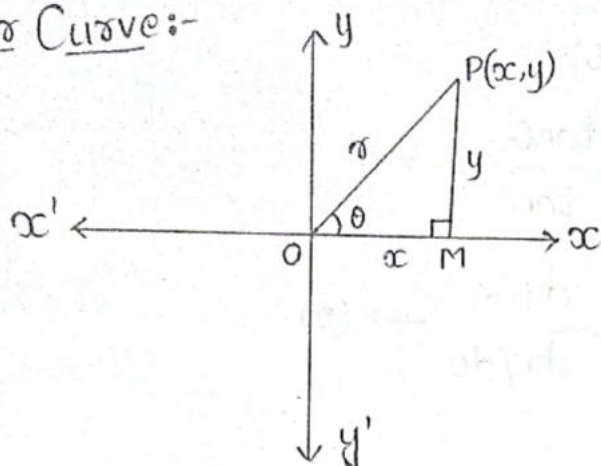


MODULE-1

Module - 1

Polar Curve:-



$$- \sin \theta = \frac{PM}{OP} = \frac{y}{r}$$

$$\Rightarrow y = r \sin \theta$$

$$- \cos \theta = \frac{OM}{OP} = \frac{x}{r}$$

$$\Rightarrow x = r \cos \theta$$

$$\Rightarrow r = \sqrt{x^2 + y^2}$$

$$\Rightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Let $P(x, y)$ be any point in xy plane such that $OP = r$ and θ be the angle between x -axis and OP . Draw PM perpendicular to x -axis, the P has a new coordinates (r, θ) is called polar coordinates.

Here, O is called pole.

OM is initial line.

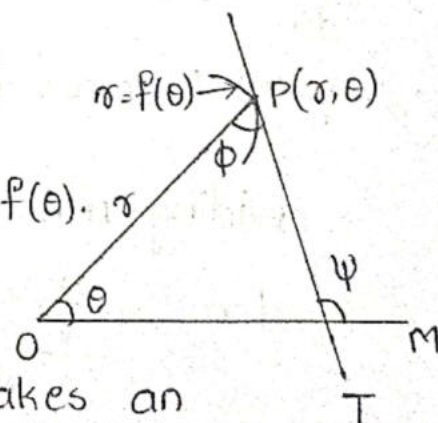
OP is radius vector.

$\Rightarrow$ P.T with usual notations $\tan \phi = r \cdot \frac{d\theta}{dr}$, (or) Find angle between radius vector and tangent.

Let O be the pole and OM be initial line.

Let $p(r, \theta)$ be any point on polar curve $r = f(\theta)$.

Let $OP = r$ be the radius vector, draw a tangent PT at the point P and which makes an angle ψ with the initial line.



Let θ be the angle between radius vector and initial line.

ϕ be the angle between radius vector and tangent.

Here we have, $\psi = \phi + \theta$

Applying $\tan$ on both sides.

$$\tan \psi = \tan(\phi + \theta)$$

$$\tan \psi = \frac{\tan \phi + \tan \theta}{1 - \tan \phi \cdot \tan \theta} \rightarrow (1)$$

Slope is given by $\tan \psi = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} \rightarrow (2)$

WKT, $x = r \cos \theta$, $y = r \sin \theta$

differentiate with respect to θ

$$\frac{dx}{d\theta} = r(-\sin \theta) + \cos \theta \frac{dr}{d\theta} \quad (\text{Let } \frac{dr}{d\theta} = r')$$

$$\Rightarrow \frac{dx}{d\theta} = r' \cos \theta - r \sin \theta$$

$$y = r \sin \theta$$

d.w. r & θ

$$\Rightarrow \frac{dy}{d\theta} = r \cos \theta + \sin \theta \frac{dr}{d\theta}$$

$$\frac{dy}{d\theta} = r \cos \theta + r' \sin \theta$$

$$\therefore \text{eq-(2) becomes } \tan \psi = \frac{r \cos \theta + r' \sin \theta}{r' \cos \theta - r \sin \theta}$$

dividing num. & deno. with $r' \cos \theta$

$$\tan \psi = \frac{\frac{r}{r'} + \tan \theta}{1 - \frac{r}{r'} \tan \theta} \rightarrow (3)$$

From eqn. (1) & (3) LHS are same

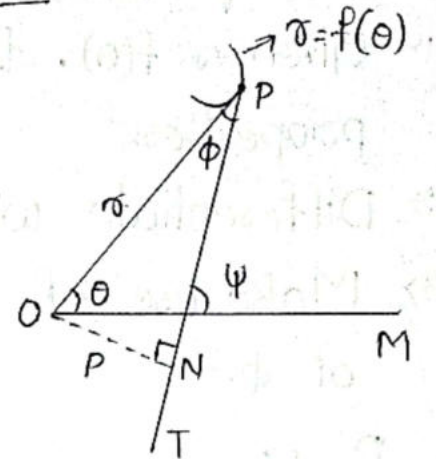
$$\therefore \tan \phi = \frac{r}{r'} = r \cdot \frac{d\theta}{dr} \Rightarrow \cot \phi = \frac{1}{r} \cdot \frac{dr}{d\theta}$$

Length of perpendicular from pole to tangent:-

Let O be the pole and OM be initial line,

$P(r, \theta)$ be any point on polar curve $r = f(\theta)$.

Draw a tangent PT at point P , draw ON perpendicular to the tangent from pole.



Let $OP = r$, $ON = p$

$$\sin \phi = \frac{ON}{OP} = \frac{p}{r}$$

$$p = r \sin \phi$$

$$\frac{1}{p} = \frac{1}{r \sin \phi}$$

$$\frac{1}{p} = \frac{1}{r} \operatorname{cosec} \phi$$

Squaring on both sides

$$\frac{1}{p^2} = \frac{1}{r^2} \operatorname{cosec}^2 \phi$$

$$\frac{1}{p^2} = \frac{1}{r^2} (1 + \cot^2 \phi)$$

$$= \frac{1}{r^2} \left(1 + \left(\frac{1}{r} \cdot \frac{dr}{d\theta} \right)^2 \right)$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} \left(\frac{dr}{d\theta} \right)^2$$

Note:-

Given $r = f(\theta) \rightarrow (1)$

W.K.T length of perpendicular from pole to tangent, $p = r \sin \phi \rightarrow (2)$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} \left(\frac{dr}{d\theta} \right)^2$$

If it possible to eliminate θ from eq-(1) and (2) or from eq-(1) & (3) then resulting equation contains p and r .
such equation is called "Pedal Equation".

Working procedure to find ϕ :-

- 1> Given $r = f(\theta)$, take log on both sides and apply logarithm properties.
- 2> Differentiate with respect to θ .
- 3> Make use of allied angles if require to get the value of ϕ .

Problems:-

1. Find the angle between radius vector and tangent, given that $r = a(1 - \cos\theta)$.

Sol:- $r = a(1 - \cos\theta)$

Take log o.b.s

$$\log r = \log (a(1 - \cos\theta))$$

$$\log r = \log a + \log (1 - \cos\theta)$$

d.w.r.to.

$$\frac{1}{r} \cdot \frac{dr}{d\theta} = 0 + \frac{0 - (-\sin\theta)}{1 - \cos\theta}$$

$$\cot \phi = \frac{\sin\theta}{1 - \cos\theta}$$

$$\cot \phi = \frac{\cancel{2} \sin \theta/2 \cdot \cos \theta/2}{\cancel{2} \sin^2 \theta/2}$$

$$\cot \phi = \cot \theta/2$$

$$\boxed{\phi = \theta/2}$$

$$2. r = a(1 + \cos \theta)$$

Take log o.b.s

$$\log r = \log a + \log (1 + \cos \theta)$$

d.w.r.to

$$\frac{1}{r} \cdot \frac{dr}{d\theta} = 0 + \frac{0 + (-\sin \theta)}{1 + \cos \theta}$$

$$\cot \phi = \frac{-\sin \theta}{1 + \cos \theta}$$

$$= \frac{-2 \sin \frac{\theta}{2} \cdot \cos \frac{\theta}{2}}{2 \cos^2 \frac{\theta}{2}}$$

$$\cot \phi = -\tan \frac{\theta}{2}$$

$$\cot \phi = \cot \left(\frac{\pi}{2} + \frac{\theta}{2} \right)$$

$$\boxed{\phi = \frac{\pi}{2} + \frac{\theta}{2}}$$

$$3. r^2 \cos 2\theta = a^2$$

Take log o.b.s

$$\log (r^2 \cos 2\theta) = \log a^2$$

$$2 \log r + \log \cos 2\theta = 2 \log a$$

$$\frac{2}{r} \cdot \frac{dr}{d\theta} + \frac{0 + (-\sin 2\theta) 2}{\cos 2\theta} = 0$$

$$2 \cot \phi = 2 \tan 2\theta$$

$$\cot \phi = \tan 2\theta$$

$$\cot \phi = \cot \left(\frac{\pi}{2} - 2\theta \right)$$

$$\boxed{\phi = \frac{\pi}{2} - 2\theta}$$

$$4. r = a \sin^3 \left(\frac{\theta}{3} \right)$$

Take log o.b.s

$$\log r = \log a + 3 \log \sin \left(\frac{\theta}{3} \right)$$

d.w.r.to

$$\frac{1}{r} \cdot \frac{dr}{d\theta} = 0 + \frac{3 \cdot \cos \left(\frac{\theta}{3} \right) \cdot \frac{1}{3}}{\sin \left(\frac{\theta}{3} \right)}$$

$$\cot \phi = \cot \left(\frac{\theta}{3} \right)$$

$$\boxed{\phi = \frac{\theta}{3}}$$

$$5. \frac{2a}{r} = 1 - \cos \theta$$

Take log o.b.s

$$\log \left(\frac{2a}{r} \right) = \log (1 - \cos \theta)$$

$$\log 2a - \log r = \log (1 - \cos \theta)$$

d.w.r.to

$$0 - \frac{1}{r} \cdot \frac{dr}{d\theta} = \frac{0 - (-\sin \theta)}{1 - \cos \theta}$$

$$-\cot \phi = \frac{2 \sin \frac{\theta}{2} \cdot \cos \frac{\theta}{2}}{2 \sin^2 \frac{\theta}{2}}$$

$$-\cot \phi = \cot \frac{\theta}{2}$$

$$\boxed{-\phi = \frac{\theta}{2}}$$

6. $r^m = a^m (\cos m\theta + \sin m\theta) \rightarrow (1)$ Find the pedal equation of the curve.

Take log a.b.s

$$\log r^m = \log a^m + \log (\cos m\theta + \sin m\theta)$$

$$m \log r = m \log a + \log (\cos m\theta + \sin m\theta)$$

d.w.r.t. θ

$$m \cdot \frac{1}{r} \cdot \frac{dr}{d\theta} = 0 + \frac{(-\sin m\theta + \cos m\theta) \cdot m}{(\cos m\theta + \sin m\theta)}$$

$$\cot \phi = \frac{\cos m\theta - \sin m\theta}{\cos m\theta + \sin m\theta}$$

$\div \cos m\theta$ in num. & deno.

$$\cot \phi = \frac{1 - \tan m\theta}{1 + \tan m\theta}$$

$$\cot \phi = -\tan \left(\frac{\pi}{4} - m\theta \right)$$

$$\cot \phi = \cot \left(\frac{\pi}{2} - \left(\frac{\pi}{4} - m\theta \right) \right)$$

$$\cot \phi = \cot \left(\frac{\pi}{4} + m\theta \right)$$

$$\boxed{\phi = \frac{\pi}{4} + m\theta}$$

$$P = r \sin \phi$$

$$= r \sin \left(\frac{\pi}{4} + m\theta \right)$$

$$= r \left(\sin \frac{\pi}{4} \cdot \cos m\theta + \cos \frac{\pi}{4} \cdot \sin m\theta \right)$$

$$= r \left(\frac{1}{\sqrt{2}} \cos m\theta + \frac{1}{\sqrt{2}} \sin m\theta \right)$$

$$P = \frac{r}{\sqrt{2}} (\cos m\theta + \sin m\theta)$$

$$\cos m\theta + \sin m\theta = \frac{P\sqrt{2}}{r} \rightarrow (2)$$

from eq-(1), $r^m = a^m (\cos m\theta + \sin m\theta)$

$$r^m = a^m \cdot \frac{P\sqrt{2}}{r}$$

$$P = \frac{r^{m+1}}{a^m \sqrt{2}}$$

$$\boxed{P \propto r^{m+1}}$$

7. Find the pedal equation of the curve $\frac{2a}{r} = 1 + \cos\theta$

Sol:- $\frac{2a}{r} = 1 + \cos\theta \longrightarrow (1)$

Take log o.b.s

$$\log 2a - \log r = \log (1 + \cos\theta)$$

d.w.r.t θ .

$$0 - \frac{1}{r} \cdot \frac{dr}{d\theta} = \frac{0 + (-\sin\theta)}{1 + \cos\theta}$$

$$\neq \cot\phi = \neq \frac{\sin\theta}{1 + \cos\theta}$$

$$= \frac{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}}{2 \cos^2 \frac{\theta}{2}}$$

$$\cot\phi = \tan \frac{\theta}{2}$$

$$\cot\phi = \cot\left(\frac{\pi}{2} - \frac{\theta}{2}\right)$$

$$\boxed{\phi = \frac{\pi}{2} - \frac{\theta}{2}}$$

WKT, $P = r \sin\phi$

$$= r \sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right)$$

$$P = r \cos\left(\frac{\theta}{2}\right) \longrightarrow (2)$$

From eq-(1) $\frac{2a}{r} = 1 + \cos\theta$.

$$\frac{2a}{r} = 2 \cos^2 \frac{\theta}{2}$$

$$\cos\left(\frac{\theta}{2}\right) = \sqrt{\frac{a}{r}}$$

eq-(2) becomes, $P = r \cdot \sqrt{\frac{a}{r}}$

$$P = \sqrt{r} \cdot \sqrt{r} \cdot \frac{\sqrt{a}}{\sqrt{r}}$$

$$P = \sqrt{a} \cdot \sqrt{r}$$

$$P \propto \sqrt{r} \quad (\because \sqrt{a} \text{ is constant})$$

8. Find the pedal equation of the curve $r^2 = a^2 \sec 2\theta$.

$$r^2 = a^2 \sec 2\theta \rightarrow (1)$$

Take log o.b.s

$$2 \log r = 2 \log a + \log \sec 2\theta$$

d.w.r.t. θ

$$2 \cdot \frac{1}{r} \cdot \frac{dr}{d\theta} = 0 + \frac{2(\sec 2\theta \cdot \tan 2\theta)}{\sec^2 2\theta}$$

$$\cot \phi = \tan 2\theta$$

$$\cot \phi = \cot \left(\frac{\pi}{2} - 2\theta \right)$$

$$\boxed{\phi = \frac{\pi}{2} - 2\theta}$$

WKT, $P = r \sin \phi$

$$= r \sin \left(\frac{\pi}{2} - 2\theta \right)$$

$$P = r \cos 2\theta \rightarrow (2)$$

from eq-(1), $r^2 = a^2 \sec 2\theta$

$$r^2 = a^2 \frac{1}{\cos 2\theta}$$

$$\cos 2\theta = \frac{a^2}{r^2}$$

eq-(2) becomes, $P = r \cdot \frac{a^2}{r^2}$

$$P \propto \frac{1}{r}$$

9. Find the pedal equation $r^n = a^n \cos n\theta$

$$r^n = a^n \cos n\theta \rightarrow (1)$$

Take log o.b.s

$$n \log r = n \log a + \log \cos n\theta$$

d.w.r.t. θ

$$n \cdot \frac{1}{r} \cdot \frac{dr}{d\theta} = 0 + \frac{-\sin n\theta \cdot n}{\cos n\theta}$$

$$\cot \phi = -\tan n\theta$$

$$\cot \phi = \cot \left(\frac{\pi}{2} + n\theta \right)$$

$$\phi = \frac{\pi}{2} + n\theta$$

WKT, $P = r \sin \phi$

$$= r \sin \left(\frac{\pi}{2} + n\theta \right)$$

$$P = r \cos n\theta \rightarrow (2)$$

From eq-(1) $r^n = a^n \cos n\theta$

$$\cos n\theta = \frac{r^n}{a^n}$$

eq-(2) becomes,

$$P = r \cdot \frac{r^n}{a^n}$$

$$P = \frac{r^{n+1}}{a^n} \Rightarrow \boxed{P \propto r^{n+1}}$$

10. P.T pedal equation of the curve $r^n = a^n \cos n\theta + b^n \sin n\theta$ in the form of $P^2 (a^{2n} + b^{2n}) = r^{2n+2}$.

Sol:- $r^n = a^n \cos n\theta + b^n \sin n\theta$

Take log o.b.s

$$n \log r = \log (a^n \cos n\theta + b^n \sin n\theta)$$

d.w.r.t. θ

$$n \cdot \frac{1}{r} \cdot \frac{dr}{d\theta} = \frac{a^n (-\sin n\theta) \cdot n + b^n (\cos n\theta) n}{a^n \cos n\theta + b^n \sin n\theta}$$

$$r \cot \phi = \frac{b^n \cos n\theta - a^n \sin n\theta}{a^n \cos n\theta + b^n \sin n\theta}$$

$$\cot \phi = \frac{b^n \cos n\theta - a^n \sin n\theta}{a^n \cos n\theta + b^n \sin n\theta}$$

$$\text{WKT, } \frac{1}{p^2} = \frac{1}{r^2} \left(1 + \left(\frac{1}{r} \cdot \frac{dr}{d\theta} \right)^2 \right)$$

$$= \frac{1}{r^2} (1 + \cot^2 \phi)$$

$$\frac{1}{p^2} = \frac{1}{r^2} \left[1 + \left(\frac{b^n \cos n\theta - a^n \sin n\theta}{a^n \cos n\theta + b^n \sin n\theta} \right)^2 \right]$$

$$= \frac{1}{r^2} \left[1 + \frac{(b^n \cos n\theta - a^n \sin n\theta)^2}{(a^n \cos n\theta + b^n \sin n\theta)^2} \right]$$

$$= \frac{1}{r^2} \left[\frac{(a^n \cos n\theta + b^n \sin n\theta)^2 + (b^n \cos n\theta - a^n \sin n\theta)^2}{(a^n \cos n\theta + b^n \sin n\theta)^2} \right]$$

$$= \frac{1}{r^2} \cdot \frac{1}{r^n} \left[\begin{aligned} &a^{2n} \cos^2 n\theta + b^{2n} \sin^2 n\theta + 2a^n \cos n\theta \cdot b^n \sin n\theta + \\ &b^{2n} \cos^2 n\theta + a^{2n} \sin^2 n\theta - 2a^n \sin n\theta \cdot b^n \cos n\theta \end{aligned} \right]$$

→ Show that the curve $r \sin^3(\theta/2) = a$, the length of the perpendicular from pole to tangent at the point $(2a, \pi/2)$ is equal to $a\sqrt{2}$.

Sol:- $r \sin^3(\theta/2) = a$

$$\log r + 3 \log \sin(\theta/2) = \log a$$

d.w.r.t. θ

$$\frac{1}{r} \cdot \frac{dr}{d\theta} + 3 \frac{(\cos \theta/2)(1/2)}{\sin^2 \theta/2} = 0$$

$$\cot \phi + \cot \theta/2 = 0$$

$$\cot \phi = -\cot \theta/2$$

$$\phi = -\theta/2$$

$$P = r \sin \phi$$

$$P = r \sin(-\theta/2)$$

$$P = 2a \sin(-\pi/4/2)$$

$$P = -2a \sin(\pi/4)$$

$$= -2a \frac{1}{\sqrt{2}} \Rightarrow P = -a\sqrt{2} \cdot \sqrt{2} \cdot \frac{1}{\sqrt{2}}$$

$$|P| = |-a\sqrt{2}|$$

$$P = a\sqrt{2}$$

→ Show that the length of cardioides $r = a(1 + \cos \theta)$ at the points $\theta = \pi/3$, $\theta = 3\pi/3$ are respectively parallel & perpendicular to the initial line.

Sol:- $r = a(1 + \cos \theta)$

$$\log r = \log a + \log(1 + \cos \theta)$$

d.w.r.t. θ

$$\frac{1}{r} \cdot \frac{dr}{d\theta} = \frac{(-\sin \theta)}{1 + \cos \theta} = \frac{-2 \sin \theta/2 \cdot \cos \theta/2}{2 \cos^2 \theta/2}$$

$$= \cot \phi = -\tan(\theta/2)$$

$$\cot \phi = \cot(\pi/2 + \theta/2)$$

$$\phi = \pi/2 + \theta/2$$

→ When $\theta = \frac{\pi}{3}$,

$$\phi = \frac{\pi}{2} + \frac{\pi}{6} \Rightarrow \phi = \frac{2\pi}{3}$$

→ at $\theta = \frac{2\pi}{3}$

$$\phi = \frac{\pi}{2} + \frac{1}{2} \cdot \frac{2\pi}{3} = \frac{5\pi}{3}$$

$$\psi = \theta + \phi$$

$$\text{at } \theta = \frac{\pi}{3} \Rightarrow \psi = \frac{\pi}{3} + \frac{2\pi}{3} = \frac{3\pi}{3} = \pi$$

$$\tan \psi = \tan \pi$$

$$\tan \psi = 0$$

Angle between two polar curves:-

Let $r = f_1(\theta)$, $r = f_2(\theta)$ be two polar curves which intersect at point P.

Let OM be initial line and OP be radius vector and 'O' be the pole.

Draw a tangent PT_1 & PT_2 at the point P to $r = f_1(\theta)$ and $r = f_2(\theta)$ respectively.

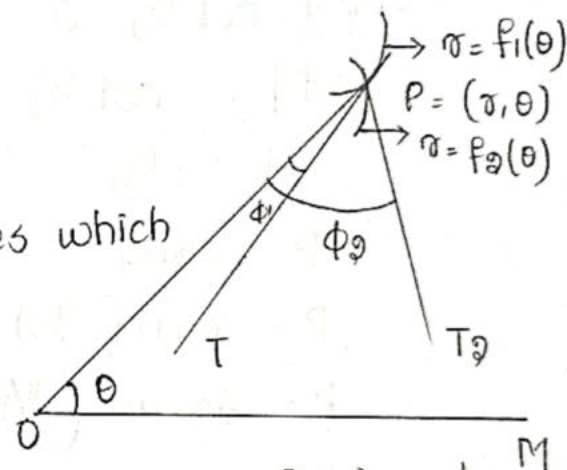
∴ Angle between radius vector and PT_1 is ϕ_1 .

Angle between radius vector and PT_2 is ϕ_2 .

∴ Acute angle between the tangents PT_1 and PT_2 is given by $|\phi_2 - \phi_1|$.

Note:- $r = f_1(\theta)$ & $r = f_2(\theta)$ are said to be orthogonal if

$$\tan \phi_1 \cdot \tan \phi_2 = 1.$$



1. Find the angle between the curves $r = 2 \sin \theta$, $r = \sin \theta + \cos \theta$

Sol:- $r = 2 \sin \theta$

Take log

$$\log r = \log(2 \sin \theta)$$

$$\log r = \log 2 + \log \sin \theta$$

d.w. r, t, θ

$$\frac{1}{r} \cdot \frac{dr}{d\theta} = 0 + \frac{\cos \theta}{\sin \theta}$$

$$\cot \phi = \cot \theta$$

$$\boxed{\phi_1 = \theta}$$

$$r = \sin \theta + \cos \theta$$

$$\log r = \log(\sin \theta + \cos \theta)$$

d.w. r, t, θ

$$\frac{1}{r} \cdot \frac{dr}{d\theta} = \frac{\cos \theta - \sin \theta}{\sin \theta + \cos \theta}$$

$$\cot \phi = \frac{1 - \tan \theta}{1 + \tan \theta}$$

$$\cot \phi = \tan\left(\frac{\pi}{4} - \theta\right)$$

$$\cot \phi = \cot\left(\frac{\pi}{2} - \left(\frac{\pi}{4} - \theta\right)\right)$$

$$\cot \phi = \cot\left(\frac{\pi}{4} + \theta\right)$$

$$\boxed{\phi_2 = \frac{\pi}{4} + \theta}$$

$\therefore$ Angle between two polar curves

$$= |\phi_2 - \phi_1| = \frac{\pi}{4} + \theta - \theta = \frac{\pi}{4}$$

2. PT angle between polar curves is $\frac{\pi}{2} + \frac{1}{2} \cos^{-1}\left(\frac{1}{3}\right)$ given
 $r = a(1 - \cos \theta)$ & $r = 2a \cos \theta$.

Sol:- $r = a(1 - \cos \theta)$

$$\log r = \log a + \log(1 - \cos \theta)$$

$$\frac{1}{r} \cdot \frac{dr}{d\theta} = \frac{-\sin \theta}{1 - \cos \theta}$$

$$\cot \phi = \frac{-2 \sin \frac{\theta}{2} \cdot \cos \frac{\theta}{2}}{2 \sin^2 \frac{\theta}{2}}$$

$$\cot \phi = \cot \frac{\theta}{2}$$

$$\boxed{\phi_1 = \frac{\theta}{2}}$$

$$r = 2a \cos \theta$$

$$\log r = \log 2a + \log \cos \theta$$

$$\frac{1}{r} \cdot \frac{dr}{d\theta} = \frac{-\sin \theta}{\cos \theta}$$

$$\cot \phi = -\tan \theta$$

$$\cot \phi = \cot\left(\frac{\pi}{2} + \theta\right)$$

$$\boxed{\phi_2 = \frac{\pi}{2} + \theta}$$

$$|\phi_2 - \phi_1| = \left| \frac{\pi}{2} + \theta - \frac{\theta}{2} \right|$$

$$= \frac{\pi}{2} + \frac{\theta}{2} \rightarrow (1)$$

Given $r = a(1 - \cos \theta)$, $r = 2a \cos \theta$

since, LHS are equal

$$\therefore 2a \cos \theta = a(1 - \cos \theta)$$

$$2 \cos \theta = 1 - \cos \theta$$

$$3 \cos \theta = 1$$

$$3 \cos \theta = 1$$

$$\theta = \cos^{-1}\left(\frac{1}{3}\right)$$

$$\therefore (1) \text{ becomes } |\phi_2 - \phi_1| = \frac{\pi}{3} + \frac{1}{3} \cos^{-1}\left(\frac{1}{3}\right)$$

3. $r^n = a^n \cos n\theta$, $r^n = b^n \sin n\theta$

Sol:- $r^n = a^n \cos n\theta$

$$n \log r = n \log a + \log \cos n\theta$$

$$\therefore \frac{1}{r} \cdot \frac{dr}{d\theta} = \frac{r(-\sin n\theta)}{\cos n\theta}$$

$$\cot \phi = -\tan n\theta$$

$$\cot \phi = \cot\left(\frac{\pi}{2} + n\theta\right)$$

$$\boxed{\phi_1 = \frac{\pi}{2} + n\theta}$$

$$r^n = b^n \sin n\theta$$

$$n \log r = n \log b + \log \sin n\theta$$

$$\therefore \frac{1}{r} \cdot \frac{dr}{d\theta} = \frac{r(\cos n\theta)}{\sin n\theta}$$

$$\cot \phi = \cot n\theta$$

$$\boxed{\phi_2 = n\theta}$$

$$|\phi_2 - \phi_1| = \left| n\theta - \left(\frac{\pi}{2} + n\theta\right) \right|$$

$$= \left| n\theta - \frac{\pi}{2} - n\theta \right| = \frac{\pi}{2}$$

4. $r = 4 \sec^2\left(\frac{\theta}{2}\right)$, $r = 9 \operatorname{cosec}\left(\frac{\theta}{2}\right)$

Sol:- $r = 4 \sec^2\left(\frac{\theta}{2}\right)$

$$\log r = \log 4 + 2 \log \sec\left(\frac{\theta}{2}\right)$$

d.w.r.t. θ

$$\frac{1}{r} \cdot \frac{dr}{d\theta} = \frac{2 \left(\sec\left(\frac{\theta}{2}\right) \cdot \tan\left(\frac{\theta}{2}\right) \right) \frac{1}{2}}{\sec^2\left(\frac{\theta}{2}\right)}$$

$$\cot \phi = \tan \frac{\theta}{2}$$

$$\cot \phi = \cot\left(\frac{\pi}{2} - \frac{\theta}{2}\right) \Rightarrow \phi_1 = \frac{\pi}{2} - \frac{\theta}{2}$$

$$r = 9 \operatorname{cosec}^2(\theta/2)$$

$$\log r = \log 9 + 2 \log \operatorname{cosec}(\theta/2)$$

d.w.r.t. θ

$$\frac{1}{r} \cdot \frac{dr}{d\theta} = \frac{2(-\operatorname{cosec} \theta/2 \cdot \cot \theta/2) \cdot \frac{1}{2}}{\operatorname{cosec}^2(\theta/2)}$$

$$\cot \phi = -\cot \theta/2$$

$$\phi_2 = -\theta/2$$

$$|\phi_2 - \phi_1| = \left| -\frac{\theta}{2} - \left(\frac{\pi}{2} - \frac{\theta}{2} \right) \right|$$

$$= \left| -\frac{\theta}{2} - \frac{\pi}{2} + \frac{\theta}{2} \right| = \left| -\frac{\pi}{2} \right| = \frac{\pi}{2}$$

$$5. \quad r = a \log \theta, \quad r = \frac{a}{\log \theta}$$

Sol:- $r = a \log \theta$

$$\log r = \log a + \log(\log \theta)$$

d.w.r.t. θ

$$\frac{1}{r} \cdot \frac{dr}{d\theta} = 0 + \frac{1}{\log \theta} \times \frac{1}{\theta}$$

$$\cot \phi = \frac{1}{\theta \log \theta}$$

$$\tan \phi_1 = \theta \log \theta$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \cdot \tan B}$$

$$\tan(\phi_1 - \phi_2) = \frac{\tan \phi_1 - \tan \phi_2}{1 + \tan \phi_1 \cdot \tan \phi_2} = \frac{\theta \log \theta - (-\theta \log \theta)}{1 + \theta \log \theta \cdot (-\theta \log \theta)}$$

$$= \frac{2\theta \log \theta}{1 - (\theta \log \theta)^2} \rightarrow (1)$$

$$r = \frac{a}{\log \theta}$$

$$\log r = \log a - \log(\log \theta)$$

d.w.r.t. θ

$$\frac{1}{r} \cdot \frac{dr}{d\theta} = 0 - \frac{1}{\log \theta} \times \frac{1}{\theta}$$

$$\cot \phi = -\frac{1}{\theta \log \theta}$$

$$\tan \phi_2 = -\theta \log \theta$$

$$r = a \log \theta, \quad r = \frac{a}{\log \theta}$$

$$a \log \theta = \frac{a}{\log \theta}$$

$$(\log \theta)^2 = 1 \Rightarrow \log \theta = 1$$

$$\boxed{\theta = e}$$

$$\tan(\phi_1 - \phi_2) = \frac{2 \cdot e \log_e e}{1 - (e \log_e e)^2} = \frac{2e}{1 - e^2}$$

$$\phi_1 - \phi_2 = \tan^{-1} \left(\frac{2e}{1 - e^2} \right)$$

Curvature and Radius of Curvature:-

Consider a curve on XOY plane. Let A be a fixed point on it and P & Q be any two point which are neighbouring points on the curve such that AP = s, AQ = s + \delta s.

Draw a tangent at the point 'P' which makes angle '\psi' with X-axis.

Draw another tangent at the point 'Q' which makes an angle \psi + \delta \psi.

Let \psi & \psi + \delta \psi be the angles made by the tangents drawn at the points P & Q with X-axis respectively.

The angle \delta \psi between tangents is called the bending of the curve which depends on \delta s.

$\frac{\delta \psi}{\delta s}$ is called as mean curvature of the curve.

The amount of the bending of the curve at P is called curvature of curve at P. and is defined mathematically as

$$\lim_{\delta s \rightarrow 0} \frac{\delta \psi}{\delta s} = \frac{d\psi}{ds} = K \quad \text{where } K \text{ is constant.}$$

∴ Curvature of the curve is a measure of rate of change of bentness.

Radius of Curvature:-

It is denoted by 'P' and it is reciprocal of the curvature.

$$P = \frac{1}{K} \text{ if } K \neq 0.$$

Note:- Curvature of straight line is zero.

Curvature of circle is constant.

ix) Radius of Curvature in Cartesian form:-

$$P = \frac{(1 + y_1^2)^{3/2}}{y_2}$$

where, y_1 & y_2 are 1st order & 2nd order differentiation respectively.

Note:- If y_2 is zero at some point then the P is infinity. So, we can represent P radius of curvature in another form.

$$P = \frac{(1 + x_1^2)^{3/2}}{x_2} \text{ where } x_1 \text{ is } \frac{dx}{dy} \text{ \& } x_2 = \frac{d^2x}{dy^2}$$

1. Find the radius of curvature for the curve $y = e^x$ at the point where it crosses Y-axis.

Sol:- Given,

Curve crosses Y-axis,

$$\therefore x = 0$$

$$y = e^x \rightarrow (1)$$

$$\therefore \text{eq. (1) becomes, } y = e^0 = 1$$

$$\text{points } (x, y) = (0, 1)$$

Diff. eq. (1) w.r.t. x

$$y_1 = e^x \rightarrow (2)$$

$$y_1 = e^0 = 1$$

D. eq. (2) w.r.t. x

$$y_2 = e^x \text{ at } (0, 1)$$

$$y_2 = c^0 = 1$$

$$\rho = \frac{(1 + y_1^2)^{3/2}}{y_2} = \frac{(1+1)^{3/2}}{1} = 2^{3/2} = 2 \cdot 2^{1/2} = 2\sqrt{2}$$

$$\boxed{\rho = 2\sqrt{2}}$$

2. Find the radius of curvature for the curve $y = ax^2 + bx + c$ at the point $x = \frac{1}{2a}(\sqrt{a^2 - 1} - b)$

Sol:- $y = ax^2 + bx + c \rightarrow (1)$

eq-(1) d.w.r. to x

$$y_1 = a \cdot 2x + b + 0 \text{ at the point } x \rightarrow (2)$$

$$y_1 = 2a \cdot \frac{1}{2a}(\sqrt{a^2 - 1} - b) + b$$

$$y_1 = \sqrt{a^2 - 1} - b + b \Rightarrow y_1 = \sqrt{a^2 - 1}$$

d (2) w.r. to x

$$y_2 = 2a(1) + 0$$

$$y_2 = 2a$$

$$\rho = \frac{(1 + y_1^2)^{3/2}}{y_2} = \frac{(1 + (\sqrt{a^2 - 1})^2)^{3/2}}{2a}$$

$$= \frac{(1 + a^2 - 1)^{3/2}}{2a} = \frac{a^{3/2}}{2a}$$

$$\boxed{\rho = \frac{a^{1/2}}{2}}$$

3. Find the radius of curvature for the curve $y = a \log \sec\left(\frac{x}{a}\right)$

Sol:- $y = a \log \sec\left(\frac{x}{a}\right)$

d.w.r. to x

$$y_1 = a \cdot \frac{1}{\sec\left(\frac{x}{a}\right)} \sec\left(\frac{x}{a}\right) \cdot \tan\left(\frac{x}{a}\right) \cdot \frac{1}{a}$$

$$y_1 = \tan\left(\frac{x}{a}\right) \text{ d.w.r. to } x$$

$$y_2 = \sec^2\left(\frac{x}{a}\right) \cdot \frac{1}{a}$$

$$\rho = \frac{(1 + y_1^2)^{3/2}}{y_2} = \frac{(1 + (\tan^2\left(\frac{x}{a}\right))^{3/2}}{\frac{1}{a} \sec^2\left(\frac{x}{a}\right)} = \frac{a (\sec^2\left(\frac{x}{a}\right))^{3/2}}{\sec^2\left(\frac{x}{a}\right)}$$

$$= \frac{a \sec^3\left(\frac{x}{a}\right)}{\sec^2\left(\frac{x}{a}\right)}$$

$$\boxed{\rho = a \sec\left(\frac{x}{a}\right)}$$

4. $y = C \cosh\left(\frac{x}{c}\right)$

d.w.r.t x

$$y_1 = \frac{d}{dx} \left(\sinh\left(\frac{x}{c}\right) \right) \cdot \frac{1}{c}$$

$$y_1 = \sinh\left(\frac{x}{c}\right)$$

d.w.r.t x

$$y_2 = \cosh\left(\frac{x}{c}\right) \cdot \frac{1}{c}$$

$$\rho = \frac{(1 + y_1^2)^{3/2}}{y_2}$$

$$= \frac{(1 + (\sinh^2\left(\frac{x}{c}\right))^{3/2}}{\frac{1}{c} \cdot \cosh\left(\frac{x}{c}\right)}$$

$$= \frac{c \cdot (\cosh^2\left(\frac{x}{c}\right))^{3/2}}{\cosh\left(\frac{x}{c}\right)}$$

$$= \frac{c (\cosh^3\left(\frac{x}{c}\right))}{\cosh\left(\frac{x}{c}\right)}$$

$$\rho = c \cdot \cosh^2\left(\frac{x}{c}\right)$$

$$y = c \cdot \cosh\left(\frac{x}{c}\right)$$

$$\frac{y}{c} = \cosh\left(\frac{x}{c}\right) \Rightarrow \rho = \frac{y^2}{c} \Rightarrow \boxed{\rho = \frac{y^2}{c}}$$

5. Define curvature & radius of curvature, find the radius of curvature of curve, $x^3 + y^3 = 3axy$, $\left(\frac{3a}{2}, \frac{3a}{2}\right)$

Sol:- $x^3 + y^3 = 3axy$
d.w.r.t. x

$$3x^2 + 3y^2 y_1 = 3a(xy_1 + y)$$

$$x^2 + y^2 y_1 = a(xy_1 + y)$$

$$x^2 + y^2 y_1 = axy_1 + ay$$

$$y^2 y_1 - axy_1 = ay - x^2$$

$$y_1(y^2 - ax) = ay - x^2$$

$$y_1 = \frac{ay - x^2}{(y^2 - ax)} \rightarrow (1)$$

at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$

$$y_1 = \frac{a\left(\frac{3a}{2}\right) - \left(\frac{3a}{2}\right)^2}{\left(\frac{3a}{2}\right)^2 - a\left(\frac{3a}{2}\right)} = \frac{-\left(\left(\frac{3a}{2}\right)^2 - a\left(\frac{3a}{2}\right)\right)}{\left(\frac{3a}{2}\right)^2 - a\left(\frac{3a}{2}\right)}$$

$$y_1 = -1$$

d. eq-(1) w.r.t. x

$$y_2 = \frac{(y^2 - ax)(ay_1 - 2x) - (ay - x^2)(2y \cdot y_1 - a)}{(y^2 - ax)^2}$$

at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$, $y_2 = \frac{\left(\left(\frac{3a}{2}\right)^2 - a\left(\frac{3a}{2}\right)\right) \cdot \left(a(-1) - 2\left(\frac{3a}{2}\right)\right) - \left(a\left(\frac{3a}{2}\right) - \left(\frac{3a}{2}\right)^2\right) \cdot \left(2\left(\frac{3a}{2}\right)(-1) - a\right)}{(y^2 - ax)^2}$

$$\begin{aligned}
 y_2 &= \frac{\left(\left(\frac{3a}{2} \right)^2 - a \left(\frac{3a}{2} \right) \right) (-4a) - \left(a \left(\frac{3a}{2} \right) - \left(\frac{3a}{2} \right)^2 \right) (-4a)}{\left(\left(\frac{3a}{2} \right)^2 - a \left(\frac{3a}{2} \right) \right)^2} \\
 &= \frac{\frac{3a}{2} \left(\frac{3a}{2} - a \right) (-4a) - \frac{3a}{2} \left(a - \frac{3a}{2} \right) (-4a)}{\left(\frac{9a^2}{4} - \frac{9a^2}{2} \right)^2} \\
 &= \frac{\frac{3a}{2} (-4a) \left[\frac{a}{2} - \left(-\frac{a}{2} \right) \right]}{\left(\frac{9a^2 - 6a^2}{4} \right)^2} = \frac{-6a^2(a)}{\left(\frac{3a^2}{4} \right)^2} = \frac{-6a^2 \cdot a}{\frac{9a^4}{16}} \\
 &= -6a^3 \times \frac{16}{9a^4} = -\frac{32}{3a}
 \end{aligned}$$

6. Find the radius of curvature for the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$.

Sol:- $\rho = \frac{(1 + y_1^2)^{3/2}}{y_2}$

$$\sqrt{x} + \sqrt{y} = \sqrt{a}$$

d.w.r. to x

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} \cdot y_1 = 0$$

$$\frac{1}{2\sqrt{y}} \cdot y_1 = -\frac{1}{2\sqrt{x}}$$

$$y_1 = -\frac{\sqrt{y}}{\sqrt{x}} \rightarrow (1)$$

d.w.r. to x

$$y_2 = - \left[\frac{\sqrt{x} \cdot \frac{1}{2\sqrt{y}} \cdot y_1 - \sqrt{y} \cdot \frac{1}{2\sqrt{x}}}{(\sqrt{x})^2} \right]$$

$$y_2 = - \left[\frac{\sqrt{x} \cdot \frac{1}{2\sqrt{y}} \left(-\frac{\sqrt{y}}{\sqrt{x}} \right) - \sqrt{y} \cdot \frac{1}{2\sqrt{x}}}{a} \right] = y_2 = - \left[\frac{-\frac{1}{2} - \frac{\sqrt{y}}{2\sqrt{x}}}{a} \right]$$

$$y' = \frac{1}{2} \left[\frac{1 + \frac{\sqrt{y}}{\sqrt{x}}}{\alpha} \right]$$

$$y' = \frac{1}{2} \left[\frac{\frac{\sqrt{x} + \sqrt{y}}{\sqrt{x}}}{\alpha} \right]$$

$$y' = \frac{1}{2} \left[\frac{\frac{\sqrt{a}}{\alpha \sqrt{x}}}{\alpha} \right]$$

$$\rho = \frac{\left(1 + \left(-\frac{\sqrt{y}}{\sqrt{x}} \right)^2 \right)^{3/2}}{\frac{1}{2} \left[\frac{\sqrt{a}}{\alpha \sqrt{x}} \right]}$$

$$= \frac{2 \left(1 + \frac{y}{x} \right)^{3/2} \alpha \sqrt{x}}{\sqrt{a}}$$

$$= \frac{2 \alpha \sqrt{x}}{\sqrt{a}} \left(\frac{x+y}{x} \right)^{3/2} = \frac{2 \alpha \sqrt{x}}{\sqrt{a}} \frac{(x+y)^{3/2}}{x^{3/2}}$$

$$\boxed{\rho = \frac{2}{\sqrt{a}} (x+y)^{3/2}}$$

7. Find the radius of curvature for the curve $x^{2/3} + y^{2/3} = a^{2/3}$

Sol:- $x^{2/3} + y^{2/3} = a^{2/3}$

d.w.r.t x

$$\frac{2}{3} x^{-1/3} + \frac{2}{3} y^{-1/3} \cdot y_1 = 0$$

$$\frac{2}{3} y^{-1/3} \cdot y_1 = -\frac{2}{3} x^{-1/3}$$

$$y_1 = \frac{-x^{-1/3}}{y^{-1/3}} \Rightarrow y_1 = -x^{-1/3} \cdot y^{1/3}$$

d.w.r.t x

$$y'' = \left[x^{-1/3} \cdot \frac{1}{3} y^{1/3-1} \cdot y_1 + y^{1/3} \cdot \left(-\frac{1}{3} \right) x^{-1/3-1} \right]$$

$$y'' = \frac{1}{3} \left[y^{1/3} \cdot x^{-4/3} - x^{-1/3} y^{-2/3} \left(-x^{-1/3} \cdot y^{1/3} \right) \right]$$

$$= \frac{1}{3} \left[y^{1/3} \cdot x^{-4/3} + x^{-1/3-1/3} \cdot y^{-2/3+1/3} \right]$$

$$= \frac{1}{3} \left[y^{1/3} \cdot x^{-4/3} + x^{-2/3} \cdot y^{-1/3} \right]$$

$$= \frac{1}{3} x^{-4/3} \cdot y^{-1/3} (x^{2/3} + y^{2/3})$$

$$y_2 = \frac{1}{3} x^{-4/3} \cdot y^{-1/3} (a^{2/3})$$

$$P = \frac{(1 + (y_1)^2)^{3/2}}{y_2} = \frac{(1 + (-x^{-1/3} \cdot y^{1/3})^2)^{3/2}}{\frac{1}{3} \cdot x^{-4/3} \cdot y^{-1/3} \cdot a^{2/3}}$$

$$P = 3 \cdot x^{4/3} \cdot y^{1/3} \cdot a^{-2/3} (1 + x^{-2/3} \cdot y^{2/3})^{3/2}$$

$$= 3x^{4/3} \cdot y^{1/3} \cdot a^{-2/3} \left(1 + \frac{y^{2/3}}{x^{2/3}} \right)^{3/2}$$

$$= 3x^{4/3} \cdot y^{1/3} \cdot a^{-2/3} \left(\frac{x^{2/3} + y^{2/3}}{x^{2/3}} \right)^{3/2}$$

$$= 3x^{4/3} \cdot y^{1/3} \cdot a^{-2/3} \frac{(a^{2/3})^{3/2}}{(x^{2/3})^{3/2}}$$

$$= 3x^{4/3} \cdot y^{1/3} \cdot a^{-2/3} \cdot \frac{a}{x}$$

$$P = 3 \cdot x^{1/3} \cdot y^{1/3} \cdot a^{1/3}$$

$$P = 3(xya)^{1/3}$$

Cubing on both side

$$P^3 \propto xya$$

$$8. \quad y = \frac{ax}{a+x} \cdot PT \left(\frac{2P}{a} \right)^{2/3} = \left(\frac{x}{y} \right)^{2/3} + \left(\frac{y}{x} \right)^{2/3}.$$

Sol:- $y = \frac{ax}{a+x} \rightarrow (1)$

d. w. r. t x

$$y_1 = a \left[\frac{(a+x)(1) - x(1)}{(a+x)^2} \right]$$

$$y_1 = a \left[\frac{a+x-x}{(a+x)^2} \right]$$

$$y_1 = a \left[\frac{a}{(a+x)^2} \right] \Rightarrow y_1 = \left[\frac{a}{a+x} \right]^2 \rightarrow (2)$$

From (1) $\frac{a}{a+x} = \frac{y}{x}$

$$y_1 = \left(\frac{y}{x} \right)^2 = \frac{y^2}{x^2}$$

D eq-(2) w. r. t x

$$y_2 = a^2 \cdot \frac{(-2)}{(a+x)^3}$$

$$y_2 = \frac{-2}{(a+x)} \cdot \left(\frac{a}{a+x} \right)^2$$

$$y_2 = \frac{-2}{(a+x)} \cdot \frac{y^2}{x^2} \Rightarrow y_2 = -2 \frac{y}{ax} \cdot \frac{y^2}{x^2} = \frac{-2y^3}{ax^3}$$

$$y_2 = \frac{-2y^3}{ax^3}$$

$$P = \frac{(1+y_1)^{3/2}}{y_2}$$

$$P = \frac{\left(1 + \left(\frac{y^2}{x^2} \right)^2 \right)^{3/2}}{\frac{-2y^3}{ax^3}}$$

$$P = \left(\frac{x^4 + y^4}{x^4} \right)^{3/2} \times \frac{ax^3}{-2y^3}$$

$$\frac{-2P}{a} = \left(\frac{x^4 + y^4}{x^4} \right)^{3/2} \times \frac{x^3}{y^3}$$

$$\frac{-2P}{a} = \frac{(x^4 + y^4)^{3/2}}{(x^4)^{3/2}} \times \frac{x^3}{y^3}$$

$$= \frac{(x^4 + y^4)^{3/2}}{x^3 \cdot y^3} \Rightarrow \frac{-2P}{a} = \left(\frac{x^4 + y^4}{x^3 \cdot y^3} \right)^{3/2}$$

$$\left(\frac{-2P}{a} \right)^{2/3} = \frac{x^4 + y^4}{x^3 y^3} = \frac{x^4 + y^4}{(xy)^3}$$

$$= \frac{x^4}{x^3 y^3} + \frac{y^4}{x^3 y^3}$$

$$\left(\frac{-2P}{a} \right)^{2/3} = \frac{x}{y^3} + \frac{y}{x^3}$$

$$\left(\frac{2P}{a} \right)^{2/3} = \left(\frac{x}{y} \right)^3 + \left(\frac{y}{x} \right)^3$$

9. Find the radius of curvature of the curve $x^3 y = a(x^2 + y^2)$

$(-2a, 2a)$

Sol:- $x^3 y = a(x^2 + y^2)$
d.w. w.r.t x

$$x^3 y_1 + 2xy_1 = a[2x + 2yy_1]$$

$$x^3 y_1 + 2xy_1 = 2ax + 2ayy_1$$

$$x^3 y_1 - 2ayy_1 = 2ax - 2xy_1$$

$$y_1 (x^3 - 2ay) = 2x(a - y)$$

$$y_1 = \frac{2x(a - y)}{(x^3 - 2ay)}$$

at $(-3a, 3a)$

$$y_1 = \frac{3(-3a)(a-(3a))}{((-3a)^2 - 3a(3a))} = \frac{-4a(-a)}{4a^2 - 4a^2}$$

$$\Rightarrow y_1 = \infty$$

Since, $y_1 = \infty$, P becomes ∞

so, WKT,
$$\rho = \frac{(1+x_1^2)^{3/2}}{x_2}$$

$$x_1^2 y = a(x^2 + y^2)$$

d.w.r.t y

$$x^2(1) + 3x \cdot x_1 y = a(2xx_1 + 2y)$$

$$x^2 + 3xx_1 y = 2axx_1 + 2ay$$

$$3axx_1 - 2xx_1 y = x^2 - 2ay$$

$$x_1(3ax - 2xy) = x^2 - 2ay$$

$$x_1 = \frac{x^2 - 2ay}{(3ax - 2xy)} \rightarrow (1) \text{ at } (-3a, 3a)$$

$$x_1 = \frac{(-3a)^2 - 2a(3a)}{(3a(-3a) - 3(-3a)(3a))} = \frac{4a^2 - 4a^2}{-4a^2 - 8a^2} = 0$$

$$\therefore x_1 = 0$$

d. eq-(1) w.r.t y

$$x_2 = \frac{(3ax - 2xy)(2xx_1 - 2a) - (x^2 - 2ay)(3ax_1 - 3x - 3x_1 y)}{(3ax - 2xy)^2}$$

at $(-3a, 3a)$ & $x_1 = 0$

$$x_2 = \frac{4a^2(-3a) - 0}{(3a(3a) - 3(-3a)(3a))^2} = \frac{-8a^3}{(-4a^2 + 8a^2)^2}$$

$$x_2 = \frac{-8a^3}{16a^4} = \frac{-1}{2a}$$

$$x_1 = 0 \text{ \& } x_2 = \frac{-1}{2a}$$

$$P = \frac{(1+x_1^2)^{3/2}}{x_2} = \frac{(1+0)^{3/2}}{-1/2a} = -2a$$

$$|P| = |-2a|$$

$$\boxed{P = 2a}$$

10. Find the radius of curvature of the curve $y^2 = \frac{4a^2(2a-x)}{x}$ where the curve meets X-axis.

Sol:- $y^2 = \frac{4a^2(2a-x)}{x} \rightarrow (1)$

Given curve meets X-axis $\Rightarrow y=0$

$$\Rightarrow 0 = \frac{4a^2(2a-x)}{x}$$

$$\Rightarrow 2a-x=0$$

$$x=2a$$

$$\therefore (2a, 0)$$

D. w. r. t x eq. (1)

$$2y \cdot y_1 = 4a^2 \left[\frac{x(0-1) - (2a-x)(1)}{x^2} \right]$$

$$2y \cdot y_1 = 4a^2 \left[\frac{-x - (2a-x)}{x^2} \right]$$

$$y_1 = \frac{2a^2}{2y} \left[\frac{-x - 2a + x}{x^2} \right]$$

at $(2a, 0)$

$$y_1 = \frac{2a^2}{0} \left[\frac{-2a - 2a + 2a}{(2a)^2} \right]$$

$$y_1 = \infty$$

Since, $y_1 = \infty$ --- P becomes ∞

Rewrite P in alternate formula.

$$P = \frac{(1 + \alpha_1^2)^{3/2}}{\alpha_2}, \quad \alpha_1 = \frac{dx}{dy}, \quad \alpha_2 = \frac{d^2x}{dy^2}$$

$$y^2 = \frac{4a^2(2a - x)}{x}$$

$$y^2 x = 8a^3 - 4a^2 x$$

$$y^2 x + 4a^2 x = 8a^3$$

$$x(y^2 + 4a^2) = 8a^3$$

$$x = \frac{8a^3}{y^2 + 4a^2}$$

d.w.r.t. y

$$\alpha_1 = 8a^3 \cdot \frac{-1}{(y^2 + 4a^2)^2} (2y)$$

$$\text{at } (2a, 0) \quad \alpha_1 = \frac{-16a^3 \cdot 0}{(0 + 4a^2)^2} = 0$$

$$\alpha_1 = 0$$

$$\alpha_2 = -16a^3 \left[\frac{\frac{d.w.r.t. \text{ to } y}{(y^2 + 4a^2)^2 (1)} - y \cdot 2(y^2 + 4a^2)(2y)}{((y^2 + 4a^2)^2)^2} \right]$$

at (2a, 0)

$$\alpha_2 = -16a^3 \left[\frac{(0 + 4a^2)^2 - 0}{(0 + 4a^2)^4} \right] = -16a^3 \left[\frac{16a^4}{16 \times 16 \times a^8} \right]$$

$$\alpha_2 = -\frac{1}{a}$$

$$P = \frac{(1 + \alpha_1^2)^{3/2}}{\alpha_2}$$

$$= \frac{(1 + 0)^{3/2}}{-\frac{1}{a}} \quad P = -a$$

$$\boxed{P = a}$$

Derive Radius of curvature in cartesian form:-

15

Note:- Derivative of arc length in cartesian form,

If the equation of the curve in the form $y = f(x)$ then derivative of arc length 's' with respect to x is given by

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}, \quad \frac{dx}{ds} = \cos \psi, \quad \frac{dy}{ds} = \sin \psi$$

Radius of curvature in cartesian form:-

Let the equation of the curve be $y = f(x)$ & 'A' be a fixed point on it. Let $P(x, y)$ is any point on the curve such that $\widehat{AP} = s$.

Let ψ be the angle made by the tangent at 'P' with X-axis.

WKT, $\tan \psi = \frac{dy}{dx} = y_1$

d.w.r.t.s

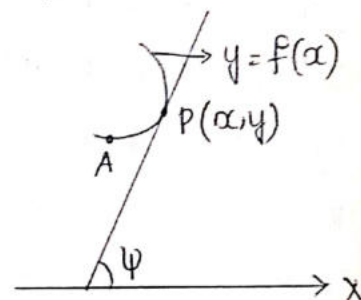
$$\sec^2 \psi \cdot \frac{d\psi}{ds} = \frac{d}{ds} \left(\frac{dy}{dx} \right) = \frac{d}{dx} \left(\frac{dy}{dx} \right) \left(\frac{dx}{ds} \right)$$

$$(1 + \tan^2 \psi) \frac{d\psi}{ds} = \frac{d^2 y}{dx^2} \cdot \frac{dx}{ds}$$

$$(1 + y_1^2) \frac{d\psi}{ds} = y_2 \cdot \frac{1}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}} = \frac{y_2}{\sqrt{1 + y_1^2}} = \frac{y_2}{\sqrt{1 + y_1^2}}$$

$$\frac{d\psi}{ds} = \frac{y_2}{\sqrt{1 + y_1^2}}$$

$$\frac{1}{\rho} = \frac{y_2}{(1 + y_1^2)^{3/2}} \Rightarrow \boxed{\rho = \frac{(1 + y_1^2)^{3/2}}{y_2}}$$



Radius of curvature in polar form:-

Note:- Derivative of arc length in polar form.

$$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{r^2 + r_1^2}$$

Prove that radius of curvature in polar form,

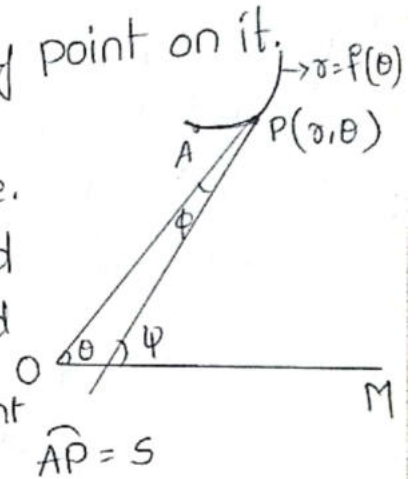
$$\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2}$$

Proof:-

Let $r = f(\theta)$ be any polar curve & $P(r, \theta)$ be any point on it.

Let $OP = r$ be the radius vector. Draw a tangent at P which makes an angle ψ with the initial line.

Let θ & ϕ be angle between radius vector and initial line also angle between radius vector and tangent respectively. Let 'A' be the fixed point



WKT, $\psi = \phi + \theta$

d.w.r.t. s

$$\frac{d\psi}{ds} = \frac{d\phi}{ds} + \frac{d\theta}{ds} = \frac{d\phi}{d\theta} \cdot \frac{d\theta}{ds} + \frac{d\theta}{ds}$$

$$\frac{d\psi}{ds} = \frac{d\theta}{ds} \left(1 + \frac{d\phi}{d\theta}\right)$$

$$\frac{1}{\rho} = \frac{d\theta}{ds} \left(1 + \frac{d\phi}{d\theta}\right)$$

$$\boxed{\rho = \frac{ds/d\theta}{1 + \frac{d\phi}{d\theta}}} \longrightarrow (1)$$

$$\text{WKT } \tan\phi = r \cdot \frac{d\theta}{dr} = \frac{r}{\frac{dr}{d\theta}}$$

$$\tan\phi = \frac{r}{r_1}$$

d.w.r.t. θ

$$\sec^2\phi \cdot \frac{d\phi}{d\theta} = \frac{r_1 \frac{dr}{d\theta} - r \cdot \frac{d}{d\theta}(r_1)}{r_1^2}$$

$$(1 + \tan^2 \phi) \cdot \frac{d\phi}{d\theta} = \frac{r_1^2 - r r_2}{r_1^2}$$

16

$$\frac{d\phi}{d\theta} = \frac{r_1^2 - r r_2}{r_1^2 (1 + \tan^2 \phi)}$$

$$\frac{d\phi}{d\theta} = \frac{r_1^2 - r r_2}{r_1^2 \left(1 + \frac{r^2}{r_1^2}\right)} = \frac{r_1^2 - r r_2}{r_1^2 \left(\frac{r_1^2 + r^2}{r_1^2}\right)}$$

add 1 both sides

$$1 + \frac{d\phi}{d\theta} = 1 + \frac{r_1^2 - r r_2}{r_1^2 + r^2}$$

$$1 + \frac{d\phi}{d\theta} = \frac{r_1^2 + r^2 + r_1^2 - r r_2}{r_1^2 + r^2}$$

$$1 + \frac{d\phi}{d\theta} = \frac{2r_1^2 + r^2 - r r_2}{r_1^2 + r^2}$$

$$1 + \frac{d\phi}{d\theta} = \frac{r^2 + 2r_1^2 - r r_2}{r_1^2 + r^2}$$

$\therefore$ (1) - becomes

$$\rho = \frac{\sqrt{r^2 + r_1^2}}{\frac{r^2 + 2r_1^2 - r r_2}{r_1^2 + r^2}} = \frac{(r^2 + r_1^2)^{1/2} \times (r_1^2 + r^2)}{r^2 + 2r_1^2 - r r_2}$$

$$\Rightarrow \boxed{\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - r r_2}}$$

1. PT $\frac{p^2}{r}$ is constant for the cardioid $r = a(1 + \cos\theta)$ where p is the radius of curvature.

Sol:-
$$p = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2}$$

$$r = a(1 + \cos\theta) \rightarrow (1)$$

d.w.r.t. θ

$$r_1 = \frac{dr}{d\theta} = a(-\sin\theta) \rightarrow (2)$$

d(2) w.r.t. θ

$$\frac{dr_1}{d\theta} = r_2 = a(-\cos\theta)$$

$$p = \frac{(r^2 + (-a\sin\theta)^2)^{3/2}}{r^2 + 2(-a\sin\theta)^2 - r(-a\cos\theta)}$$

$$= \frac{(a^2(1 + \cos\theta)^2 + a^2\sin^2\theta)^{3/2}}{a^2(1 + \cos\theta)^2 + 2a^2\sin^2\theta + a(1 + \cos\theta)(a\cos\theta)}$$

$$= \frac{(a^2)^{3/2} (1 + \cos^2\theta + \sin^2\theta)^{3/2}}{a^2(1 + \cos^2\theta + 2\sin^2\theta + \cos\theta + \cos^2\theta)}$$

$$= \frac{(a^2)^{3/2} (1 + \cos^2\theta + \sin^2\theta)^{3/2}}{a^2(1 + \cos^2\theta + 2\sin^2\theta + \cos\theta + \cos^2\theta)}$$

$$= \frac{a^3 (1 + \cos^2\theta + \sin^2\theta)^{3/2}}{a^2(1 + \cos^2\theta + 2\sin^2\theta + \cos\theta + \cos^2\theta)}$$

$$= \frac{a^3 (1 + 1 + \cos^2\theta + \sin^2\theta)^{3/2}}{a^2(1 + \cos^2\theta + 2\sin^2\theta + \cos\theta + \cos^2\theta)}$$

$$= \frac{a^3 (2 + \cos^2\theta + \sin^2\theta)^{3/2}}{a^2(1 + \cos^2\theta + 2\sin^2\theta + \cos\theta + \cos^2\theta)}$$

$$= \frac{a^3 (2 + \cos^2\theta + \sin^2\theta)^{3/2}}{a^2(1 + \cos^2\theta + 2\sin^2\theta + \cos\theta + \cos^2\theta)}$$

$$= \frac{a^3 (2 + \cos^2\theta + \sin^2\theta)^{3/2}}{a^2(1 + \cos^2\theta + 2\sin^2\theta + \cos\theta + \cos^2\theta)}$$

$$= \frac{(2)^{3/2} a (1 + \cos\theta)^{3/2}}{(1 + 2 + 3\cos\theta)}$$

$$= \frac{(2)^{3/2} a (1 + \cos\theta)^{3/2}}{3(1 + \cos\theta)}$$

$$= \frac{a\sqrt{2}}{3} (1 + \cos\theta)^{1/2}$$

from eq - (1)

$$= \frac{a\sqrt{3}}{3} \left(\frac{r}{a}\right)^{1/3}$$

$$p = \frac{a\sqrt{3}}{3} \frac{\sqrt{r}}{\sqrt{a}}$$

$$p = \sqrt{a} \cdot \sqrt{a} \cdot \frac{\sqrt{3}}{3} \cdot \frac{\sqrt{r}}{\sqrt{a}}$$

$$p = \frac{\sqrt{a} \sqrt{3}}{3} \cdot \sqrt{r}$$

Squaring o.b.s

$$p^2 = \frac{8a}{9} r$$

$$\frac{p^2}{r} = \frac{8a}{9}$$

$$\frac{p^2}{r} = \text{constant}$$

$$p^2 \propto r$$

2. PT for the curve $r(1-\cos\theta) = 2a$, p^2 varies as r^3 .

Sol:-
$$p = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2}$$

$$r(1-\cos\theta) = 2a \quad \text{--- (1)}$$

d.w.r.t. θ

$$r(1-\cos\theta) = 2a$$

$$r(0 - (-\sin\theta)) + (1-\cos\theta)r_1 = 0$$

$$r(\sin\theta) + (1-\cos\theta)r_1 = 0$$

$$(1-\cos\theta)r_1 = -r\sin\theta \rightarrow (2)$$

d.w.r.t. θ (2)

$$(1-\cos\theta)r_2 + (0 - (-\sin\theta))r_1 = -[r(\cos\theta) + \sin\theta r_1]$$

$$(1-\cos\theta)r_2 + r_1 \sin\theta = -r\cos\theta - \sin\theta r_1$$

$$\frac{\partial a}{\partial r} \cdot r_2 = -r \cos \theta - r_1 \sin \theta - r_1 \sin \theta$$

from eq. (2)

$$(1 - \cos \theta) r_1 = -r \sin \theta$$

$$r_1 = \frac{-r \sin \theta}{1 - \cos \theta}$$

$$r_1 = \frac{-r \sin \theta}{\partial a / \partial r}$$

$$\boxed{r_1 = -\frac{r^2}{\partial a} \sin \theta}$$

$$r_2 = (-r \cos \theta - 2r_1 \sin \theta) \frac{r}{\partial a}$$

$$r_2 = (-r \cos \theta - 2 \sin \theta \left(-\frac{r^2}{\partial a} \right) \sin \theta) \frac{r}{\partial a}$$

$$r_2 = -\frac{r^3}{\partial a} \cos \theta + \frac{r^3}{\partial a^2} \sin^2 \theta$$

$$P = \left(r^2 + \left(-\frac{r^3}{\partial a} \sin \theta \right)^2 \right)^{3/2}$$

$$\frac{r^2 + 2 \left(-\frac{r^3}{\partial a} \sin \theta \right)^2 - r \left(-\frac{r^3}{\partial a} \cos \theta + \frac{r^3}{\partial a^2} \sin^2 \theta \right)}{\quad \times \quad}$$

$$2. r(1 - \cos \theta) = \partial a$$

$$\log r + \log(1 - \cos \theta) = \log \partial a$$

d.w.r.to θ

$$\frac{1}{r} \cdot \frac{dr}{d\theta} + \frac{\sin \theta}{1 - \cos \theta} = 0$$

$$\frac{1}{r} \cdot \frac{dr}{d\theta} + \frac{2 \sin \theta \cdot \cos \theta}{2 \sin^2 \theta} = 0$$

$$\frac{1}{r} \cdot \frac{dr}{d\theta} = -\cot \theta$$

$$\frac{1}{r} \cdot r_1 = -\cot \theta$$

$$r_1 = -r \cot \theta \quad \text{d.w.r.to } \theta$$

$$r_2 = - \left[r (-\operatorname{cosec}^2 \frac{\theta}{2}) \frac{1}{2} + \cot(\frac{\theta}{2}) r_1 \right]$$

$$= \frac{r}{2} \operatorname{cosec}^2(\frac{\theta}{2}) - \cot(\frac{\theta}{2}) (-r \cot \frac{\theta}{2})$$

$$r_2 = \frac{r}{2} \operatorname{cosec}^2(\frac{\theta}{2}) + r \cdot \cot^2(\frac{\theta}{2})$$

$$P = \frac{(r_1 + r_2)^{3/2}}{r_1 + 2r_2 - r r_2}$$

$$= \frac{(r_1 + (-r \cot \frac{\theta}{2})^2)^{3/2}}{r_1 + 2(-r \cot \frac{\theta}{2})^2 - r(r \cot^2 \frac{\theta}{2})}$$

$$= \frac{(r_1 + r^2 \cot^2 \frac{\theta}{2})^{3/2}}{r_1 + 2r^2 \cot^2 \frac{\theta}{2} - r^2 (\frac{1}{2} \operatorname{cosec}^2 \frac{\theta}{2} + \cot^2 \frac{\theta}{2})}$$

$$= \frac{(r_1 + r^2 \cot^2 \frac{\theta}{2})^{3/2}}{r_1 + 2r^2 \cot^2 \frac{\theta}{2} - r^2 (\frac{1}{2} \operatorname{cosec}^2 \frac{\theta}{2} + \cot^2 \frac{\theta}{2})}$$

$$= \frac{(r^2)^{3/2} (1 + \cot^2 \frac{\theta}{2})^{3/2}}{r^2 (1 + 2\cot^2 \frac{\theta}{2} - \frac{1}{2} \operatorname{cosec}^2 \frac{\theta}{2} - \cot^2 \frac{\theta}{2})}$$

$$= \frac{(r^2)^{3/2} (1 + \cot^2 \frac{\theta}{2})^{3/2}}{r^2 (1 + 2\cot^2 \frac{\theta}{2} - \frac{1}{2} \operatorname{cosec}^2 \frac{\theta}{2} - \cot^2 \frac{\theta}{2})}$$

$$= \frac{r (\operatorname{cosec}^2 \frac{\theta}{2})^{3/2}}{1 + \cot^2(\frac{\theta}{2}) - \frac{1}{2} \operatorname{cosec}^2 \frac{\theta}{2}}$$

$$= \frac{r \operatorname{cosec}^3 \frac{\theta}{2}}{\operatorname{cosec}^2 \frac{\theta}{2} - \frac{1}{2} \operatorname{cosec}^2 \frac{\theta}{2}}$$

$$= \frac{r \operatorname{cosec}^3(\frac{\theta}{2})}{\operatorname{cosec}^2 \frac{\theta}{2} (1 - \frac{1}{2})}$$

$$= \frac{r \operatorname{cosec}(\frac{\theta}{2})}{\frac{1}{2}} = 2r \operatorname{cosec}(\frac{\theta}{2})$$

WKT, $r(1 - \cos \theta) = 2a$

$$r \cdot 2 \sin^2 \frac{\theta}{2} = 2a$$

$$\sin^2 \frac{\theta}{2} = \frac{a}{r}$$

$$\sin \frac{\theta}{2} = \sqrt{\frac{a}{r}}$$

$$\cos \sec \left(\frac{\theta}{2} \right) = \sqrt{\frac{r}{a}}$$

$$P = 2r \frac{\sqrt{r}}{\sqrt{a}}$$

Squaring o.b.s

$$P^2 = 4r^2 \cdot \frac{r}{a}$$

$$P^2 = \left(\frac{4}{a} \right) r^3$$

$$P^2 = C r^3$$

$$\boxed{P^2 \propto r^3}$$

where $\frac{4}{a} = \text{constant}$

3. Find P for the curve $r = a e^{\theta \cot \alpha}$

Sol:- $r = a e^{\theta \cot \alpha} \rightarrow (1)$

Take log

$$\log r = \log a + \log e^{\theta \cot \alpha}$$

d.w.r.t θ

$$\frac{1}{r} \cdot \frac{dr}{d\theta} = 0 + \cot \alpha (1)$$

$$r_1 = r \cot \alpha$$

d.w.r.t θ

$$r_2 = \cot \alpha \cdot r_1$$

$$= \cot \alpha (r \cot \alpha) = r \cot^2 \alpha$$

$$P = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - r r_2}$$

$$= \frac{(r^2 + (r \cot \alpha)^2)^{3/2}}{r^2 + 2(r \cot \alpha)^2 - r(r \cot^2 \alpha)}$$

$$= \frac{(r^2 + r^2 \cot^2 \alpha)^{3/2}}{r^2 + 2r^2 \cot^2 \alpha - r^2 \cot^2 \alpha}$$

$$= \frac{(r^2 (1 + \cot^2 \alpha))^{3/2}}{r^2 (1 + 2\cot^2 \alpha - \cot^2 \alpha)}$$

$$= \frac{(r^2)^{3/2} (1 + \cot^2 \alpha)^{3/2}}{r^2 (1 + \cot^2 \alpha)}$$

$$= \frac{(r^2)^{3/2} (1 + \cot^2 \alpha)^{3/2}}{r^2 (1 + \cot^2 \alpha)}$$

$$= \frac{(r^2)^{3/2} (1 + \cot^2 \alpha)^{3/2}}{r^2 (1 + \cot^2 \alpha)}$$

$$\frac{r(\operatorname{cosec}^3 \alpha)^{3/2}}{(1+\cot^2 \alpha)} = \frac{r \operatorname{cosec}^3 \alpha}{\operatorname{cosec}^2 \alpha} = \boxed{p = r \operatorname{cosec} \alpha}$$

19

4. Find p of the curve $r^n = a^n \cos n\theta$

Sol:- $r^n = a^n \cos n\theta \rightarrow (1)$

$$n \log r = n \log a + \log \cos n\theta$$

d.w.r.t. θ

$$n \cdot \frac{1}{r} \cdot r_1 = 0 + \frac{(-\sin n\theta) n}{\cos n\theta}$$

$$\frac{1}{r} \cdot r_1 = -\frac{\sin n\theta}{\cos n\theta}$$

$$r_1 = -r \cdot \tan n\theta$$

$$\boxed{r_1 = -r \cdot \tan n\theta}$$

d.w.r.t. θ

$$r_2 = -[r_1 \cdot \tan n\theta + r \cdot \sec^2 n\theta \cdot n]$$

$$= -[(-r \cdot \tan n\theta) \cdot \tan n\theta + nr \sec^2 n\theta]$$

$$= -[nr \sec^2 n\theta - r \tan^2 n\theta]$$

$$= -nr \sec^2 n\theta + r \tan^2 n\theta$$

$$p = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2rr_1 - rr_2} = \frac{(r^2 + (-r \cdot \tan n\theta)^2)^{3/2}}{r^2 + 2r(-r \cdot \tan n\theta) - r(-nr \sec^2 n\theta + r \tan^2 n\theta)}$$

$$= \frac{(r^2 + r^2 \tan^2 n\theta)^{3/2}}{r^2 + 2r^2 \tan^2 n\theta + nr^2 \sec^2 n\theta - r^2 \tan^2 n\theta}$$

$$= \frac{(r^2)^{3/2} (1 + \tan^2 n\theta)^{3/2}}{r^2 (1 + 2 \tan^2 n\theta + n \sec^2 n\theta - \tan^2 n\theta)}$$

$$= \frac{r (\sec^n \theta)^{3/2}}{(1 + \tan^n \theta + n \sec^n \theta)} = \frac{r \sec^n \theta}{(\sec^n \theta + n \sec^n \theta)} = \frac{r \sec^n \theta}{\sec^n \theta (1+n)}$$

$$p = \frac{r \sec^n \theta}{(1+n)}$$

Given, $r^n = a^n \cos n\theta$

$$\cos n\theta = \frac{a^n}{r^n}$$

$$\frac{1}{\cos n\theta} = \frac{a^n}{r^n}$$

$$\sec n\theta = \frac{a^n}{r^n} \Rightarrow p = \frac{r}{(1+n)} \cdot \frac{a^n}{r^n}$$

$$p = c \cdot \frac{1}{r^{n-1}} \quad \text{where } c = \frac{a^n}{(1+n)}$$

$$p \propto \frac{1}{r^{n-1}}$$

5. $r^n = a \sin n\theta$

Sol:- $r^n = a \sin n\theta \rightarrow (1)$

$$n \log r = \log a + \log \sin n\theta$$

d.w. r.t. θ

$$n \cdot \frac{1}{r} \cdot \frac{dr}{d\theta} = 0 + \frac{\cos n\theta}{\sin n\theta} \cdot r$$

$$\boxed{r_1 = r \cdot \cot n\theta}$$

d.w. r.t. θ

$$r_2 = (r \cdot (-\operatorname{cosec}^2 n\theta) + r_1 \cot n\theta)$$

$$r_2 = (-n r \operatorname{cosec}^2 n\theta + (r \cdot \cot n\theta) (\cot n\theta))$$

$$r_2 = (r \cdot \cot^2 n\theta - n r \operatorname{cosec}^2 n\theta)$$

$$p = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - r r_2} = \frac{(r^2 + (r \cot n\theta)^2)^{3/2}}{r^2 + 2(r \cot n\theta)^2 - r(r \cot^2 n\theta - n r \operatorname{cosec}^2 n\theta)}$$

$$\begin{aligned}
 &= \frac{(r^n + r^n \cot^n \theta)^{3/2}}{r^n + 2r^n \cot^n \theta - r^n \cot^n \theta + nr^n \operatorname{cosec}^2 \theta} \\
 &= \frac{(r^n)^{3/2} (1 + \cot^n \theta)^{3/2}}{r^n (1 + 2\cot^n \theta - \cot^n \theta + n \operatorname{cosec}^2 \theta)} \\
 &= \frac{r (\operatorname{cosec}^2 \theta)^{3/2}}{(1 + \cot^n \theta + n \operatorname{cosec}^2 \theta)} = \frac{r \operatorname{cosec}^3 \theta}{(\operatorname{cosec}^2 \theta + n \operatorname{cosec}^2 \theta)} \\
 &= \frac{r \operatorname{cosec}^3 \theta}{\operatorname{cosec}^2 \theta (1+n)} \\
 \rho &= \frac{r \operatorname{cosec} \theta}{(n+1)}
 \end{aligned}$$

$$\Rightarrow r^n = a \sin^n \theta$$

$$\sin^n \theta = \frac{r^n}{a}$$

$$\frac{1}{\sin^n \theta} = \frac{a}{r^n}$$

$$\operatorname{cosec}^n \theta = \frac{a}{r^n}$$

$$\rho = \frac{r}{(1+n)} \cdot \frac{a}{r^n}$$

$$\rho = C \cdot \frac{1}{r^{n-1}}$$

$$\boxed{\rho \propto \frac{1}{r^{n-1}}}$$

$$\text{where, } C = \frac{a}{(1+n)}$$

$$6. r^n = a^n \sin n\theta$$

Take log on both sides

$$n \log r = n \log a + \log \sin n\theta$$

d.w.r.t. θ

$$n \cdot \frac{1}{r} \cdot \frac{dr}{d\theta} = 0 + \frac{\cos n\theta \cdot n}{\sin n\theta}$$

$$r \cdot r_1 = r \cot n\theta \cdot r$$

$$\boxed{r_1 = r \cot n\theta}$$

d.w.r.t. θ

$$r_2 = (r(-\operatorname{cosec} n\theta) \cdot n + r_1 \cot n\theta)$$

$$= (-nr \operatorname{cosec} n\theta + (r \cot n\theta) \cot n\theta)$$

$$r_2 = (r \cot^2 n\theta - nr \operatorname{cosec} n\theta)$$

$$\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2} = \frac{(r^2 + (r \cot n\theta)^2)^{3/2}}{r^2 + 2(r \cot n\theta)^2 - r(r \cot^2 n\theta - nr \operatorname{cosec} n\theta)}$$

$$= \frac{(r^2 + r^2 \cot^2 n\theta)^{3/2}}{r^2 + 2r^2 \cot^2 n\theta - r^2 \cot^2 n\theta + nr^2 \operatorname{cosec} n\theta}$$

$$= \frac{(r^2)^{3/2} (1 + \cot^2 n\theta)^{3/2}}{r^2 (1 + 2\cot^2 n\theta - \cot^2 n\theta + n \operatorname{cosec} n\theta)}$$

$$= \frac{r (\operatorname{cosec} n\theta)^{3/2}}{(1 + \cot^2 n\theta + n \operatorname{cosec} n\theta)}$$

$$= \frac{r \operatorname{cosec}^3 n\theta}{(\operatorname{cosec}^2 n\theta + n \operatorname{cosec} n\theta)} = \frac{r \operatorname{cosec}^2 n\theta}{\operatorname{cosec} n\theta (1+n)}$$

$$\rho = \frac{r \operatorname{cosec} n\theta}{(1+n)}$$

Given, $r^n = a^n \sin n\theta$

21

$$\sin n\theta = \frac{r^n}{a^n}$$

$$\frac{1}{\sin n\theta} = \frac{a^n}{r^n}$$

$$\operatorname{cosec} n\theta = \frac{a^n}{r^n}$$

$$p = \frac{r}{(1+n)} \cdot \frac{a^n}{r^n} \Rightarrow p = \frac{1}{r^{n-1}} C \text{ where } C = \frac{a^n}{(1+n)}$$

$$p \propto \frac{1}{r^{n-1}}$$

Radius of Curvature in Pedal form:-

Radius of curvature of in pedal form is given by,

$$\rho = r \cdot \frac{dr}{dp}$$

1. Find the pedal form of radius of curvature. $r(1+\cos\theta) = a$

Sol:- $r(1+\cos\theta) = a \rightarrow (1)$

Take log o.b.s

$$\log r + \log(1+\cos\theta) = \log a$$

d.w.r.t. θ

$$\frac{1}{r} \cdot \frac{dr}{d\theta} + \frac{0 + (-\sin\theta)}{1+\cos\theta} = 0$$

$$\frac{1}{r} \cdot \frac{dr}{d\theta} - \frac{\cancel{r} \sin \theta/2 \cdot \cos \theta/2}{\cancel{r} \cos^2 \theta/2} = 0$$

$$\cot \phi - \tan \theta/2 = 0$$

$$\cot \phi = \tan \theta/2$$

$$\cot \phi = \cot \left(\frac{\pi}{2} - \theta/2 \right)$$

$$\phi = \frac{\pi}{2} - \frac{\theta}{2}$$

$$P = r \sin \phi$$

$$P = r \sin \left(\frac{\pi}{2} - \frac{\theta}{2} \right)$$

$$P = r \cos \left(\frac{\theta}{2} \right)$$

from eq-(1)

$$r(1 + \cos \theta) = a$$

$$r \cdot 2 \cos^2 \frac{\theta}{2} = a$$

$$\cos^2 \frac{\theta}{2} = \frac{a}{2r}$$

$$\cos \frac{\theta}{2} = \sqrt{\frac{a}{2r}}$$

$$P = r \cdot \frac{\sqrt{a}}{\sqrt{2} \cdot \sqrt{r}}$$

$$P = \frac{\sqrt{r} \cdot \sqrt{a}}{\sqrt{2}}$$

$$P = \frac{\sqrt{a}}{\sqrt{2}} \cdot \sqrt{r}$$

d.w.r.to r

$$\frac{dP}{dr} = \frac{\sqrt{a}}{\sqrt{2}} \cdot \frac{1}{2\sqrt{r}}$$

WKT Radius of curvature in pedal form is given by

$$\rho = r \cdot \frac{dr}{dp}$$

$$\rho = \frac{r \cdot 2\sqrt{2} \cdot \sqrt{r}}{\sqrt{a}}$$

$$\rho = C \cdot r\sqrt{r}$$

squaring obs

$$\rho^2 = C^2 \cdot r^3$$

$$\boxed{\rho^2 \propto r^3}$$

$$\text{where } C = \frac{2\sqrt{2}}{\sqrt{a}} (\text{constant})$$

Q. PT P is $\frac{a^3}{3r}$ for the curve $r^3 \sec 2\theta = a^3$ in pedal form. 22

Sol:- $r^3 \sec 2\theta = a^3 \rightarrow (1)$

$$3 \log r + \log \sec 2\theta = 3 \log a$$

d.w.r.t. θ

$$3 \cdot \frac{1}{r} \cdot \frac{dr}{d\theta} + \frac{(\sec 2\theta \cdot \tan 2\theta) \cdot 2}{\sec^2 2\theta} = 0$$

$$3 \cot \phi + \tan 2\theta \cdot 2 = 0$$

$$3 \cot \phi = -\tan 2\theta \cdot 2$$

$$\cot \phi = \cot \left(\frac{\pi}{2} + 2\theta \right)$$

$$\boxed{\phi = \frac{\pi}{2} + 2\theta}$$

$$P = r \sin \phi$$

$$P = r \sin \left(\frac{\pi}{2} + 2\theta \right)$$

$$P = r \cos 2\theta$$

from eq-(1) $r^3 \sec 2\theta = a^3$

$$r^3 \cdot \frac{1}{\cos 2\theta} = a^3$$

$$\cos 2\theta = \frac{r^3}{a^3}$$

$$P = r \cdot \frac{r^3}{a^3} = P = \frac{r^3}{a^3}$$

d.w.r.t. 'r'

$$\frac{dP}{dr} = \frac{1}{a^3} \cdot 3r^2$$

$$P = r \cdot \frac{dr}{dP}$$

$$P = r \cdot \frac{a^3}{3r^2} \Rightarrow \boxed{P = \frac{a^3}{3r}}$$

3. Find the radius of curvature for the curve $r = ae^{\theta \cot \alpha}$ in pedal form.

Sol:- $r = ae^{\theta \cot \alpha} \rightarrow (1)$

$$\log r = \log a + \log e^{\theta \cot \alpha}$$

$$\log r = \log a + \theta \cot \alpha \cdot \log e. \quad (\because \log e = 1)$$

d.w.r.t. θ

$$\frac{1}{r} \cdot \frac{dr}{d\theta} = 0 + \cot \alpha (1).$$

$$\cot \phi = \cot \alpha$$

$$\phi = \alpha$$

$$P = r \sin \phi$$

$$P = r \sin \alpha$$

d.w.r.t. r

$$\frac{dP}{dr} = \sin \alpha \cdot 1$$

$$\frac{dr}{dP} = \frac{1}{\sin \alpha}$$

$$P = r \cdot \frac{dr}{dP}$$

$$P = r \cdot \frac{1}{\sin \alpha} \Rightarrow P = r \cdot C$$

$$\boxed{P \propto r}$$

4. Find the radius of curvature for the curve $r = a(1 - \cos \theta)$ in pedal form.

Sol:- $r = a(1 - \cos \theta) \rightarrow (1)$

$$\log r = \log a + \log (1 - \cos \theta)$$

d.w.r.t. θ

$$\frac{1}{r} \cdot \frac{dr}{d\theta} = 0 + \frac{0 - (-\sin \theta)}{(1 - \cos \theta)}$$

$$\cot \phi = \frac{2 \sin \theta / 2 \cos \theta / 2}{2 \sin^2 \theta / 2}$$

$$\cot \phi = \cot \theta/2$$

$$\boxed{\phi = \theta/2}$$

$$p = r \sin \phi$$

$$p = r \sin \theta/2$$

$$\text{from eq. (1), } r = a(1 - \cos \theta)$$

$$r = a \cdot 2 \sin^2 \theta/2$$

$$\sin \theta/2 = \frac{\sqrt{r}}{\sqrt{2a}}$$

$$p = r \cdot \frac{\sqrt{r}}{\sqrt{2a}}$$

d.w.r.t. r

$$\frac{dp}{dr} = \frac{1}{\sqrt{2a}} \cdot \frac{1}{2\sqrt{r}}$$

$$\frac{dr}{dp} = 2\sqrt{r} \cdot \sqrt{a} \cdot \sqrt{r}$$

$$p = r \cdot \frac{dr}{dp}$$

$$p = r \cdot 2\sqrt{r} \cdot \sqrt{a} \cdot \sqrt{r}$$

sq. o.b.s

$$p^2 = r^2 \cdot 8ar$$

$$p^2 = 8ar^3 \Rightarrow p^2 = Cr^3$$

$$\boxed{p^2 \propto r^3}$$

$$5. r^n = a^n \cos n\theta$$

$$\text{Sol:- } r^n = a^n \cos n\theta \rightarrow (1)$$

$$n \log r = n \log a + \log \cos n\theta$$

d.w.r.t. θ

$$n \cdot \frac{1}{r} \cdot \frac{dr}{d\theta} = 0 + \frac{(-\sin n\theta)n}{\cos n\theta}$$

$$r' \cot \phi = -r \cdot \frac{\sin n\theta}{\cos n\theta}$$

$$\cot \phi = \cot \left(\frac{\pi}{2} + n\theta \right)$$

$$\phi = \frac{\pi}{2} + n\theta$$

$$P = r \sin \phi$$

$$P = r \sin \left(\frac{\pi}{2} + n\theta \right)$$

$$P = r \cos n\theta$$

from eq. (1) $r^n = a^n \cos n\theta$

$$\cos n\theta = \frac{r^n}{a^n}$$

$$P = r \cdot \frac{r^n}{a^n} \Rightarrow P = \frac{1}{a^n} \cdot r^{n+1}$$

d.w.r.t 'r'

$$\frac{dP}{dr} = \frac{1}{a^n} (n+1) r^{n+1-1}$$

$$\frac{dP}{dr} = \frac{1}{a^n} (n+1) r^n$$

$$\frac{dr}{dP} = \frac{a^n}{r^n (n+1)}$$

$$\rho = r \cdot \frac{dr}{dP}$$

$$\rho = r \cdot \frac{a^n}{(n+1)} \cdot \frac{1}{r^n}$$

$$\rho = \frac{1}{r^{n-1}} \cdot \frac{a^n}{(n+1)}$$

$$\boxed{\rho \propto \frac{1}{r^{n-1}}}$$

$$6. \frac{2a}{r} = 1 + \cos \theta$$

24

Sol:- $\frac{2a}{r} = 1 + \cos \theta \rightarrow (1)$

$$\log 2a - \log r = \log(1 + \cos \theta)$$

d.w.r.t. θ

$$0 - \frac{1}{r} \cdot \frac{dr}{d\theta} = \frac{1}{(1 + \cos \theta)} \times (0 + (-\sin \theta))$$

$$+ \cot \phi = \frac{r \sin \frac{\theta}{2} \cdot \cos \frac{\theta}{2}}{r \cos^2 \frac{\theta}{2}}$$

$$\cot \phi = \tan \frac{\theta}{2}$$

$$\cot \phi = \cot \left(\frac{\pi}{2} - \frac{\theta}{2} \right)$$

$$\phi = \frac{\pi}{2} - \frac{\theta}{2}$$

$$P = r \sin \phi$$

$$P = r \sin \left(\frac{\pi}{2} - \frac{\theta}{2} \right)$$

$$P = r \cos \left(\frac{\theta}{2} \right)$$

from eq. (1)

$$\frac{2a}{r} = 1 + \cos \theta \Rightarrow \frac{2a}{r} = 2 \cos^2 \frac{\theta}{2}$$

$$\cos \frac{\theta}{2} = \sqrt{\frac{a}{r}}$$

$$P = r \cdot \frac{\sqrt{a}}{\sqrt{r}} = P = \sqrt{r} \cdot \sqrt{a}$$

d.w.r.t. r

$$\frac{dP}{dr} = \sqrt{a} \cdot \frac{1}{2\sqrt{r}}$$

$$\frac{dr}{dP} = \frac{2\sqrt{r}}{\sqrt{a}}$$

$$P = r \cdot \frac{\sqrt{r}}{\sqrt{a}}$$

$$P = C \cdot r\sqrt{r} \quad \text{where } C = \frac{2}{\sqrt{a}}$$

s.o.b.s

$$P^3 = r^3 \cdot r \cdot C^3$$

$$P^3 \propto r^3$$

$$7. r^m = a^m (\cos m\theta + \sin m\theta)$$

$$\text{Sol:- } r^m = a^m (\cos m\theta + \sin m\theta) \rightarrow (1)$$

$$m \log r = m \log a + \log (\cos m\theta + \sin m\theta)$$

d.w.r.t. θ

$$m \cdot \frac{1}{r} \cdot \frac{dr}{d\theta} = 0 + \frac{(-\sin m\theta)m + (\cos m\theta)m}{(\cos m\theta + \sin m\theta)}$$

$$r \cdot \cot \phi = r \cdot \frac{(\cos m\theta - \sin m\theta)}{(\cos m\theta + \sin m\theta)}$$

divide $\cos m\theta$ in nume & deno.

$$\cot \phi = \frac{1 - \tan m\theta}{1 + \tan m\theta}$$

$$\cot \phi = \tan \left(\frac{\pi}{4} - m\theta \right)$$

$$\cot \phi = \cot \left(\frac{\pi}{4} - \left(\frac{\pi}{4} - m\theta \right) \right)$$

$$\phi = \frac{\pi}{4} + m\theta$$

$$P = r \sin \phi$$

$$P = r \sin \left(\frac{\pi}{4} + m\theta \right)$$

$$= r \left(\sin \frac{\pi}{4} \cdot \cos m\theta + \cos \frac{\pi}{4} \cdot \sin m\theta \right)$$

$$= r \left(\frac{1}{\sqrt{2}} \cos m\theta + \frac{1}{\sqrt{2}} \sin m\theta \right)$$

$$P = \frac{1}{\sqrt{2}} (\cos m\theta + \sin m\theta)$$

$$\text{from eq. (1)} \quad r^m = a^m (\cos m\theta + \sin m\theta)$$

$$\cos m\theta + \sin m\theta = \frac{r^m}{a^m}$$

$$P = \frac{r}{\sqrt{2}} \cdot \frac{r^m}{a^m} \Rightarrow P = \frac{1}{a^m \sqrt{2}} \cdot r^{m+1}$$

d.w.r.t. r

$$\frac{dP}{dr} = \frac{1}{\sqrt{2} a^m} (m+1) r^{m+1-1}$$

$$\frac{dp}{dr} = \frac{1}{\sqrt{2} a^m} (m+1) r^m$$

25

$$\frac{dr}{dp} = \frac{\sqrt{2} a^m}{(m+1)} \cdot \frac{1}{r^m}$$

$$p = r \cdot \frac{dr}{dp}$$

$$p = r \cdot \frac{\sqrt{2} a^m}{(m+1)} \cdot \frac{1}{r^m}$$

$$p = \frac{1}{r^{m-1}} \cdot C$$

$$p \propto \frac{1}{r^{m-1}}$$

Radius of Curvature in parametric form:-

if $x = x(t)$, $y = y(t)$

The radius of curvature in parametric form is given by

$$p = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - x''y'}$$

1. Find radius of curvature for the cycloid at any point given

$$x = a(\theta + \sin\theta), y = a(1 - \cos\theta)$$

Sol:- $x = a(\theta + \sin\theta)$ $y = a(1 - \cos\theta)$

WKT radius of curvature of in parametric form is given by

$$p = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - x''y'}$$

$$x = a(\theta + \sin\theta)$$

d.w.r.t. θ

$$x' = a(1 + \cos\theta)$$

d.w.r.t. θ

$$x'' = a(-\sin\theta)$$

$$y = a(1 - \cos\theta)$$

d.w.r.t. θ

$$y' = a(0 - (-\sin\theta))$$

$$y' = a \sin\theta$$

d.w.r.t. θ

$$y'' = a \cos\theta$$

$$\begin{aligned}
 \text{Consider, } x'^2 + y'^2 &= (a(1+\cos\theta))^2 + (a\sin\theta)^2 \\
 &= a^2 [1 + \cos^2\theta + 2\cos\theta + \sin^2\theta] \\
 &= a^2 [2 + 2\cos\theta] \\
 &= 2a^2 [1 + \cos\theta]
 \end{aligned}$$

$$\begin{aligned}
 \text{Consider, } x'y'' - x''y' &= a(1+\cos\theta)(a\cos\theta) - (-a\sin\theta)a\sin\theta \\
 &= a^2(\cos\theta + \cos^2\theta) + a^2\sin^2\theta \\
 &= a^2(\cos\theta + \cos^2\theta + \sin^2\theta) = a^2(1+\cos\theta)
 \end{aligned}$$

$$P = \frac{(2a^2[1+\cos\theta])^{3/2}}{a^2(1+\cos\theta)}$$

$$= \frac{(2a^2)^{3/2} (1+\cos\theta)^{3/2}}{a^2(1+\cos\theta)}$$

$$= \frac{2\sqrt{2} a^3 (1+\cos\theta)^{1/2}}{a^2}$$

$$= 2\sqrt{2} a (1+\cos\theta)^{1/2}$$

$$= 2\sqrt{2} a \sqrt{2\cos^2\theta/2}$$

$$= 2\sqrt{2} a \sqrt{2} \cos\theta/2$$

$$P = 4a\cos\theta/2$$

Q. Find the radius of curvature at any point ellipse 26

$$x = a \cos \theta, \quad y = b \sin \theta$$

Sol:- $x = a \cos \theta$

d.w.r.t θ

$$x' = a(-\sin \theta)$$

d.w.r.t θ

$$x'' = -a \cos \theta$$

$$y = b \sin \theta$$

d.w.r.t θ

$$y' = b \cos \theta$$

d.w.r.t θ

$$y'' = -b \sin \theta$$

$$P = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - x''y'}$$

Consider, $x'^2 + y'^2 = (-a \sin \theta)^2 + (b \cos \theta)^2 = a^2 \sin^2 \theta + b^2 \cos^2 \theta$

Consider, $x'y'' - x''y' = -a \sin \theta (-b \sin \theta) - (-a \cos \theta)(b \cos \theta)$
 $= ab \sin^2 \theta + ab \cos^2 \theta$
 $= ab(\sin^2 \theta + \cos^2 \theta)$
 $= ab$

$$P = \frac{a^2 \sin^2 \theta + b^2 \cos^2 \theta}{ab}$$

$$= \frac{a^2 \sin^2 \theta}{ab} + \frac{b^2 \cos^2 \theta}{ab}$$

$$= \frac{a \sin^2 \theta}{b} + \frac{b \cos^2 \theta}{a}$$

$$P = \frac{a}{b} \sin^2 \theta + \frac{b}{a} \cos^2 \theta$$

3. PT the radius of curvature at any point of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ is times the length of perpendicular from the origin to the tangent at that point.

Sol:- Write the equation,

Consider $x^{2/3} + y^{2/3} = a^{2/3} \rightarrow (1)$

eq-(1) in parametric form

$$x = a \cos^3 t$$

d.w.r. to t

$$x' = -3a \cos^2 t \sin t$$

d.w.r. to t

$$x'' = -3a [\cos t \cdot \sin t (-\sin t) + \cos^2 t \cdot \cos t]$$

$$x'' = -3a (-\cos t \sin^2 t + \cos^3 t)$$

$$x'' = -3a \cos t (-\sin^2 t + \cos^2 t)$$

$$y = a \sin^3 t$$

d.w.r. to t

$$y' = 3a \sin^2 t \cdot \cos t$$

d.w.r. to t

$$y'' = 3a (\cos t \cdot \cos^2 t - \sin^2 t \cdot \sin t)$$

$$= 3a (\cos^3 t - \sin^3 t)$$

$$y'' = 3a \sin t (\cos^2 t - \sin^2 t)$$

$$\text{Consider, } x'^2 + y'^2 = (-3a \cos^2 t \sin t)^2 + (3a \sin^2 t \cos t)^2$$

$$= 9a^2 (\cos^4 t \sin^2 t + \sin^4 t \cos^2 t)$$

$$= 9a^2 \cos^2 t \sin^2 t (\cos^2 t + \sin^2 t)$$

$$= 9a^2 \cos^2 t \sin^2 t \Rightarrow (3a \cos t \sin t)^2$$

$$\text{Consider, } x'y'' - x''y'$$

$$= (-3a \cos^2 t \sin t) [3a (\cos^3 t - \sin^3 t)] -$$

$$[-3a (-\cos t \sin^2 t + \cos^3 t)] (3a \sin^2 t \cos t)$$

$$= -9a^2 [(\cos^2 t \sin t) (\cos^3 t - \sin^3 t) - (-\cos t \sin^2 t + \cos^3 t) (\sin^2 t \cos t)]$$

$$= -9a^2 \cos t \sin t [\cos t (\cos^3 t - \sin^3 t) - (-\cos t \sin^2 t + \cos^3 t) \sin t]$$

$$-9a^3 \cos t \cdot \sin t \left[\cos t \cdot \sin t (\cos^2 t - \sin^2 t) - (-\sin^3 t + \cos^3 t) \cos t \cdot \sin t \right] \quad \text{Q 7}$$

$$= -9a^3 \cos^3 t \cdot \sin^2 t \left[\cos^2 t - \sin^2 t + \sin^3 t - \cos^3 t \right]$$

$$= -9a^3 \cos^3 t \cdot \sin^2 t \left[\sin^2 t + \cos^2 t \right]$$

$$= -9a^3 \cos^3 t \sin^2 t = x'y'' - x''y'$$

$$P = \frac{\left((3a \cdot \cos t \cdot \sin t)^2 \right)^{3/2}}{- (3a \cos t \cdot \sin t)^2}$$

$$P = |-3a \cos t \cdot \sin t|$$

$$P = 3a \cos t \cdot \sin t$$

WKT equation of tangent at (x_1, y_1)

$$y - y_1 = m(x - x_1)$$

$$\text{at } (x_1, y_1) = (a \cos^3 t, a \sin^3 t)$$

$$\text{slope} = m = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y_1}{x_1}$$

$$= \frac{3a \sin^2 t \cdot \cos t}{-3a \cos^2 t \cdot \sin t} = -\tan t$$

$$\Rightarrow y - a \sin^3 t = -\tan t (x - a \cos^3 t)$$

$$y - a \sin^3 t = -x \tan t + a \cdot \frac{\sin t}{\cos t} \cdot \cos^3 t$$

$$y - a \sin^3 t = -x \tan t + a \sin t \cdot \cos^2 t$$

$$y = -x \tan t + a \sin t \cdot \cos^2 t + a \sin^3 t$$

$$= -x \tan t + a \sin t (\cos^2 t + \sin^2 t)$$

$$y = -x \tan t + a \sin t$$