) = 2[y(0)\cdot y_2(0) + y_1^2(0)]$ $y_3(0) = 2$
$y_4 = 2[y\cdot y_3 + y_2 y_1 + y_1 y_2 + y_1 y_2]$	$y_4(0) = 2[y(0)\cdot y_3(0) + y_2(0)\, y_1(0) +$ $y_1(0) y_2(0) + y_1(0)\, y_2(0)]$ $y_4(0) = 0.$

$$\tan x = 0 + x1(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(0)$$

$$\tan x = x1 + \frac{x^3}{3}$$

5. Expand $\log(1+\sin x)$ as MS

Sol:-

Function	Value at $x=0$
$y = \log(1+\sin x)$ d.w.r.t x	$y(0) = \log(1+\sin(0))$ $y(0) = \log 1 \Rightarrow y(0) = 0$
$y_1 = \frac{1}{1+\sin x} \times (0+\cos x)$	$y_1(0) = \frac{\cos(0)}{1+\sin(0)} = \frac{1}{1+0}$
$y_1 = \frac{\cos x}{\quad}$	$y_1(0) = 1$

$$y_2 = \frac{(1+\sin x)(-\sin x) - \cos x \cdot \cos x}{(1+\sin x)^2}$$

$$= \frac{-\sin x - \sin^2 x - \cos^2 x}{(1+\sin x)^2}$$

$$= \frac{-(\sin x + \sin^2 x + \cos^2 x)}{(1+\sin x)^2}$$

$$= \frac{-\cancel{(\sin x + 1)}}{(1+\sin x)^{\cancel{2}}} = \frac{-1}{1+\sin x}$$

d.w.r.t x

$$y_2(0) = \frac{-1}{(1+\sin(0))}$$

$$= \frac{-1}{1+0} = -1$$

$$y_2(0) = -1$$

$$y_3 = -\left(\frac{-1}{(1+\sin x)^2} \times \cos x\right)$$

$$= \frac{\cos x}{(1+\sin x)^2}$$ d.w.r.t x

$$y_3(0) = \frac{\cos(0)}{(1+\sin(0))^2}$$

$$y_3(0) = 1$$

$$y_4 = \frac{(1+\sin x)^2(-\sin x) - \cos x \, 2(1+\sin x)\cos x}{((1+\sin x)^2)^2}$$

$$y_4 = \frac{(1+\sin x)^2(-\sin x) - 2(1+\sin x)\cos^2 x}{((1+\sin x)^2)^2}$$

$$y_4 = \frac{-\sin x - \sin^2 x - 2\cos^2 x}{(1+\sin x)^3}$$

$$y_4(0) = \frac{\sin(0) - \sin^2(0) - 2\cos^2(0)}{(1+\sin(0))^3}$$

$$= \frac{-2}{(1+0)^3} = -2$$

$$\log(1+\sin x) = 0 + x_1(1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(1) + \frac{x^4}{4!}(-2)$$

$$\log(1+\sin x) = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} \cdots\cdots$$

6. Expand $\log(1+x)$ as MS

Sol:-

Function	Value at $x=0$
$y = \log(1+x)$ d.w.r.t x	$y(0) = \log(1+0)$ $\because \log 1 = 0$ $y(0) = 0$
$y_1 = \frac{1}{1+x}$ d.w.r.t x	$y_1(0) = \frac{1}{1+0}$ $y_1(0) = 1$
$y_2 = \frac{-1}{(1+x)^2}$ d.w.r.t. x	$y_2(0) = \frac{-1}{(1+0)^2} = -1$ $y_2(0) = -1$
$y_3 = \frac{(-1)(-2)}{(1+x)^3}$ d.w.r.t. x.	$y_3(0) = \frac{2}{(1+0)^3}$ $y_3(0) = 2$
$y_4 = \frac{(-1)(-2)(-3)}{(1+x)^4}$.	$y_4 = \frac{2(-3)}{(1+0)^4} = -6$

$$\log(1+x) = 0 + x(1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(-6)$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \text{-----}$$

Hence reduce that $\log\sqrt{\frac{1+x}{1-x}} = x + \frac{x^3}{3} + \text{-----}$

$$\log\left(\frac{1+x}{1-x}\right)^{1/2} = \frac{1}{2}\left(\log(1+x) - \log(1-x)\right)$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \text{-----}$$

Indeterminate forms

If an expression $f(x)$ at $x=a$ assumes forms like $\frac{0}{0}, \frac{\infty}{\infty}, 0\times\infty, \infty-\infty, \infty^0, 1^\infty$ do not represent any value are called indeterminate form.

L' Hospital's Rule:-

If $f(x)$ and $g(x)$ are two functions such that

i) $\lim_{x\to a} f(x)=0$ & $\lim_{x\to a} g(x)=0$

ii) $f'(x)$ and $g'(x)$ exists and $g'(a)$ is not equals to '0' then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)} \qquad (g'(a)\neq 0)$$

if $f'(a)=0$ & $g'(a)=0$, then $\lim_{x\to a}\frac{f'(x)}{g'(x)} = \lim_{x\to a}\frac{f''(x)}{g''(x)}$

These process is continued until the indeterminate form

This rule is applicable only for $\frac{0}{0}, \frac{\infty}{\infty}$ indeterminate form, to all other indeterminate forms rearrange to get $\frac{0}{0}, \frac{\infty}{\infty}$ and then apply L'H rule.

Note:- Standard limit :-

i) $\lim_{x\to 0}\frac{\sin x}{x} = 1 = \lim_{x\to 0}\frac{x}{\sin x}$

ii) $\lim_{x\to 0}\frac{\tan x}{x} = 1 = \lim_{x\to 0}\frac{x}{\tan x}$

1. Evalute $\lim_{x\to 0}\frac{a^x-b^x}{x}$

Sol:- Let $K = \lim_{x\to 0}\frac{a^x-b^x}{x}$

apply L'H rule $\left(\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}\right)$

$$K = \lim_{x\to 0}\frac{a^x\log a - b^x\log b}{1}$$

Apply limit

$K = a^0 \log a - b^0 \log b$

$K = \log a - \log b$

$K = \log\left(\frac{a}{b}\right)$

2. Evalute $\lim_{x \to \frac{\pi}{2}} \frac{\log\left(x - \frac{\pi}{2}\right)}{\tan x}$ $\left(\frac{\log\left(\frac{\pi}{2} - \frac{\pi}{2}\right)}{\tan\frac{\pi}{2}} = \frac{-\infty}{\infty}\right)$

Sol:- Apply L'H rule

$$K = \lim_{x \to \frac{\pi}{2}} \frac{\frac{1}{x - \frac{\pi}{2}} (1)}{\sec^2 x} = \lim_{x \to \frac{\pi}{2}} \frac{\cos^2 x}{x - \frac{\pi}{2}}$$

Apply L'H rule

$$K = \lim_{x \to \frac{\pi}{2}} \frac{2\cos x(-\sin x)}{1 - 0}$$

Apply limit

$$K = \frac{-\cancel{2} \sin \cancel{2} \cdot \frac{\pi}{\cancel{2}}}{1}$$

$$\boxed{K = 0}$$

<u>Indeterminate of the form $0^0, \infty^0, 0^\infty, 1^\infty$</u> :-

1. Evaluate $\lim_{x \to 0} \left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$

<u>Sol</u>:- $K = \lim_{x \to 0} \left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$

Take log b.s

$$\log K = \lim_{x \to 0} \log\left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$$

$$= \lim_{x \to 0} \frac{1}{x^2} \log\left(\frac{\tan x}{x}\right)$$

$$= \lim_{x \to 0} \frac{\log\left(\frac{\tan x}{x}\right)}{x^2}$$

Apply L'H Rule

$$= \lim_{x \to 0} \frac{\frac{1}{\left(\frac{\tan x}{x}\right)}}{2x} \times \left(\frac{x \sec^2 x - \tan x (1)}{x^2} \right)$$

$$= \frac{1}{2} \cancel{\lim_{x \to 0} \frac{x}{\tan x}}^{\,1} \cdot \lim_{x \to 0} \frac{x \sec^2 x - \tan x}{x^3}$$

Apply L'H rule

$$= \frac{1}{\cancel{2}} \lim_{x \to 0} \frac{x \cancel{2} \sec x \cdot \sec x \cdot \tan x + \cancel{\sec^2 x (1)} - \cancel{\sec^2 x}}{3x^2}$$

$$= \frac{1}{3} \lim_{x \to 0} \frac{\sec^2 x \cdot \tan x}{x}$$

$$= \frac{1}{3} \cancel{\lim_{x \to 0} \frac{\tan x}{x}}^{\,1} \cdot \lim_{x \to 0} \sec^2 x$$

$$\log k = \frac{1}{3} \sec^2(0)$$

$$\log k = \frac{1}{3}$$

$$\boxed{k = e^{1/3}}$$

2. $\lim_{x \to \frac{\pi}{2}} (\sin x)^{\tan x}$

Sol:- Let $k = \lim_{x \to \frac{\pi}{2}} (\sin x)^{\tan x}$

Take log o.b.s

$$\log k = \lim_{x \to \frac{\pi}{2}} \log (\sin x)^{\tan x}$$

$$= \lim_{x \to \frac{\pi}{2}} \tan x \log (\sin x)$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\log (\sin x)}{\cot x}$$

Apply L'H Rule

$$= \lim_{x \to \frac{\pi}{2}} \frac{\frac{1}{\sin x} \times \cos x}{-\operatorname{cosec}^2 x}$$

$$= -\lim_{x \to \frac{\pi}{2}} \frac{\cot x}{\operatorname{cosec}^2 x}$$

Apply limit

$$= \frac{-\cot\left(\frac{\pi}{2}\right)}{\operatorname{cosec}^2\left(\frac{\pi}{2}\right)} = \frac{0}{1} = 0$$

$\log K = 0$

$K = e^0 \Rightarrow \boxed{K = 1}$

3. $\lim_{x \to \frac{\pi}{2}} (\cos x)^{\pi/2 - x}$

Sol:- $K = \lim_{x \to \frac{\pi}{2}} (\cos x)^{\pi/2 - x}$

Take log

$$\log K = \lim_{x \to \frac{\pi}{2}} \left(\frac{\pi}{2} - x\right) \log(\cos x)$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\log \cos x}{\frac{1}{\left(\frac{\pi}{2} - x\right)}} = \left[\frac{\infty}{\infty}\right]$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\frac{1}{\cos x}(-\sin x)}{\frac{-1}{\left(\frac{\pi}{2} - x\right)^2}(-1)} = \lim_{x \to \frac{\pi}{2}} \frac{\frac{-\sin x}{\cos x}}{\frac{1}{\left(\frac{\pi}{2} - x\right)^2}}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{-\tan x}{\frac{1}{\left(\frac{\pi}{2} - x\right)^2}}$$

$$= -\lim_{x \to \frac{\pi}{2}} \frac{1}{\cot x} \times \left(\frac{\pi}{2} - x\right)^2$$

$$= -\lim_{x \to \frac{\pi}{2}} \frac{\left(\frac{\pi}{2} - x\right)^2}{\cot x}$$

Apply LH Rule

$$\lim_{x \to \frac{\pi}{2}} \frac{-2\left(\frac{\pi}{2}-x\right)(-1)}{-\operatorname{cosec}^2 x}$$

Apply limit

$$\frac{2\left(\frac{\pi}{2}-\frac{\pi}{2}\right)}{\operatorname{cosec}^2\left(\pi/2\right)} = \frac{-0}{1} = 0$$

$\log k = 0$

$k = e^0$

$\boxed{k = 1}$

4. $\lim_{x \to a} \left(2 - \frac{x}{a}\right)^{\tan\left(\frac{\pi x}{2a}\right)}$

Sol:- Let $k = \lim_{x \to a} \left(2 - \frac{x}{a}\right)^{\tan\left(\frac{\pi x}{2a}\right)}$

Take log o.b.s

$$\log k = \lim_{x \to a} \tan\left(\frac{\pi x}{2a}\right) \log\left(2 - \frac{x}{a}\right)$$

$$= \lim_{x \to a} \frac{\log\left(2-\frac{x}{a}\right)}{\cot\left(\frac{\pi x}{2a}\right)}$$

Apply LH rule

$$\lim_{x \to a} \frac{\frac{1}{\left(2-\frac{x}{a}\right)} \times \frac{-1}{a}}{-\operatorname{cosec}^2\left(\frac{\pi x}{2a}\right)\left(\frac{\pi}{2a}\right)}$$

$$\frac{2}{\pi} \lim_{x \to a} \frac{1}{\operatorname{cosec}^2\left(\frac{\pi x}{2a}\right)\left(2-\frac{x}{a}\right)}$$

$$\frac{2}{\pi} \lim_{x \to a} \frac{\sin^2\left(\frac{\pi x}{2a}\right)}{\left(2-\frac{x}{a}\right)}$$

Apply limit

$$\frac{2}{\pi} \cdot \frac{\sin^2\left(\frac{\pi a}{2a}\right)}{2 - \frac{a}{a}} = \frac{2}{\pi} \frac{\sin\frac{\pi}{2}}{1}$$

$$\log k = \frac{2}{\pi} \times 1$$

$$\log k = \frac{2}{\pi}$$

$$k = e^{2/\pi}$$

5. Evalute $\lim_{x \to 0} \left(\frac{a^x + b^x}{2}\right)^{\frac{1}{x}}$

Sol:- Let $k = \lim_{x \to 0} \left(\frac{a^x + b^x}{2}\right)^{\frac{1}{x}}$

Take log o.b.s

$$\log k = \lim_{x \to 0} \frac{1}{x} \log\left[\frac{a^x + b^x}{2}\right]$$

$$= \lim_{x \to 0} \frac{\log\left[\frac{a^x + b^x}{2}\right]}{x}$$

Apply LH Rule

$$= \lim_{x \to 0} \frac{\frac{1 \times (a^x \log a + b^x \log b)}{\left[\frac{a^x + b^x}{\cancel{2}}\right] \cancel{2}}}{x(1)}$$

$$= \lim_{x \to 0} \frac{a^x \log a + b^x \log b}{a^x + b^x}$$

Apply limit

$$= \frac{a^0 \log a + b^0 \log b}{a^0 + b^0} = \frac{\log a + \log b}{2}$$

$$\log k = \frac{\log(ab)}{2} = \log(ab)^{\frac{1}{2}}$$

$$\log k = \log(ab)^{1/2}$$

$$k = \sqrt{ab}$$

6. Evalute $\lim_{x \to 0} \left[\frac{a^x + b^x + c^x}{3}\right]^{\frac{1}{x}}$

Sol:- Let $k = \lim_{x \to 0} \left[\frac{a^x + b^x + c^x}{3}\right]^{\frac{1}{x}}$

Take log o.b.s

$$\log k = \lim_{x \to 0} \frac{1}{x} \left\{ \log \left[\frac{a^x + b^x + c^x}{3}\right]^{\frac{1}{x}}\right.$$

$$\log k = \lim_{x \to 0} \frac{\log \left(\frac{a^x + b^x + c^x}{3}\right)}{x}$$

$$\log\left(\frac{m}{n}\right) = \log m - \log n$$

$$= \lim_{x \to 0} \frac{\log(a^x + b^x + c^x) - \log 3}{x}$$

Apply LH Rule

$$= \lim_{x \to 0} \frac{\frac{1 \times a^x \log a + b^x \log b + c^x \log c}{a^x + b^x + c^x}}{1}$$

$$= \lim_{x \to 0} \frac{a^x \log a + b^x \log b + c^x \log c}{a^x + b^x + c^x}$$

Apply limit

$$\frac{a^0 \log a + b^0 \log b + c^0 \log c}{a^0 + b^0 + c^0} = \frac{1}{3} \cdot \log a + \log b + \log c$$

$$\log K = \frac{1}{3}\log abc$$

$$\log K = \log (abc)^{1/3}$$

$$K = \sqrt[3]{abc}$$

7. Evalute $\lim_{x\to 0}\left(\frac{a^x+b^x+c^x+d^x}{4}\right)^{\frac{1}{x}}$

Sol: Let $K = \lim_{x\to 0}\left(\frac{a^x+b^x+c^x+d^x}{4}\right)^{\frac{1}{x}}$

Take log o.b.s

$$\log K = \lim_{x\to 0}\frac{1}{x}\log\left(\frac{a^x+b^x+c^x+d^x}{4}\right)$$

$$= \lim_{x\to 0}\frac{\log\left(\frac{a^x+b^x+c^x+d^x}{4}\right)}{x}$$

Apply LH Rule

$$\lim_{x\to 0}\frac{\frac{a^x\log a+b^x\log b+c^x\log c+d^x\log d}{a^x+b^x+c^x+d^x}}{1} - 0$$

$$\lim_{x\to 0}\frac{a^x\log a+b^x\log b+c^x\log c+d^x\log d}{a^x+b^x+c^x+d^x}$$

Apply limit

$$\log K = \frac{a^0\log a+b^0\log b+c^0\log c+d^0\log d}{a^0+b^0+c^0+d^0}$$

$$= \frac{\log a+\log b+\log c+\log d}{4}$$

$$\log K = \log (abcd)^{\frac{1}{4}}$$

$$K = (abcd)^{\frac{1}{4}}$$

8. Evalute $\lim\limits_{x\to 0} \left(\frac{\tan x}{x}\right)^{\frac{1}{x}}$

Sol:- Let $K = \lim\limits_{x\to 0} \left(\frac{\tan x}{x}\right)^{\frac{1}{x}}$

Take log o.b.s

$$\log K = \lim_{x\to 0} \frac{1}{x} \log\left(\frac{\tan x}{x}\right)$$

$$\log K = \lim_{x\to 0} \frac{\log\left(\frac{\tan x}{x}\right)}{x} = \lim_{x\to 0} \frac{\log \tan x - \log x}{x}$$

Apply LH Rule

$$\lim_{x\to 0} \frac{\frac{1 \times \sec^2 x}{\tan x} - \frac{1}{x}(1)}{1}$$

$$\lim_{x\to 0} \frac{x\sec^2 x - \tan x}{x\tan x}$$

Multiply x & $\div$ x in denominator

$$= \lim_{x\to 0} \frac{x\sec^2 x - \tan x}{x^2 \frac{\tan x}{x}}$$

$$= \lim_{x\to 0} \frac{x\sec^2 x - \tan x}{x^2}$$

Apply LH Rule

$$\lim_{x\to 0} \frac{x \cdot 2\sec x \cdot \sec x \cdot \tan x + \sec^2 x(1) - \sec^2 x}{2x}$$

$$\lim_{x\to 0} \frac{\cancel{2x} \cdot \sec^2 x \cdot \tan x}{\cancel{2x}}$$

Apply limit

$\log K = \sec^2(0) \cdot \tan(0)$

$\log K = 0$

$K = e^0 = 1$

$\boxed{K = 1}$

9. $K = \lim_{x \to \infty} \left(\frac{ax+1}{ax-1}\right)^x$

Take log o.b.s

$\log K = \lim_{x \to \infty} \log\left(\frac{ax+1}{ax-1}\right)^x$

$= \lim_{x \to \infty} x \log\left(\frac{ax+1}{ax-1}\right)$

$= \lim_{x \to \infty} x \log\left(\frac{\cancel{x}\left(a+\frac{1}{x}\right)}{\cancel{x}\left(a-\frac{1}{x}\right)}\right)$

$\lim_{x \to \infty} \frac{\log \frac{\left(a+\frac{1}{x}\right)}{\left(a-\frac{1}{x}\right)}}{\frac{1}{x}}$

Apply LH Rule

$\lim_{x \to \infty} \frac{\log\left(a+\frac{1}{x}\right) - \log\left(a-\frac{1}{x}\right)}{\frac{1}{x}}$

$\lim_{x \to \infty} \frac{\frac{1}{\left(a+\frac{1}{x}\right)}\left(0-\frac{1}{x^2}\right) - \frac{1}{\left(a-\frac{1}{x}\right)}\left(0-\left(-\frac{1}{x^2}\right)\right)}{-\frac{1}{x^2}}$

$$= \lim_{x\to\infty} \frac{1}{\left(a+\frac{1}{x}\right)}\left(\frac{-1}{x^2}\right) - \frac{1}{\left(a-\frac{1}{x}\right)}\left(\frac{1}{x^2}\right)$$

$$\lim_{x\to\infty} \frac{\cancel{\left(\frac{-1}{x^2}\right)}\left(\frac{1}{a+\frac{1}{x}} - \frac{1}{a-\frac{1}{x}}\right)}{\cancel{\left(\frac{-1}{x^2}\right)}}$$

Apply limit

$$\frac{1}{a+\frac{1}{\infty}} + \frac{1}{a-\frac{1}{\infty}} = \frac{1}{a}+\frac{1}{a}$$

$$\log K = \frac{2}{a}$$

$$\boxed{K = e^{2/a}}$$

10. $K = \lim_{x\to 0} (1-x^2)^{\frac{1}{\log(1-x)}}$

<u>Sol:-</u> $\log K = \lim_{x\to 0} \frac{1}{\log(1-x)} \log(1-x^2)$

$$= \lim_{x\to 0} \frac{\log(1-x^2)}{\log(1-x)}$$

Apply LH Rule

$$\lim_{x\to 0} \frac{\frac{1}{(1-x^2)} \times (0-2x)}{\frac{1}{(1-x)} \times (0-1)} = \frac{\frac{-2x}{(1-x^2)}}{\frac{-1}{(1-x)}}$$

$$= \frac{+2x}{(1-x^2)} \cdot \frac{(1-x)}{+1}$$

$$= \frac{2x-2x^2}{(1-x^2)}$$

Partial Differentiation:-

Partial differentiation of a function deals with differentiation of function of many independent variables.

Partial derivatives:-

Let U be the function two independent variables then partial derivatives of U is denoted by $\frac{\delta U}{\partial x}$.

Similarly, partial derivative of U with respective y is denoted by $\frac{\partial U}{\partial y}$ and is defined as,

i.e. if $u = f(x,y)$

$$\Rightarrow \frac{\partial u}{\partial x} = U_x = \lim_{\partial x \to 0} \frac{f(x+\partial x, y) - f(x,y)}{\partial x}$$

$$\Rightarrow \frac{\partial u}{\partial y} = U_y = \lim_{\partial y \to 0} \frac{f(x, y+\partial y) - f(x,y)}{\partial y}$$

Higher Order derivatives:-

1) $\frac{\partial^2 u}{\partial x^2} = U_{xx}$ 2) $\frac{\partial^2 u}{\partial y^2} = U_{yy}$ 3) $\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = U_{xy}$ 4) $\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = U_{yx}$

Note:- $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

Partial derivative can be extended to the function which contains more than one independent variable.

Problems:-

1. Find $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ $u = x^2 + y^2 \longrightarrow (1)$

Sol:-

d.(1) p.w.r. to x

$\frac{\partial u}{\partial x} = 2x + 0$ (Treat all y terms as constant)

$U = \frac{\partial u}{\partial x} = 2x$

d(1) p. w. r to y (Treat all x terms as constant)

$$U_y = \frac{\partial u}{\partial y} = 0 + 2y \Rightarrow \frac{\partial u}{\partial y} = 2y$$

2. $Z = 2xy \rightarrow (1)$

Sol:- d(1) p. w. r to x

$$Z_x = \frac{\partial z}{\partial x} = 2y(1) \text{ (Treat all 'y' terms as constant)}$$

$$Z_x = \frac{\partial z}{\partial x} = 2y$$

d(1) p. w. r. to y

$$Z_y = \frac{\partial z}{\partial y} = 2x(1) \text{ (Treat all } x \text{ terms as constant)}$$

$$Z_y = \frac{\partial z}{\partial y} = 2x$$

3. $Z = e^{x+y}$

Sol:- $Z = e^x \cdot e^y \longrightarrow (1)$

d(1) p. w. r. to 'x' (Treat y terms as constant)

$$Z_x = \frac{\partial z}{\partial x} = e^y \cdot e^x$$

d(1) p. w. r. to 'y' (Treat x terms as constant)

$$Z_y = \frac{\partial z}{\partial y} = e^x \cdot e^y$$

4. $U = e^x \cdot \sin 2x \cdot \cos y \rightarrow (1)$

Sol:- d(1) p. w. r. to 'x' (Treat all 'y' terms as constant)

$$U_x = \frac{\partial u}{\partial x} = \cos y \left[e^x \cdot \cos 2x(2) + \sin 2x \cdot e^x\right]$$

d(1) p. w. r. to 'y' (Treat all 'x' terms as constant)

$$U_y = \frac{\partial u}{\partial y} = e^x \sin 2x\,[-\sin y] \Rightarrow U_y = \frac{\partial u}{\partial y} = -e^x \cdot \sin 2x \cdot \sin y$$

5. $u = \frac{\tan x . \sec y}{x+y} \rightarrow (1)$

Sol:- $u_x = \frac{\partial u}{\partial x} = \sec y . \frac{\partial}{\partial x}\left(\frac{\tan x}{x+y}\right)$

$= \sec y \left[\frac{(x+y)\sec^2 x - \tan x\,(1+0)}{(x+y)^2}\right]$

$= \sec y \left[\frac{(x+y)\sec^2 x - \tan x}{(x+y)^2}\right]$

$u_y = \frac{\partial u}{\partial y} = \tan x \frac{\partial}{\partial y}\left(\frac{\sec y}{x+y}\right)$

$= \tan x \left[\frac{(x+y)\sec y . \tan y - \sec y\,(1)}{(x+y)^2}\right]$

6. $u = x^3 - 3xy^2 + x + e^x \cos y + 1$, P.T $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Sol:- $u = x^3 - 3xy^2 + x + e^x \cos y + 1 \rightarrow (1)$

$u_x = \frac{\partial u}{\partial x} = 3x^2 - 3y^2(1) + 1 + \cos y\, e^x + 0$

$u_{xx} = \frac{\partial^2 u}{\partial x^2} = 6x - 0 + 0 + \cos y\, e^x$

$u_{xx} = 6x + \cos y\, e^x$

d (1) p. w. r. to 'y'

$u_y = \frac{\partial u}{\partial y} = 0 - 3x.2y + 0 + e^x(-\sin y) + 0$

$u_y = -3x.2y - e^x \sin y$

$u_{yy} = \frac{\partial^2 u}{\partial y^2} = -\left[6x(1) + e^x \cos y\right] = -6x - e^x \cos y$

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x + e^x \cos y - 6x - e^x \cos y$

$= 0$

7. $u = \tan^{-1}\left(\frac{y}{x}\right)$

d.p. w. r. to x, y seperately.

$$\frac{\partial u}{\partial x} = y \cdot \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{-1}{x^2}$$

$$= \frac{-y}{x^2} \cdot \frac{1}{\frac{x^2+y^2}{x^2}} = \frac{-y}{\cancel{x^2}} \cdot \frac{\cancel{x^2}}{x^2+y^2}$$

$$\frac{\partial u}{\partial x} = \frac{-y}{x^2+y^2}.$$

$$\frac{\partial u}{\partial y} = \frac{1}{x} \cdot \frac{1}{1+\left(\frac{y}{x}\right)^2}$$

$$= \frac{1}{x} \cdot \frac{1}{\frac{x^2+y^2}{x^2}} = \frac{1}{\cancel{x}} \cdot \frac{x^{\cancel{2}}}{x^2+y^2}$$

$$\frac{\partial u}{\partial y} = \frac{x}{x^2+y^2}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = -y\left[\frac{-1}{(x^2+y^2)^2}(2x+0)\right]$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{2xy}{(x^2+y^2)^2}$$

$$\Rightarrow \frac{\partial^2 u}{\partial y^2} = x\left[\frac{-1}{(x^2+y^2)^2}(0+2y)\right]$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{-2xy}{(x^2+y^2)^2}$$

Total Derivative:-

If $u=f(x,y)$ where $x=x(t)$ & $y=y(t)$ then we can express u as function of 't' alone by substituteing the values of x & y in $f(x,y)$. Then we find ordinary derivative differentiation, $\frac{du}{dt}$ is called total derivative.

i.e.

$$u \longrightarrow (x,y) \longrightarrow t$$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

1. If $u=x^3y^2$, $x=e^t$, $y=\log t$ then find $\frac{du}{dt}$ and also verify the result by taking direct substitution.

Sol:-

$$u \longrightarrow (x,y) \longrightarrow t$$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

Consider, $u=x^3y^2$

D.P. w.r. to 'x'

$$\frac{\partial u}{\partial x} = y^2(3x^2)$$

D.P. w.r. to 'y'

$$\frac{\partial u}{\partial y} = x^3(2y)$$

Consider,

$$x=e^t$$

d.w.r.to t

$$\frac{dx}{dt} = e^t$$

$$y=\log t$$

d.w.r. to t

$$\frac{dy}{dt} = \frac{1}{t}$$

$\therefore$ (1) becomes,

$$\frac{du}{dt} = (3x^2y^2)e^t + 2x^3y\left(\frac{1}{t}\right)$$

but $x = e^t$, $y = \log t$

$$\frac{du}{dt} = 3(e^t)^2(\log t)^2 . e^t + 2(e^t)^3 . \log t\left(\frac{1}{t}\right)$$

$$\frac{du}{dt} = 3e^{3t}(\log t)^2 + 2e^{3t}\left(\frac{\log t}{t}\right)$$

$$u = x^3y^2$$

$$u = (e^t)^3(\log t)^2$$

$$u = e^{3t}(\log t)^2$$

d.w.r.to 't'

$$\frac{du}{dt} = e^{3t}(2)\log t\,(1/t) + (\log t)^2 e^{3t}.3$$

$\therefore$ Result is verified by direct substitution.

2. $Z = u^2+v^2$, $u = at^2$, $V = 2at$ find $\frac{dz}{dt}$

Sol:-

$$Z \longrightarrow (u,v) \longrightarrow t$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial u}.\frac{du}{dt} + \frac{\partial z}{\partial v}.\frac{dv}{dt}$$

Consider, $Z = u^2+v^2$

d.p.w.r. to u.

$$\frac{\partial z}{\partial u} = 2u+0 \Rightarrow \frac{\partial z}{\partial u} = 2u$$

d.p.w.r. to v.

$$\frac{\partial z}{\partial v} = 0+2V \Rightarrow \frac{\partial z}{\partial v} = 2V$$

Consider, $u = at^2$

d.w.r.to. t

$$\frac{du}{dt} = a2t \Rightarrow \frac{du}{dt} = 2at$$

$$V = 2at$$

$$\frac{dv}{dt} = 2a$$

$\therefore$ (1) becomes, $\frac{dz}{dt} = 2u.2at + 2v.2a$

but $u = at^2$, $V = 2at$

$$= 2(at^2).2at + 2(2at).2a$$

$$= 4a^2t^3 + 8a^2t$$

3. If $u = \sin^{-1}\left(\frac{x}{y}\right)$, $x = e^t$, $y = t^2$ find $\frac{du}{dt}$

$$u \longrightarrow (x,y) \longrightarrow t$$

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$

Consider, $u = \sin^{-1}\left(\frac{x}{y}\right)$ d.p. w.r.t x

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}} \cdot \frac{1}{y}(1) = \frac{1}{\sqrt{\frac{y^2-x^2}{y^2}}} \cdot \frac{1}{y}$$

$$= \frac{\cancel{y}}{\sqrt{y^2-x^2}} \cdot \frac{1}{\cancel{y}}$$

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{y^2-x^2}}$$

~~Consider,~~ $\cancel{x = e^t}$ d.p.w.r. to y.

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\, x\left(\frac{-1}{y^2}\right) = \frac{1}{\sqrt{\frac{y^2-x^2}{y^2}}} \cdot \frac{-x}{y^2}$$

$$= \frac{\cancel{y}}{\sqrt{y^2-x^2}} \cdot \frac{-x}{y^{\cancel{2}}}$$

$$= \frac{-x}{y\sqrt{y^2-x^2}}$$

Consider, $x = e^t$

d.w.r.to.t

$$\frac{dx}{dt} = e^t$$

$$y = t^2$$

d.w.r.to t

$$\frac{dy}{dt} = 2t$$

$$= \frac{1}{\sqrt{y^2-x^2}} \cdot e^t + \left(\frac{-x}{y}\right)\frac{1}{\sqrt{y^2-x^2}} \cdot 2t$$

$$= \frac{1}{\sqrt{y^2-x^2}}\left(e^t - \frac{x}{y}2t\right)$$

$$= \frac{1}{\sqrt{(t^2)^2-(e^t)^2}}\left(e^t - \frac{e^t}{t^{\cancel{2}}}2\cancel{t}\right)$$

$$= \frac{e^t}{\sqrt{t^4 - e^{2t}}}\left(1 - \frac{2}{t}\right)$$

4. If $u = f(x,y,z)$ & $x = x(t), y = y(t), z = z(t)$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt}$$

Find $\frac{\partial u}{dt}$ at $t=0$ if $u = e^{x^2+y^2+z^2}$ & $x = t^2+1, y = t\cos t, z = \sin t$

sol:- $u \rightarrow (x,y,z) \longrightarrow t$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt}$$

consider, $u = e^{x^2+y^2+z^2}$

$$u = e^{x^2}\cdot e^{y^2}\cdot e^{z^2}$$

d.p.w.r. to x

$$\frac{\partial u}{\partial x} = e^{y^2}\cdot e^{z^2}\cdot e^{x^2}\,2x \longrightarrow (1)$$

d.p.w.r. to y

$$\frac{\partial u}{\partial y} = e^{z^2}\cdot e^{x^2}\cdot e^{y^2}\,2y \longrightarrow (2)$$

d.p.w.r. to z

$$\frac{\partial u}{\partial z} = e^{x^2}\cdot e^{y^2}\cdot e^{z^2}\cdot 2z \longrightarrow (3)$$

consider,

$$x = t^2+1$$

d.w.r. to t

$$\frac{dx}{dt} = 2t+0 \Rightarrow \frac{dx}{dt} = 2t$$

$y = t\cos t$

d.w.r. to t

$$\frac{dy}{dt} = t(-\sin t) + \cos(1)$$

$$\frac{dy}{dt} = \cos t - t\sin t$$

$z = \sin t$

d.w.r to t

$$\frac{dz}{dt} = \cos t$$

$$\frac{du}{dt} = e^{y^2}.e^{z^2}.e^{x^2}(2x).2t + e^{z^2}.e^{x^2}.e^{y^2}(2y).\cos t - t\sin t + e^{x^2}.e^{y^2}.e^{z^2}(2z)\cos t$$

$$= e^{x^2+y^2+z^2}(2t \cdot 2x + 2y(\cos t - t\sin t + \frac{2z}{\cos t})$$

$$= 2e^{x^2+y^2+z^2}(2xt + y(\cos t - t\sin t) + \frac{z}{\cos t})$$

$$= 2e^{(t^2+1)+(t\cos t)^2+(\sin t)^2}(0+y+z)$$

$$= 2e^1(0 + t\cos t + \sin t)$$

$$= 2e^1(0)$$

$$= 2e^1(0)$$

$$\frac{du}{dt} = 0$$

Total derivative of composite function (two variables)

If u is function of x & y that is $u = f(x,y)$ and x, y are functions of 'r' and 's' then the total derivative of u is given by

$$u \longrightarrow (x,y) \longrightarrow (r,s)$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}.\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}.\frac{\partial y}{\partial r} \quad \& \quad \frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}.\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}.\frac{\partial y}{\partial s}$$

5. $z = f(x,y)$ $x = r\cos\theta$, $y = r\sin\theta$ PT

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \left(\frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2\right)$$

sol:-

$$z \longrightarrow (x,y) \longrightarrow (r,\theta)$$

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial r} \quad \& \quad \frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

$x = r\cos\theta$

D.P. w.r to 'r'

$$\frac{\partial x}{\partial r} = \cos\theta(1)$$

D.P. w.r to 'θ'

$$\frac{\partial x}{\partial \theta} = -r\sin\theta$$

$y = r\sin\theta$

D.P. w.r. to 'r'

$$\frac{\partial y}{\partial r} = \sin\theta(1)$$

D.P. w.r to θ

$$\frac{\partial y}{\partial \theta} = r\cos\theta$$

$$\frac{\partial z}{\partial r} = \left(\frac{\partial z}{\partial x}\right)(\cos\theta) + \frac{\partial z}{\partial y}(\sin\theta)$$

$$\frac{\partial z}{\partial \theta} = \left(\frac{\partial z}{\partial x}\right)(-r\sin\theta) + \frac{\partial z}{\partial y}(r\cos\theta)$$

consider,

$$\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$$

$$= \left(\left(\frac{\partial z}{\partial x}\right)\cos\theta + \left(\frac{\partial z}{\partial y}\right)\sin\theta\right)^2 + \frac{1}{r^2}\left(\left(\frac{\partial z}{\partial r}\right)(-r\sin\theta) + \frac{\partial z}{\partial y}(r\cos\theta)\right)$$

$$LHS = \left(\frac{\partial z}{\partial x}\right)^2\cos^2\theta + \left(\frac{\partial z}{\partial y}\right)^2\sin^2\theta + 2\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}\sin\theta\cdot\cos\theta +$$

$$\frac{1}{r^2}\left[\left(\frac{\partial z}{\partial x}\right)^2 r^2\sin^2\theta + \left(\frac{\partial z}{\partial y}\right)^2 r^2\cos^2\theta - 2rr\frac{\partial z}{\partial x}\sin\theta.\frac{\partial z}{\partial y}\cos\theta\right]$$

$$= \left(\frac{\partial z}{\partial x}\right)^2\cos^2\theta + \left(\frac{\partial z}{\partial y}\right)^2\sin^2\theta + 2\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}\sin\theta.\cos\theta +$$

$$\left(\frac{\partial z}{\partial x}\right)^2\sin^2\theta + \left(\frac{\partial z}{\partial y}\right)^2\cos^2\theta - 2\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}\sin\theta.\cos\theta$$

6. If $z = f(x,y)$, $x = u-v$, $y = uv$.

i) $(u+v)\dfrac{\partial z}{\partial x} = u.\dfrac{\partial z}{\partial u} - v.\dfrac{\partial z}{\partial v}$

ii) $(u+v)\dfrac{\partial z}{\partial y} = \dfrac{\partial z}{\partial u} + \dfrac{\partial z}{\partial v}$

sol:- $z \rightarrow (x,y) \longrightarrow (u,v)$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\left(\frac{\partial x}{\partial u}\right) + \frac{\partial z}{\partial y}\left(\frac{\partial y}{\partial u}\right) \rightarrow (1) \quad \& \quad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\left(\frac{\partial x}{\partial v}\right) + \frac{\partial z}{\partial y}\left(\frac{\partial y}{\partial v}\right) \longrightarrow (2)$$

consider: $x = u-v$

diff p.w.r to x & y seperately

$$\frac{\partial x}{\partial u} = 1-0 \Rightarrow \frac{\partial x}{\partial u} = 1 \; ; \; \frac{\partial x}{\partial v} = 0-1 \rightarrow \frac{\partial x}{\partial v} = -1$$

$y = uv$

diff p.w.r. to x & y seperately

$$\frac{\partial y}{\partial u} = v(1) \qquad \frac{\partial y}{\partial v} = u(1)$$

$\therefore$ (1) & (2) becomes

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}(1) + \frac{\partial z}{\partial y}(v)$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}(-1) + \frac{\partial z}{\partial y}(u)$$

Consider: $u\dfrac{\partial z}{\partial u} - v.\dfrac{\partial z}{\partial v}$

$$= u\left[\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}v\right] - v\left[-\frac{\partial z}{\partial x} + u.\frac{\partial z}{\partial y}\right]$$

$$= u.\frac{\partial z}{\partial x} + uv\frac{\partial z}{\partial y} + v.\frac{\partial z}{\partial x} - uv\frac{\partial z}{\partial y} = u\frac{\partial z}{\partial x} + v\frac{\partial z}{\partial x} = \frac{\partial z}{\partial x}(u+v)$$

ii) $(u+v)\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$

eq-(1) $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}(v)$

eq-(2) $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial y}u - \frac{\partial z}{\partial x}$

consider;

RHS

$= \cancel{\frac{\partial z}{\partial x}} + \frac{\partial z}{\partial y}(v) + \frac{\partial z}{\partial y}(u) - \cancel{\frac{\partial z}{\partial x}}$

$= \frac{\partial z}{\partial y}(u+v)$

$=$ LHS

7. If $z = f(x,y)$, $x = e^u \cos v$, $y = e^4 \sin v$

PT $\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 = \left[e^{2u}\left(\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right)\right]$

sol:- $z \rightarrow (x,y) \longrightarrow (u,v)$

$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u} \rightarrow (1)$, $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v} \longrightarrow (2)$

$x = e^u \cos v$

d.p.w.r.to u & v seperately

$\frac{\partial x}{\partial u} = \cos v . e^4$

$\frac{\partial x}{\partial v} = -e^4 \sin v$

eq-(1) becomes

$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}(e^4 \cos v) + \frac{\partial z}{\partial y}(e^4 \sin v)$

$y = e^4 \sin v$

d.p.w.r to u & v seperately

$\frac{\partial y}{\partial u} = e^4 \sin v$

$\frac{\partial y}{\partial v} = e^4 \cos v$

eq-(2)

$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}(e^4 \sin v) + \frac{\partial z}{\partial y}(e^4 \cos v)$

$= \frac{\partial z}{\partial y}(e^4 \cos v) - \frac{\partial z}{\partial x}(e^4 \sin v)$

Consider, LHS,

$$\left(\frac{\partial z}{\partial u}\right)^2+\left(\frac{\partial z}{\partial v}\right)^2=\left(\frac{\partial z}{\partial x}(e^u\cos v)+\frac{\partial z}{\partial y}(e^u\sin v)\right)^2+\left(\frac{\partial z}{\partial y}(e^u\cos v)-\frac{\partial z}{\partial x}(e^u\sin v)\right)^2$$

$$=\left(\frac{\partial z}{\partial x}\right)^2e^{2u}\cos^2 v+\left(\frac{\partial z}{\partial y}\right)^2\cdot e^{2u}\sin^2 v+2\frac{\partial z}{\partial x}\,e^u\cancel{\cos u\cdot\frac{\partial z}{\partial y}}\,e^u\sin u+$$

$$\left(\frac{\partial z}{\partial y}\right)^2e^{2u}\cos^2 v+\left(\frac{\partial z}{\partial x}\right)^2e^{2u}\sin^2 v-2\frac{\partial z}{\partial x}\,e^u\cancel{\sin v\cdot\frac{\partial z}{\partial y}}\,e^u\cos v$$

$$=e^{2u}\left[\left(\frac{\partial z}{\partial x}\right)^2\cos^2 v+\left(\frac{\partial z}{\partial y}\right)^2\sin^2 v+\left(\frac{\partial z}{\partial y}\right)^2\cos^2 v+\left(\frac{\partial z}{\partial x}\right)^2\sin^2 v\right]$$

$$=e^{2u}\left[\sin^2 v\left(\left(\frac{\partial z}{\partial y}\right)^2+\left(\frac{\partial z}{\partial x}\right)^2\right)+\cos^2 v\left(\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right)\right]$$

$$=e^{2u}\left[\left(\frac{\partial z}{\partial y}\right)^2+\left(\frac{\partial z}{\partial x}\right)^2(\sin^2 v+\cos^2 v)\right]$$

$$=e^{2u}\left[\left(\frac{\partial z}{\partial y}\right)^2+\left(\frac{\partial z}{\partial x}\right)^2\right]$$

= RHS

8. PT $x\cdot\frac{\partial u}{\partial x}+y\cdot\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=0$ if $u=f\left(\frac{x}{y},\frac{y}{z},\frac{z}{x}\right)$

Sol:- $u=f\left(\frac{x}{y},\frac{y}{z},\frac{z}{x}\right)$ put $\frac{x}{y}=p,\ \frac{y}{z}=q,\ \frac{z}{x}=r$

$u=f(p,q,r)$, $p=\frac{x}{y}$, $q=\frac{y}{z}$, $r=\frac{z}{x}$

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial x}+\frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial x}+\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x}\longrightarrow(1)$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial y}+\frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial y}+\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y}\longrightarrow(2)$$

$$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial z}+\frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial z}+\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z}\longrightarrow(3)$$

$p=\frac{x}{y}$

9. If $u = f(x-y, y-z, z-x)$

PT $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

sol:- $u = f(x-y, y-z, z-x)$ Put $x-y=a$, $y-z=b$, $z-x=c$

$u = f(a,b,c)$, $a = x-y$; $b = y-z$; $c = z-x$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial a}\cdot\frac{\partial a}{\partial x} + \frac{\partial u}{\partial b}\cdot\frac{\partial b}{\partial x} + \frac{\partial u}{\partial c}\cdot\frac{\partial c}{\partial x} \longrightarrow (1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial a}\cdot\frac{\partial a}{\partial y} + \frac{\partial u}{\partial b}\cdot\frac{\partial b}{\partial y} + \frac{\partial u}{\partial c}\cdot\frac{\partial c}{\partial y} \longrightarrow (2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial a}\cdot\frac{\partial a}{\partial z} + \frac{\partial u}{\partial b}\cdot\frac{\partial b}{\partial z} + \frac{\partial u}{\partial c}\cdot\frac{\partial c}{\partial z} \longrightarrow (3)$$

$a = x-y$

d.p. w.r to x, y, z seperately

$$\frac{\partial a}{\partial x} = 1, \quad \frac{\partial a}{\partial y} = -1, \quad \frac{\partial a}{\partial z} = 0$$

$b = y-z$ d.p.w.r.to x, y, z seperately

$$\frac{\partial b}{\partial x} = 0, \quad \frac{\partial b}{\partial y} = 1, \quad \frac{\partial b}{\partial z} = -1$$

$c = z-x$ d.p.w.r.t x, y, z seperately

$$\frac{\partial c}{\partial x} = -1, \quad \frac{\partial c}{\partial y} = 0, \quad \frac{\partial c}{\partial z} = 1$$

eq-(1) becomes

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial a}(1) + \frac{\partial u}{\partial b}(0) + \frac{\partial u}{\partial c}(-1)$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial a} - \frac{\partial u}{\partial c}$$

eq $\rightarrow$ (2) becomes

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial a}(-1) + \frac{\partial u}{\partial b}(1) + \frac{\partial u}{\partial c}(0)$$

D.P w.r. to x, y, z seperately

$$\frac{\partial p}{\partial x} = \frac{1}{y}, \quad \frac{\partial p}{\partial y} = x\left(\frac{-1}{y^2}\right), \quad \frac{\partial p}{\partial z} = 0$$

$$q = \frac{y}{z}$$

D.P. w.r to x, y, z seperately

$$\frac{\partial q}{\partial x} = 0, \quad \frac{\partial q}{\partial y} = \frac{1}{z}, \quad \frac{\partial q}{\partial z} = \frac{-y}{z^2}$$

$$r = \frac{z}{x}$$

D.P. w.r. to x, y, z seperately

$$\frac{\partial r}{\partial x} = \frac{-z}{x^2}, \quad \frac{\partial r}{\partial y} = 0, \quad \frac{\partial r}{\partial z} = \frac{1}{x}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p} \cdot \frac{1}{y} + \frac{\partial u}{\partial q}(0) + \frac{\partial u}{\partial r}\left(\frac{-z}{x^2}\right)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\left(\frac{-x}{y^2}\right) + \frac{\partial u}{\partial q}\left(\frac{1}{z}\right) + \frac{\partial u}{\partial r}(0)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}(0) + \frac{\partial u}{\partial q}\left(\frac{-y}{z^2}\right) + \frac{\partial u}{\partial r}\left(\frac{1}{x}\right)$$

consider: $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z \cdot \frac{\partial u}{\partial z}$

$$= x \cdot \frac{\partial u}{\partial p}\frac{1}{y} - \frac{z}{x} \cdot \frac{\partial u}{\partial r} - \frac{x}{y} \cdot \frac{\partial u}{\partial p} + \frac{y}{z} \cdot \frac{\partial u}{\partial q} - \frac{y}{z} \cdot \frac{\partial u}{\partial q} + \frac{z}{x} \cdot \frac{\partial u}{\partial r} = 0$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial b} - \frac{\partial u}{\partial a}$$

eq → (3) becomes

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial a}(0) + \frac{\partial u}{\partial b}(-1) + \frac{\partial u}{\partial c}(1)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial c} - \frac{\partial u}{\partial b}$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \cancel{\frac{\partial u}{\partial a}} - \cancel{\frac{\partial u}{\partial c}} + \cancel{\frac{\partial u}{\partial b}} - \cancel{\frac{\partial u}{\partial a}} + \cancel{\frac{\partial u}{\partial c}} - \cancel{\frac{\partial u}{\partial b}}$$

$$= 0$$

10. If $u = f(2x-3y, 3y-4z, 4z-2x)$ PT $\frac{1}{2}u_x + \frac{1}{3}u_y + \frac{1}{4}u_z = 0$

sol: Put $p = 2x-3y$, $q = 3y-4z$, $r = 4z-2x$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial x} + \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} \longrightarrow (1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial y} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial y} + \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} \longrightarrow (2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial z} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial z} + \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z} \longrightarrow (3)$$

$p = 2x-3y$

d.p.w.r. to x, y, z seperately.

$$\frac{\partial p}{\partial x} = 2(1) - 0 = \frac{\partial p}{\partial x} = 2, \quad \frac{\partial p}{\partial y} = -3, \quad \frac{\partial p}{\partial z} = 0$$

$q = 3y-4z$

d.p.w.r.t x, y, z seperately

$$\frac{\partial q}{\partial x} = 0, \quad \frac{\partial q}{\partial y} = 3, \quad \frac{\partial q}{\partial z} = -4$$

$r = 4z - 2x$

dp.w.r.t x,y,z seperately

$$\frac{\partial r}{\partial x} = -2, \quad \frac{\partial r}{\partial y} = 0, \quad \frac{\partial r}{\partial z} = 4$$

eq – (1) becomes

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p} \cdot 2 + \frac{\partial u}{\partial q} \cdot 0 + \frac{\partial u}{\partial r} \cdot -2$$

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial p} - 2 \cdot \frac{\partial u}{\partial r}$$

$$\frac{\partial u}{\partial y} = -3\frac{\partial u}{\partial p} + 4\frac{\partial u}{\partial r}$$

consider $\frac{1}{2}u_x + \frac{1}{3}u_y + \frac{1}{4}u_z$

$$= \frac{1}{2}\left(2\frac{\partial u}{\partial p} - 2\frac{\partial u}{\partial r}\right) + \frac{1}{3}\left(3\frac{\partial u}{\partial q} - 3\frac{\partial u}{\partial p}\right) + \frac{1}{4}\left(4\frac{\partial u}{\partial r} - 4\frac{\partial u}{\partial q}\right)$$

$$= \cancel{\frac{\partial u}{\partial p}} - \cancel{\frac{\partial u}{\partial r}} + \cancel{\frac{\partial u}{\partial q}} - \cancel{\frac{\partial u}{\partial p}} + \cancel{\frac{\partial u}{\partial r}} - \cancel{\frac{\partial u}{\partial q}}$$

$= 0$

Maxima and Minima:-

A function $f(x,y)$ is said to have

i) Maximum value at the point (a,b) if there exists neighbourhood of a point $(a+h, b+k)$ such that $f(a,b) > f(a+h, b+k)$

ii) Minimum value at the point (a,b) if $f(a,b) < f(a+h, b+k)$

iii) Extreme value: if $f(a,b)$ has maximum value (or) minimum value then we say that as extreme value.

Note:- $f(a,b)$ is said to be extreme value of $f(x,y)$ if it is a minimum value (or) maximum value.

Working procedure:-

1. Find the stationary points x,y such that $f_x = 0, f_y = 0$.
2. Find the second order partial derivative f_{xx}, f_{xy}, f_{yy} and take $A = f_{xx}$, $B = f_{xy}$, $C = f_{yy}$
3. If $AC - B^2$ is greater than '0' zero and A is less than '0' then the function has maximum value. (or) $AC - B^2 > 0$, $A < 0$

$$(f_{xx} f_{yy} - f_{xy}^2 > 0, f_{xx} < 0)$$

- $AC - B^2 > 0$ & $A > 0$ then the function has minimum value

4. If $AC - B^2$ is less than '0' (or) $AC - B^2 = 0$ (or) $A = 0$ then (x,y) is called saddle point.

1. Find extreme value of the function $f(x,y) = x^3+y^3-3x-12y+20$

Sol:- $f(x,y) = x^3+y^3-3x-12y+20 \longrightarrow (1)$

D.(1) p.w.r.to x,y seperately.

$f(x) = 3x^2+0-3-0+0$

$f_x = 3x^2-3$

$f_y = 0+3y^2-0-12+0$

$f_y = 3y^2-12$

put $f_x = 0$ & $f_y = 0$

$3x^2-3=0$ $\quad$ $3y^2-12=0$

$3x^2=3$ $\quad$ $3y^2=12$

$x^2=1$ $\quad$ $y^2=4$

$x = \pm 1$ $\quad$ $y = \pm 2$

W.k.T (x,y) is representation of point

$(1,2)$, $(1,-2)$, $(-1,2)$, $(-1,-2)$

$f_x = 3x^2-3$ $\quad$ $f_y = 3y^2-12$

$A = f_{xx} = 6x$ $\quad$ $C = f_{yy} = 6y$

$B = f_{xy} = 0$

(f_{xy} = diff of x w.r.to y)

Points	$AC-B^2 = 6x(6y)-0 = 36xy$	$A=6x$	conclusion
$(1,2)$	$72>0$	$6>0$	Minimum value
$(1,-2)$	$-72<0$		saddle point
$(-1,2)$	$36(-1)(2)=-72<0$		saddle point
$(-1,-2)$	$36(-1)(-2)=72>0$	$6(-1)=-6<0$	Maximum value

eq - (1) becomes

$f(1,2) = (1)^3 + (2)^3 - 3(1) - 12(2) + 20$

$= 1 + 8 - 3 - 24 + 20$

$= 2$

$f(-1,-2) = (-1)^3 + (-2)^3 - 3(-1) - 12(-2) + 20$

$= -1 - 8 + 3 + 24 + 20$

$= 38$

$\therefore$ The function attains maximum value at $(-1,-2)$ and the maximum value is 38.

The function attains minimum value at $(1,2)$ and the minimum value is 2.

3. Find the maxima and minima value of the function.

$f(x,y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$

sol:- $f(x,y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$

~~f(x~~ $f_x = 3x^2 + 3y^2 - 30x - 0 + 72$

$= 3x^2 + 3y^2 - 30x + 72$

$f_y = 0 + 6xy - 0 - 30y + 0$

$= 6xy - 30y$

Put $f_x = 0$

$3x^2 - 3y^2 - 30x + 72 = 0 \longrightarrow (2)$

put $y = 0$ in eq-(2)

$3x^2 - 30x + 72 = 0$

$3(x^2 - 10x + 24) = 0$

$x^2 - 6x - 4x + 24 = 0$

$x(x-6) - 4(x-6) = 0$

$f_y = 0$

$6xy - 30y = 0$

$6y(x-5) = 0$

$y = 0$ (or) $x = 5$

$x - 4 = 0$; $x - 6 = 0$

$\boxed{x = 4}$, $\boxed{x = 6}$

$\therefore$ $(4,0)$ & $(6,0)$ are stationary point at $y=0$

at $x=5$ (2) becomes

$$3(5)^2 + 3y^2 - 30(5) + 72 = 0$$

$$75 + 3y^2 - 150 + 72 = 0$$

$$3y^2 - 3 = 0$$

$$y^2 = 1$$

$$y = \pm 1$$

$\therefore$ at $x=5$ stationary points are

$(5,1)$ & $(5,-1)$

$f_x = 3x^2 + 3y^2 - 30 + 72$

$A = f_{xx} = 6x + 0 - 30 + 0 = 0$

$A = 6x - 30$

$B = f_{xy} = 0 + 6y + 0 + 0$

$B = 6y$

$C = f_{yy} = 6x - 30$

$C = 6x - 30$

Points	$AC-B^2=(6x-30)(6x-30)-(6y)^2$	$A=6x-30$	Conclusion
$(4,0)$	$36>0$	$-6<0$	Maximum value.
$(6,0)$	$36>0$	$6>0$	Minimum value
$(5,1)$	$-36<0$		Saddle point
$(5,-1)$	$-36<0$		Saddle point.

$f(4,0)=x^3+3xy^2-15x^2-15y^2+72x$

$=(4)^3+3(4)(0)-15(4)^2-0+72(1)$

$=64+0-240+288$

$=352-240$

$=112$

$f(6,0)=(6)^3+0-15(6)^2-0+72(6)$

$=216-540+432$

$=648-540$

$=108$

4 Find the extreme value of the function $f(x,y)=xy(1-x-y)$

sol:- $f=xy(1-x-y)$

$f=xy-x^2y-xy^2 \longrightarrow (1)$

D.eq $-(1)$ p w.r.to x,y seperately

$f_x=y-2xy-y^2$ & $f_y=0$

put $f_x=0$

$x-x^2-2xy=0$

$y^2-2xy-y^2=0$

$x(1-x-2y)=0$

$y(1-2x-y)=0$

$\boxed{x=0}$ (or) $1-x-2y=0$

$\boxed{y=0}$ (or) $1-2x-y=0$

$x+2y=1$

$2x+y=1$

taking $y=0$ in $x=0$ $(0,0)$

$y=0$ in $1-x-2y=0 \rightarrow x=1$ $(1,0)$

put $x=0$ in $y=0$ $(0,0)$

$x=0$ in $1-2x-y=0 \Rightarrow y=1$ $(0,1)$

$\therefore$ stationary points are $(0,0)$ $(1,0)$ $(0,1)$

step 2:- $f_x = y-2xy-y^2 \longrightarrow (2)$ $\quad$ $f_y = x-x^2-2xy \longrightarrow (3)$

D.P. w.r. to x $\quad$ D. (3) p w.r. to y

$A = f_{xx} = 0-2y-0$ $\quad$ $C = f_{yy} = 0-0-2x$

$A = -2y$ $\quad$ $C = -2x$

D. eq — (2) p w.r. to y

$B = f_{xy} = 1-2x-2y$

$B = 1-2x-2y$

points	$AC-B^2 = (-2y)(-2x)-(1-2x-2y)^2$	$A=-2y$	conclusion
$(0,0)$	$-1<0$	0	saddle point
$(1,0)$	$-1<0$	0	saddle point
$(0,1)$	$0-(1-0,-2)^2=-1<0$	$-2<0$	saddle point
$(1/3, 1/3)$	$1/3>0$		Maximum point

conclusion = $1/27$ is maximum value.

5. Find the extreme value of the function $x^3+y^3-3axy, a>0$

sol:- $f(x,y) = x^3+y^3-3axy \longrightarrow (1)$

D.(1) p w.r.to x, y seperately

$f_x = 3x^2+0-3ay$

take $f_x = 0$

$3x^2-3ay = 0 \div 3$

$x^2-ay=0$

$\boxed{x^2=ay}$

&

$f_y = 0+3y^2-3ax$

$f_y=0$

$3y^2-3ax = 0 \div 3$

$y^2-ax=0$

$\boxed{y^2=ax}$

squaring on both sides

$x^4 = a^2y^2$

$x^4 = a^2ax$

$x^4-a^3x = 0$

$x(x^3-a^3)=0$

$x=0, x^3-a^3=0$

$(a-b)(a^2-ab+b^2) = a^3-b^3$

$(x-a)(x^2-xa+a^2) = x^3-a^3$

$\boxed{x=a}$ $x^2-xa+a^2=0$

$a=1, b=-a, c=a^2$

$x = \dfrac{-a \pm \sqrt{b^2-4ac}}{2a}$

$x = \dfrac{-1 \pm \sqrt{(-a)^2-4(1)a^2}}{2(1)}$

$x = \dfrac{-1 \pm \sqrt{-3a^2}}{2} = \dfrac{-1 \pm ia\sqrt{3}}{2}$

$x=a$ is one root

take $x=0$ in $y^2=ax \Rightarrow y=0 (0,0)$

$x=a$ in $y^2=ax \rightarrow y^2=a^2$

$y = \pm a$

(a,a) $(a,-a)$

$\therefore$ Stationary points are $(0,0)$ (a,a) $(a,-a)$

$f_x = 3x^2-3ay \longrightarrow (2)$

D.2.p.w.r.to x

$A = f_{xx} = 6x-0$

D(2).p.w.r.to y

$B = f_{xy} = 0-3a$

$f_y = 3y^2-3ax$

$C = f_{yy} = 6y$

Points	$AC-B^2 = 36xy - 9a^2$	$A = 6x$	conclusion
$(0,0)$	$0-9a^2 < 0$	-	Saddle point
(a,a)	$36a^2-9a^2 = 27a^2 > 0$	$6a>0$	Minimum value
$(a,-a)$	$-36a^2-9a^2 = -45a^2 < 0$	-	Saddle point.

$f(x,y) = x^3+y^3-3axy$

$f(a,a) = a^3+a^3-3a(a)(a)$

$= 2a^3-3a^3$

$= -a^3$

$\therefore$ Minimum value that the function attain at (a,a) is $-a^3$

Jacobian

If U & V are any two functions of x & y then Jacobian of (u,v) w.r.to (x,y) is denoted by $J\left(\frac{u,v}{x,y}\right)$ (or) $\frac{\partial(u,v)}{\partial(x,y)}$ & is given by

$$J\left(\frac{U,V}{x,y}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial(u,v)}{\partial(x,y)}$$

similarly,

$$J\left(\frac{u,v,w}{x,y,z}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Properties:

1) If U & V are functions of r, s and r, s are functions of x, y

$$(u,v) \longrightarrow (r,s) \longrightarrow (x,y)$$

then,

$$J\left(\frac{u,v}{x,y}\right) = \frac{\partial(u,v)}{\partial(r,s)} \times \frac{\partial(r,s)}{\partial(x,y)}$$

2) If U & V are functions of x & y and if

$$J = \frac{\partial(u,v)}{\partial(x,y)} \quad \& \quad J' = \frac{\partial(x,y)}{\partial(u,v)}$$

then $\boxed{J \cdot J' = 1}$

Problems:-

1) If $u = e^x \cos y$, $v = e^x \sin y$, find $\frac{\partial(u,v)}{\partial(x,y)}$

sol:- wkt,

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \longrightarrow (1)$$

consider: $u = e^x \cos y$

D.P. w.r.to x & y respectively seperately,

$$\frac{\partial u}{\partial x} = \cos y e^x, \quad \frac{\partial u}{\partial y} = -e^x \sin y$$

Consider = $v = e^x \sin y$

D.P. w.r to x & y seperately

$$\frac{\partial v}{\partial x} = \sin y e^x, \quad \frac{\partial v}{\partial y} = e^x \cos y$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} e^x\cos y & -e^x\sin y \\ e^x\sin y & e^x\cos y \end{vmatrix}$$

$$= e^x e^x \begin{vmatrix} \cos y & -\sin y \\ \sin y & \cos y \end{vmatrix}$$

$$= e^{2x}(\cos^2 y + \sin^2 y) = e^{2x}$$

2. $u = x+y+z$
$v = y+z$ find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$
$w = z$

sol:- $$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 1(1-0) - 1(0-0) + 1(0-0)$$

$$= \frac{\partial(u,v,w)}{\partial(x,y,z)} = 1$$

3. If $u = \sqrt{x_1 x_2}$, $v = \sqrt{x_2 x_3}$, $w = \sqrt{x_1 x_3}$ then find $J\left(\frac{u,v,w}{x_1,x_2,x_3}\right)$

Sol:- $u = \sqrt{x_1} \cdot \sqrt{x_2}$, $v = \sqrt{x_2} \cdot \sqrt{x_3}$, $w = \sqrt{x_1} \cdot \sqrt{x_3}$

$$\frac{\partial(u,v,w)}{\partial(x_1,x_2,x_3)} = \begin{vmatrix} \frac{\partial u}{\partial x_1} & \frac{\partial u}{\partial x_2} & \frac{\partial u}{\partial x_3} \\ \frac{\partial v}{\partial x_1} & \frac{\partial v}{\partial x_2} & \frac{\partial v}{\partial x_3} \\ \frac{\partial w}{\partial x_1} & \frac{\partial w}{\partial x_2} & \frac{\partial w}{\partial x_3} \end{vmatrix}$$

$$= \begin{vmatrix} \sqrt{x_2}\frac{1}{2\sqrt{x_1}} & \sqrt{x_1}\frac{1}{2\sqrt{x_2}} & 0 \\ 0 & \sqrt{x_3}\frac{1}{2\sqrt{x_2}} & \sqrt{x_2}\frac{1}{2\sqrt{x_3}} \\ \sqrt{x_3}\frac{1}{2\sqrt{x_1}} & 0 & \sqrt{x_1}\frac{1}{2\sqrt{x_3}} \end{vmatrix}$$

$$= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \left[\frac{\sqrt{x_2}}{\sqrt{x_1}} \left(\frac{\cancel{\sqrt{x_3}}}{\sqrt{x_2}} \cdot \frac{\sqrt{x_1}}{\cancel{\sqrt{x_3}}} - 0 \right) - \frac{\sqrt{x_1}}{\sqrt{x_2}} \left(0 - \frac{\sqrt{x_2}}{\cancel{\sqrt{x_3}}} \cdot \frac{\cancel{\sqrt{x_3}}}{\sqrt{x_1}} \right) + 0 \cancel{\left(0 - \frac{\sqrt{x_3}}{\sqrt{x_2}} \cdot \frac{\sqrt{x_3}}{\sqrt{x_1}} \right)}^{0} \right]$$

$$= \frac{1}{8} \left[\frac{\cancel{\sqrt{x_2}}}{\cancel{\sqrt{x_1}}} \cdot \frac{\cancel{\sqrt{x_1}}}{\cancel{\sqrt{x_2}}} + \frac{\cancel{\sqrt{x_1}}}{\cancel{\sqrt{x_2}}} \cdot \frac{\cancel{\sqrt{x_2}}}{\cancel{\sqrt{x_1}}} \right]$$

$$= \frac{1}{8}[1+1]$$

$$= \frac{1}{8} \times 2$$

$$= \frac{1}{4}$$

4. If $x = \rho\cos\phi$, $y = \rho\sin\phi$ $z = z$ find $J\left(\frac{x, y, z}{\rho, \phi, z}\right)$

sol:- $$\frac{\partial(x,y,z)}{\partial(\rho,\phi,z)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\phi & -\rho\sin\phi & 0 \\ \sin\phi & \rho\cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \cos\phi\,(\rho\cos\phi - 0) + \rho\sin\phi\,(\sin\phi - 0) + 0(0-0)$$

$$= \rho\cos^2\phi + \rho\sin^2\phi + 0$$

$$= \rho(\sin^2\phi + \cos^2\phi)$$

$$= \rho$$

$$\rightarrow \frac{\partial(x,y,z)}{\partial(\rho,\phi,z)} = \rho$$

5. If $x = r\sin\theta\cdot\cos\phi$, $y = r\sin\theta\cdot\sin\phi$, $z = r\cos\theta$

PT, $J\left(\frac{x, y, z}{r, \theta, \phi}\right) = r^2\sin\theta$

sol:- $$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$\begin{vmatrix} \sin\theta\cdot\cos\phi & r\cos\phi\cos\theta & -r\sin\theta\sin\phi \\ \sin\theta\cdot\sin\phi & r\sin\phi\cos\theta & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$= \sin\theta\cos\phi\,(0+r^2\sin^2\theta\cdot\cos\phi) - r\cos\phi\cos\theta(0-r\sin\theta\cdot\cos\theta\cdot\cos\phi)$
$\quad - r\sin\theta\cdot\sin\phi\,(-r\sin^2\theta\sin\phi - r\sin\phi\cdot\cos^2\theta)$

$= r^2\sin^3\theta\cos^2\phi + r^2\cos^2\phi\cdot\cos^2\theta\cdot\sin\theta + r^2\sin^3\theta\cdot\sin^2\phi + r^2\sin^2\phi\cdot\sin\theta\cdot\cos^2\theta$

$= r^2\sin\theta(\sin^2\theta\cdot\cos^2\phi + \cos^2\phi\cdot\cos^2\theta + \sin^2\theta\cdot\sin^2\phi + \sin^2\phi\cdot\cos^2\theta)$

$= r^2\sin\theta(\sin^2\theta(\sin^2\phi+\cos^2\phi) + \cos^2\theta(\sin^2\phi+\cos^2\phi))$

$= r^2\sin\theta(\sin^2\theta+\cos^2\theta)$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta$$

6. If $x=u^2-v^2,\ y=v^2-w^2,\ z=w^2-u^2$ find $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$.

sol:- $$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$= \begin{vmatrix} 2u & -2v & 0 \\ 0 & 2v & -2w \\ -2u & 0 & -2w \end{vmatrix}$$

$= 2u(2v\cdot 2w-0) + 2v(0-(2w\cdot 2u)) - 0$

$= 8uvw - 8uvw$

$= 0$

7. If $u=\frac{yz}{x}$, $v=\frac{zx}{y}$, $w=\frac{xy}{z}$ PT $\frac{\partial(u,v,w)}{\partial(x,y,z)}=4$

sol:-
$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{-yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y}(1) & \frac{-zx}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & \frac{-xy}{z} \end{vmatrix}$$

$$= \frac{1}{x^2}\cdot\frac{1}{y^2}\cdot\frac{1}{z^2} \begin{vmatrix} -yz & zx & yx \\ zy & -zx & xy \\ yz & zx & -xy \end{vmatrix}$$

$$= \frac{1}{x^2y^2z^2}\left[(-yz)(x^2yz - x^2yz) - zx(-xy^2z - xy^2z) + xy[xz^2y + xz^2y)\right]$$

$$= \frac{1}{x^2y^2z^2}\left[0 + zx(2xy^2z) + xy(2xz^2y)\right]$$

$$= \frac{2}{x^2y^2z^2}\,xyz(zxy + xyz)$$

$$= \frac{2}{x^2y^2z^2}\,xyz(2zxy)$$

$$= \frac{2}{\cancel{x^2y^2z^2}}\,\cancel{2x^2y^2z^2}$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = 4$$