

BCS405A - Module 3

Relations and functions

Relations and Functions: Cartesian Products and Relations, Functions – Plain and One-to-One, Onto Functions. The Pigeon-hole Principle, Function Composition and Inverse Functions.

Properties of Relations, Computer Recognition – Zero-One Matrices and Directed Graphs, Partial Orders – Hasse Diagrams, Equivalence Relations and Partitions.

Textbook 1: Ch 5:5.1 to 5.3, 5.5, 5.6, Ch 7:7.1 to 7.4

3.1 Cartesian product of sets

Cartesian product:

Let A and B be two sets. Then the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$ is called the Cartesian product of A and B.

$$A \times B = \{(a, b) | a \in A \text{ and } b \in B\}, B \times A = \{(b, a) | b \in B \text{ and } a \in A\}$$

$$A \times A = \{(a, b) | a \in A \text{ and } b \in A\}, A \times B \neq B \times A \text{ and } A^2 = A \times A$$

Cardinality:

$$\text{If } n(A) = |A|, n(B) = |B| \text{ then } |A \times B| = |A||B|, |B \times A| = |B||A|, |A \times A| = |A|^2$$

Equality:

$$(a, b) = (c, d) \Rightarrow a = c \text{ and } b = d$$

1. Find x and y in each of the following cases: (i) $(2x, x + y) = (6, 1)$

(ii) $(x + 2, 4) = (5, 2x + y)$ (iii) $(y - 2, 2x + 1) = (x - 1, y + 2)$ (iv) $(x, y) = (x^2, y^2)$

(i) $x = 3, y = -2$

(ii) $x = 3, y = -2$

(iii) Solve $x - y = -1, 2x - y = 1$ we get $x = 0, y = -1$.

(iv) Solve $x(x - 1) = 0, y(y - 1) = 0$ we get $x = 0 \text{ or } 1, y = 0 \text{ or } 1$.

2. Given $A = \{a, b\}$ and $B = \{1, 2, 3\}$ find $A \times B, B \times A, A \times A, B \times B$.

$$A \times B = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$$

$$B \times A = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$$

$$A \times A = \{(a, a), (a, b), (b, a), (b, b)\}$$

$$B \times B = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$$

3. Given $A = \{1, 2\}, B = \{a, b, c\}, C = \{3, 4\}$ find $A \times B \times C, B \times C \times A$.

$$A \times B \times C = \{(1, a, 3), (1, a, 4), (1, b, 3), (1, b, 4), (1, c, 3), (1, c, 4), \\ (2, a, 3), (2, a, 4), (2, b, 3), (2, b, 4), (2, c, 3), (2, c, 4)\}$$

$$B \times C \times A = \left\{ \begin{array}{l} (a, 3, 1), (a, 3, 2), (a, 4, 1), (a, 4, 2), \\ (b, 3, 1), (b, 3, 2), (b, 4, 1), (b, 4, 2), \\ (c, 3, 1), (c, 3, 2), (c, 4, 1), (c, 4, 2) \end{array} \right\}$$

4. Given $A = \{1, 3, 5\}, B = \{2, 3\}, C = \{4, 6\}$

Find $A \cap (B \times C), (A \times B) \cup (B \times C), (A \times B) \cap (B \times A), (A \times B) \cap (B \times C)$.

$$A \cap (B \times C)$$

$$= \{1, 3, 5\} \cap \{(2, 4), (2, 6), (3, 4), (3, 6)\}$$

$$= \phi$$

$$(A \times B) \cup (B \times C)$$

$$= \{(1, 2), (1, 3), (3, 2), (3, 3), (5, 2), (5, 3)\} \cup \{(2, 4), (2, 6), (3, 4), (3, 6)\}$$

$$= \{(1, 2), (1, 3), (3, 2), (3, 3), (5, 2), (5, 3), (2, 4), (2, 6), (3, 4), (3, 6)\}$$

$$(A \times B) \cap (B \times A)$$

$$= \{(1, 2), (1, 3), (3, 2), (3, 3), (5, 2), (5, 3)\} \cap \{(2, 1), (2, 3), (2, 5), (3, 1), (3, 3), (3, 5)\}$$

$$= \{(3, 3)\}$$

$$(A \times B) \cap (B \times C)$$

$$= \{(1, 2), (1, 3), (3, 2), (3, 3), (5, 2), (5, 3)\} \cap \{(2, 4), (2, 6), (3, 4), (3, 6)\}$$

$$= \phi$$

5. Given $A = \{\alpha, \beta, \gamma\}$, $B = \{\theta, \eta\}$, $C = \{\lambda, \mu, \gamma\}$.

Find $A \times C$, $(A \cup B) \times C$, $A \cup (B \times C)$, $A \times (B \cup C)$

$$A \times C = \{(\alpha, \lambda), (\alpha, \mu), (\alpha, \gamma), (\beta, \lambda), (\beta, \mu), (\beta, \gamma), (\gamma, \lambda), (\gamma, \mu), (\gamma, \gamma)\}$$

$$\begin{aligned}(A \cup B) \times C &= \{\alpha, \beta, \gamma, \theta, \eta\} \times \{\lambda, \mu, \gamma\} \\ &= \left\{ \begin{array}{l} (\alpha, \lambda), (\alpha, \mu), (\alpha, \gamma), (\beta, \lambda), (\beta, \mu), (\beta, \gamma), (\gamma, \lambda), \\ (\gamma, \mu), (\gamma, \gamma), (\theta, \lambda), (\theta, \mu), (\theta, \gamma), (\eta, \lambda), (\eta, \mu), (\eta, \gamma) \end{array} \right\}\end{aligned}$$

$$\begin{aligned}A \cup (B \times C) &= \{\alpha, \beta, \gamma\} \cup \{(\theta, \lambda), (\theta, \mu), (\theta, \gamma), (\eta, \lambda), (\eta, \mu), (\eta, \gamma)\} \\ &= \{\alpha, \beta, \gamma, (\theta, \lambda), (\theta, \mu), (\theta, \gamma), (\eta, \lambda), (\eta, \mu), (\eta, \gamma)\}\end{aligned}$$

$$\begin{aligned}A \times (B \cup C) &= \{\alpha, \beta, \gamma\} \times \{\theta, \eta, \lambda, \mu, \gamma\} \\ &= \left\{ \begin{array}{l} (\alpha, \theta), (\alpha, \eta), (\alpha, \lambda), (\alpha, \mu), (\alpha, \gamma), (\beta, \theta), (\beta, \eta), \\ (\beta, \lambda), (\beta, \mu), (\beta, \gamma), (\gamma, \theta), (\gamma, \eta), (\gamma, \lambda), (\gamma, \mu), (\gamma, \gamma) \end{array} \right\}\end{aligned}$$

6. For any non-empty sets A, B and C Prove the following:

$$A \times (B \cap C) = (A \times B) \cap (A \times C), \quad A \times (B - C) = (A \times B) - (A \times C)$$

$$\begin{aligned}(x, y) \in A \times (B \cap C) &\Leftrightarrow x \in A \text{ and } y \in B \cap C \\ &\Leftrightarrow x \in A \text{ and } (y \in B \text{ and } y \in C) \\ &\Leftrightarrow (x \in A \text{ and } y \in B) \text{ and } (x \in A \text{ and } y \in C) \\ &\Leftrightarrow x \in A \times B \text{ and } x \in A \times C \\ &\Leftrightarrow x \in (A \times B) \cap (A \times C)\end{aligned}$$

Therefore, $A \times (B \cap C) = (A \times B) \cap (A \times C)$

$$\begin{aligned}(x, y) \in A \times (B - C) &\Leftrightarrow x \in A \text{ and } y \in B - C \\ &\Leftrightarrow x \in A \text{ and } (y \in B \text{ and } y \notin C) \\ &\Leftrightarrow (x \in A \text{ and } y \in B) \text{ and } (x \in A \text{ and } y \notin C) \\ &\Leftrightarrow x \in A \times B \text{ and } x \notin A \times C \\ &\Leftrightarrow x \in (A \times B) - (A \times C)\end{aligned}$$

Therefore, $A \times (B - C) = (A \times B) - (A \times C)$

7. Find $A \cap B$, $B \cap C$, $\overline{A \cup C}$, $\overline{B \cup C}$ if $A, B, C \subseteq Z \times Z$ with

$$A = \{(x, y) | y = 5x - 1\}, B = \{(x, y) | y = 6x\}, C = \{(x, y) | 3x - y = -7\}$$

(i) $(x, y) \in A \cap B \Rightarrow y = 5x - 1$ and $y = 6x$

$$\Rightarrow 5x - 1 = 6x$$

$$\Rightarrow x = -1 \text{ and } y = -6$$

Therefore, $A \cap B = \{(-1, -6)\}$

(ii) $(x, y) \in B \cap C \Rightarrow y = 6x$ and $3x - y = -7$

$$\Rightarrow 6x = 3x - 7$$

$$\Rightarrow x = -7/3 \text{ and } y = -14 \text{ or } 0$$

Therefore, $B \cap C = \left\{\left(-\frac{7}{3}, -14\right), \left(-\frac{7}{3}, 0\right)\right\}$

(iii) $(x, y) \in \overline{A \cup C} \Rightarrow (x, y) \in A \cap C$

$$\Rightarrow y = 5x - 1 \text{ and } 3x - y = -7$$

$$\Rightarrow 5x - 1 = 3x + 7$$

$$\Rightarrow x = 4 \text{ and } y = 19$$

Therefore, $\overline{A \cup C} = \{(4, 19)\}$

(iv) $(x, y) \in \overline{B \cup C} \Rightarrow (x, y) \in \overline{B \cap C}$

$$\Rightarrow (x, y) \notin B \cap C$$

$$\Rightarrow Z \times Z - (x, y) \in B \cap C$$

$$\Rightarrow Z \times Z - (y = 6x \text{ and } 3x - y = -7)$$

$$\Rightarrow Z \times Z - (6x = 3x + 7)$$

$$\Rightarrow Z \times Z - \left(x = \frac{7}{3} \text{ and } y = 14\right)$$

Therefore, $\overline{B \cup C} = Z \times Z - \left\{\left(\frac{7}{3}, 14\right)\right\}$

3.2 Relations

Introduction:

- ❖ Let A and B be two sets. Then a subset of $A \times B$ is called a relation from A to B .
- ❖ A subset of $A \times A$ is called a binary relation on A .
- ❖ $(a, b) \in R \Leftrightarrow aRb$
- ❖ Number of relations from A to B = no. of subsets of $A \times B = 2^{mn}$, ($|A| = m, |B| = n$)

1. Let A and B be sets with $|B| = 3$. If there are 4096 relations from A to B find $|A|$.

By data, $|A \times B| = 2^{mn} = 4096, n = 3$

$$4096 = 2^{12} = 2^{3m}, m = |A| = 4$$

2. Let $A = \{1, 2, 3\}, B = \{2, 4, 5\}$. Determine (i) $|A \times B|$ (ii) Number of relations from A to B (iii) Number of binary relations on A . (iv) Number of relations from A to B that contain exactly 5 ordered pairs.

(i) $|A \times B| = mn = 3 \times 3 = 9$

(ii) Number of relations from A to $B = 2^{mn} = 2^{3 \times 3} = 512$

(iii) Number of binary relations on $A = 2^{mm} = 2^{3 \times 3} = 2^9 = 512$

(iv) Number of relations from A to B that contain exactly 5 ordered pairs = ${}^9C_5 = 126$

3. Binary relation R on N is defined recursively by

$(0, 0) \in R$, If $(s, t) \in R$ then $(s + 1, t + 7) \in R$ find R a set of ordered pairs.

$$(0, 0) \in R \Rightarrow (0 + 1, 0 + 7) \in R$$

$$(1, 7) \in R \Rightarrow (1 + 1, 7 + 7) \in R$$

$$(2, 14) \in R \Rightarrow (2 + 1, 14 + 7) \in R$$

$$(3, 21) \in R \Rightarrow (3 + 1, 21 + 7) \in R$$

$$\text{Therefore, } R = \{(0, 0), (1, 7), (2, 14), (3, 21), (4, 28), \dots\}$$

4. Let $R \subseteq N \times N$ where $(m, n) \in R$ if $n = 5m + 2$. Give a recursive definition for R .

R is defined by $(0, 2) \in R$, if $(m, n) \in R$ then $(m + 1, n + 5) \in R$.

3.3 Zero-one matrices and digraphs

Matrix of a relation:

Let $A = \{a_1, a_2, \dots, a_m\}$ and $B = \{b_1, b_2, \dots, b_n\}$ then $A \times B = \{(a_i, b_j) | 1 \leq i \leq m, 1 \leq j \leq n\}$.

Let R be a relation from A to B which is a subset of $A \times B$.

Then $M(R) = \begin{pmatrix} m_{11} & \cdots & m_{1n} \\ \vdots & \ddots & \vdots \\ m_{m1} & \cdots & m_{mn} \end{pmatrix}$ where $m_{ij} = \begin{cases} 1, & \text{if } (a_i, b_j) \in R \\ 0, & \text{if } (a_i, b_j) \notin R \end{cases}$

is called the matrix of the relation R (or) Zero-one matrix for R .

Example:

Let $A = \{0, 1, 2\}$, $B = \{p, q\}$. Let R be a relation from A to B defined by $R = \{(0, p), (1, q), (2, p)\}$.

Then the matrix of the relation R is $M(R) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Digraph of a relation:

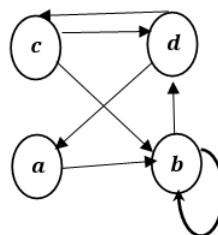
Let R be a binary relation on a finite set A . Draw a small circle for each element of A . These circles are called vertices or nodes. Draw an arrow from a vertex x to vertex y iff $(x, y) \in R$. These arrows are called edges. The pictorial representation of R is called a digraph or directed graph.

Example:

Let $A = \{a, b, c, d\}$

$R = \{(a, b), (b, b), (b, d), (c, b), (c, d), (d, a), (d, c)\}$

then the digraph R is



In-degree and out degree:

No. of edges terminating at a vertex is called in-degree.

No. of edges leaving a vertex is called out-degree.

Vertices	a	b	c	d
In - Degree	1	3	1	2
Out - Degree	1	2	2	2

Composite relations:

If A, B and C are sets with $R_1 \subseteq A \times B$ and $R_2 \subseteq B \times C$, then the composition relation $R_1 \circ R_2$ is a relation from A to C defined by $R_1 \circ R_2 = \{(x, z) | x \in A, z \in C\}$.

Problems:

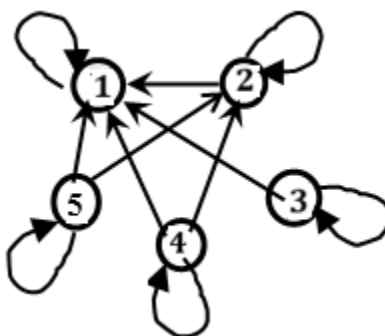
1. Let $A = \{1, 2, 3, 4, 6\}$ and R be a relation on A defined by aRb iff a is a multiple of b .

(i) Write down R as a set of ordered pairs (ii) Write matrix of R

(iii) Draw the digraph of R .

(i) $\{(1, 1), (2, 1), (3, 1), (4, 1), (6, 1), (2, 2), (4, 2), (6, 2), (3, 3), (6, 3), (4, 4), (6, 6)\}$

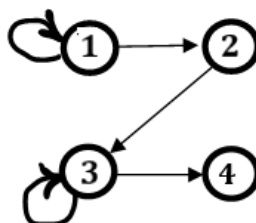
(ii) $M = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & 6 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 6 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix} \end{matrix}$ (iii)



2. Let $R = \{(1, 1), (1, 2), (2, 3), (3, 3), (3, 4)\}$ be a relation on the set $A = \{1, 2, 3, 4\}$.

(i) Write matrix of R (ii) Draw the digraph of R .

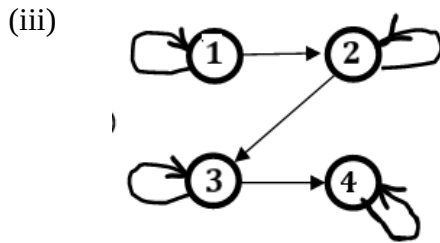
(i) $M = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{matrix}$ (ii)



3. Let $A = \{1, 2, 3, 4\}$. Let R be a relation on A defined by xRy iff ' x divides y '. Write down (i) Write down R as a set of ordered pairs (ii) Write matrix of R (iii) Draw the digraph of R (iv) In-degrees and out-degrees of all vertices of the digraph.

(i) $R = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$

(ii)
$$\begin{matrix} 1 & \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ 2 & \\ 3 & \\ 4 & \end{matrix}$$



(iv)

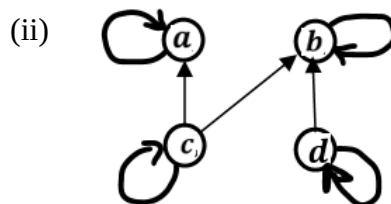
Vertices	1	2	3	4
In - Degree	1	2	2	2
Out - Degree	2	2	2	1

4. Let $A = \{a, b, c, d\}$ and R be a relation on A that has the matrix

$$M_R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

- (i) Write down R as a set of ordered pairs (ii) Draw the digraph of R (iii) List in-degrees and out-degrees of all vertices of the digraph.

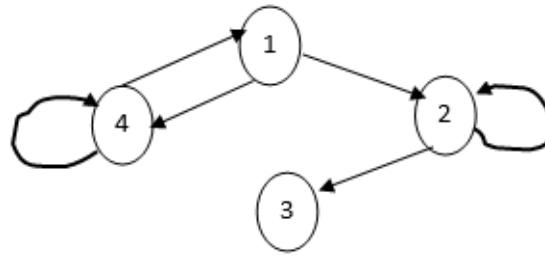
(i) $R = \{(a, a), ((b, b), (c, a), (c, b), (c, c), (d, b), (d, d))\}$



(iii)

Vertices	a	b	c	d
In - Degree	2	3	1	1
Out - Degree	1	1	2	3

5. Find the relation represented by the digraph given below. Also write down its matrix.



Relation represented by the digraph is $R = \{(1, 1), (1, 2), (2, 2), (2, 3), (4, 1), (4, 4)\}$

Corresponding matrix is $M_R = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$

6. For $A = \{1, 2, 3, 4\}$ and $R = \{(1, 2), (1, 3), (2, 4), (4, 4)\}$. Find $R^2, R^3, R^\infty, M_R, (M_R)^T$.

$$R^2 = \{(1, 4), (2, 4), (4, 4)\}$$

$$R^3 = \{(1, 4), (2, 4), (4, 4)\}$$

$$R^\infty = \{(1, 4), (1, 3), (2, 4), (4, 4)\}$$

$$M_R = \begin{pmatrix} 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } (M_R)^T = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

7. Let $A = \{1, 2, 3, 4\}$ and R, S are relations on A defined by

$$R = \{(1, 2), (1, 3), (2, 4), (4, 4)\}, S = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 3), (2, 4)\}.$$

Find RoS, SoR, R^2, S^2 . Write down their matrices.

$$RoS = \{(1, 3), (1, 4)\}$$

$$SoR = \{(1, 2), (1, 3), (1, 4), (2, 4)\}$$

$$R^2 = \{(1, 4), (2, 4), (4, 4)\}$$

$$S^2 = \{(1, 1), (1, 2), (1, 3), (1, 4)\}$$

8. Let $A = \{1, 2, 3, 4\}$, $B = \{w, x, y, z\}$ and $C = \{p, q, r, s\}$. Consider

$R_1 = \{(1, x), (2, w), (3, z)\}$ a relation from A to B,

$R_2 = \{(w, p), (z, q), (y, s), (x, p)\}$ a relation from B to C.

(i) Write relation matrices $M(R_1)$, $M(R_2)$, $M(R_1 \circ R_2)$

(ii) Verify $M(R_1) \times M(R_2) = M(R_1 \circ R_2)$.

$$R_1 = \{(1, x), (2, w), (3, z)\}, R_2 = \{(w, p), (z, q), (y, s), (x, p)\}$$

$$R_1 \circ R_2 = \{(1, p), (2, p), (3, q)\}$$

$$M(R_1) = \begin{matrix} & \begin{matrix} w & x & y & z \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{matrix}, \quad M(R_2) = \begin{matrix} & \begin{matrix} p & q & r & s \end{matrix} \\ \begin{matrix} w \\ x \\ y \\ z \end{matrix} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \end{matrix},$$

$$M(R_1 \circ R_2) = \begin{matrix} & \begin{matrix} p & q & r & s \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{matrix}, \quad M(R_1) \times M(R_2) = \begin{matrix} & \begin{matrix} p & q & r & s \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{matrix}$$

Therefore, $M(R_1) \times M(R_2) = M(R_1 \circ R_2)$.

3.4 Properties of relations

Introduction: A relation R on a set A is said to be

- ❖ Reflexive if $(a, a) \in R, \forall a \in A$
- ❖ Symmetric if $(a, b) \in R \Rightarrow (b, a) \in R$
- ❖ Transitive if $(a, b) \in R, (b, c) \in R \Rightarrow (a, c) \in R$
- ❖ Ir-reflexive if $(a, a) \notin R$, for any $a \in A$
- ❖ Anti-symmetric if $(a, b) \in R, (b, a) \in R \Rightarrow a = b$
- ❖ Asymmetric if it is neither symmetric nor anti-symmetric.

Problems:

1. Let $A = \{1, 2, 3\}$. Determine the nature of the following relations on A .

$$R_1 = \{(1, 2), (2, 1), (1, 3), (3, 1)\}, \quad R_2 = \{(1, 1), (2, 2), (3, 3), (2, 3)\},$$

$$R_3 = \{(1, 1), (2, 2), (3, 3)\}, \quad R_4 = \{(1, 1), (2, 2), (3, 3), (2, 3), (3, 2)\},$$

$$R_5 = \{(1, 1), (2, 3), (3, 3)\}, \quad R_6 = \{(2, 3), (3, 4), (2, 4)\}, \quad R_7 = \{(1, 3), (3, 2)\}$$

	Reflexive	Symmetric	Transitive	Ir-reflexive	Anti symmetric	Asymmetric
R_1	X	✓	X	✓	X	X
R_2	✓	X	✓	X	X	✓
R_3	✓	✓	✓	X	X	X
R_4	✓	✓	✓	X	X	X
R_5	X	X	✓	X	X	✓
R_6	X	X	✓	✓	✓	X
R_7	X	X	X	✓	✓	X

2. Let $A = \{1, 2, 3, 4\}$, $R = \{(1, 3), (1, 1), (3, 1), (1, 2), (3, 3), (4, 4)\}$ be a relation on A. Determine whether R is reflexive, symmetric, transitive, irreflexive or antisymmetric.

R is irreflexive because $(1, 1) \in R$ but $(2, 2) \notin R$.

R is asymmetric because $(1, 3), (3, 1), (1, 2) \in R$ but $(2, 1) \notin R$.

R is not transitive because $(3, 1), (1, 2) \in R$ but $(3, 2) \notin R$.

3. Let A be a finite set which consists of n elements. Determine the relations on A which are (i) reflexive (ii) symmetric (iii) antisymmetric.

Total number of reflexive relations on A is 2^{n^2-n}

Total number of symmetric relations on A is $2^{\frac{(n^2+n)}{2}}$

Total number of anti symmetric relations on A is $2^n \times 3^{\frac{n^2-n}{2}}$

4. Show that the relation represented by the matrix $M_R = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ is transitive.

$R = \{(a, a), (a, b), (a, c), (b, c), (c, c)\}$ is a transitive relation.

5. If $A = \{1, 2, 3, 4\}$ and $R = \{(1, 2), (2, 2), (3, 4), (4, 1)\}$ verify the relation R on A is whether symmetric or asymmetric or antisymmetric.

$R = \{(1, 2), (2, 2), (3, 4), (4, 1)\}$ is an asymmetric relation.

Because $(2, 2), (1, 2) \in R$ but $(2, 1) \notin R$

3.5 Equivalence relations and partitions

❖ Equivalence relations:

A relation R on a set A is said to be an equivalence relation on A if it is reflexive, symmetric and transitive.

❖ Equivalence classes:

Let R be an equivalence relation on A and $a \in A$. Then $[a] = \{x \in A \mid (x, a) \in R\}$ is called the equivalence class of a .

❖ Partition of a set:

Let A be a nonempty set and $A_1, A_2, \dots, A_n$ be nonempty subsets of A such that

$$(i) A_1 \cup A_2 \cup \dots \cup A_n = A$$

(ii) $A_1, A_2, \dots, A_n$ are disjoint then $P = \{A_1, A_2, \dots, A_n\}$ is called a partition of A .

Problems:

1. Let $A = \{1, 2, 3, 4\}$ and $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 3), (3, 3), (4, 4)\}$ be a relation on A . Verify that R is an equivalence relation.

$$(1, 1), (2, 2), (3, 3), (4, 4) \in R$$

$$(a, a) \in R, \forall a \in A,$$

R is **reflexive**.

$$(1, 2), (2, 1) \in R \Rightarrow (1, 1) \in R$$

$$(3, 4), (4, 3) \in R \Rightarrow (3, 3) \in R$$

$$(a, b) \in R, (b, c) \in R \Rightarrow (a, c) \in R, \forall a, b, c \in A$$

R is **transitive**.

$$(1, 2) \in R \Rightarrow (2, 1) \in R$$

$$(3, 4) \in R \Rightarrow (4, 3) \in R$$

$$(a, b) \in R \Rightarrow (b, a) \in R, \forall a, b \in A$$

R is **symmetric**.

Therefore, R is an **equivalence relation**.

2. Let $A = \{1, 2, 3, 4\}$ and $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 1), (3, 3), (1, 3), (4, 1), (4, 4)\}$ be a relation on A .

Is R an equivalence relation?

$$(1, 1), (2, 2), (3, 3), (4, 4) \in R$$

$$(a, a) \in R, \forall a \in A,$$

R is reflexive.

$$(4, 1), (1, 2) \in R, \text{ but } (4, 2) \notin R$$

R is not transitive.

Therefore, R is not an equivalence relation.

3. A relation R on a set $A = \{a, b, c\}$ is represented $M_R = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Determine whether R is an equivalence relation.

$$R = \{(a, a), (b, b), (c, c), (a, c)\}$$

$$(a, c) \in R, \text{ but } (c, a) \notin R$$

R is not symmetric.

Therefore, R is not an equivalence relation.

4. For a fixed integer $n > 1$, prove that congruent modulo n is an equivalence relation.

$a \equiv b \pmod{n} \Rightarrow a - b$ is a multiple of n .

For any $a \in \mathbb{Z}$,

$$a - a = 0$$

$$\Rightarrow a - a = 0n$$

$$\Rightarrow a \equiv a \pmod{n}$$

$$\Rightarrow (a, a) \in R$$

R is reflexive.

For any $a, b \in \mathbb{Z}$,

$$(a, b) \in R$$

$$\Rightarrow a \equiv b \pmod{n}$$

$$\Rightarrow a - b \text{ is a multiple of } n$$

$$\Rightarrow b - a \text{ is a multiple of } n$$

$$\Rightarrow b \equiv a \pmod{n}$$

$$\Rightarrow (b, a) \in R$$

R is symmetric.

For any $a, b, c \in \mathbb{Z}$,

$$(a, b), (b, c) \in R$$

$$\Rightarrow a \equiv b \pmod{n}, b \equiv c \pmod{n},$$

$$\Rightarrow a \equiv c \pmod{n}$$

$$\Rightarrow a - c \text{ is a multiple of } n$$

$$\Rightarrow a \equiv c \pmod{n}$$

$$\Rightarrow (a, c) \in R$$

R is transitive.

Therefore, R is an equivalence relation.

5. If $A = A_1 \cup A_2 \cup A_3$, where $A_1 = \{1, 2\}$, $A_2 = \{2, 3, 4\}$, $A_3 = \{5\}$. Define relation R on A by xRy if x and y are in the same subset A_i for $1 \leq i \leq 3$. Is R an equivalence relation?

$$A = \{1, 2, 3, 4, 5\}$$

$$R = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (1, 2), (2, 1), (2, 3), (3, 2), (2, 4), (4, 2), (3, 4), (4, 3)\}$$

$$(1, 1), (2, 2), (3, 3), (4, 4) \in R$$

$$(a, a) \in R, \forall a \in A,$$

R is **reflexive**.

$$(1, 2), (2, 1) \in R \Rightarrow (1, 1) \in R$$

$$(2, 3), (3, 2) \in R \Rightarrow (2, 2) \in R$$

$$(2, 3), (3, 4) \in R \Rightarrow (2, 4) \in R$$

$$(a, b) \in R, (b, c) \in R \Rightarrow (a, c) \in R, \forall a, b, c \in A$$

R is **transitive**.

$$(1, 2) \in R \Rightarrow (2, 1) \in R$$

$$(3, 4) \in R \Rightarrow (4, 3) \in R$$

$$(2, 3) \in R \Rightarrow (3, 2) \in R$$

$$(a, b) \in R \Rightarrow (b, a) \in R, \forall a, b \in A$$

R is **symmetric**.

Therefore, R is an **equivalence relation**.

6. Let $A = \{1, 2, 3, 4, 5\} \times \{1, 2, 3, 4, 5\}$ and on A by $(x_1, y_1)R(x_2, y_2)$ if $x_1 + y_1 = x_2 + y_2$. Verify that R is an equivalence relation.
- (ii) Determine the equivalence classes $[(1, 3)]$, $[(2, 4)]$, $[(1, 1)]$.

For any $(a, b) \in A \times A$,

$$a + b = a + b$$

$$\Rightarrow (a, b)R(a, b), \forall a \in A$$

R is reflexive.

$$(a, b)R(c, d)$$

$$\Rightarrow a + b = c + d$$

$$\Rightarrow c + d = a + b$$

$$\Rightarrow (c, d)R(a, b)$$

R is symmetric.

$$(a, b)R(c, d) \text{ and } (c, d)R(e, f)$$

$$\Rightarrow a + b = c + d \text{ and } c + d = e + f$$

$$\Rightarrow a + b = e + f$$

$$\Rightarrow (a, b)R(e, f)$$

R is transitive.

Therefore, R is an **equivalence relation**.

7. Let $A = \{1, 2, 3, 4, 5\} \times \{1, 2, 3, 4, 5\}$ and on A by $(x_1, y_1)R(x_2, y_2)$ if $x_1 + y_1 = x_2 + y_2$. Determine the equivalence classes $[(1, 3)]$, $[(2, 4)]$, $[(1, 1)]$.

$$[(1, 3)] = \{(x, y) \mid (x, y)R(1, 3)\}$$

$$= \{(x, y) \mid x + y = 1 + 3\}$$

$$= \{(2, 2), (3, 1), (1, 3)\}$$

$$[(2, 4)] = \{(x, y) \mid (x, y)R(2, 4)\}$$

$$= \{(x, y) \mid x + y = 2 + 4\}$$

$$= \{(1, 5), (5, 1), (3, 3), (2, 4), (4, 2)\}$$

$$[(1, 1)] = \{(x, y) \mid (x, y)R(1, 1)\}$$

$$= \{(x, y) \mid x + y = 1 + 1\} = \{(1, 1)\}$$

8. For $A = \{a, b, c, d, x, y, z\}$ define an equivalence relation on A. Hence find equivalence classes. Also find a partition of A.

Equivalence relation:

$$A = \{a, b, c, d, x, y, z\}$$

$$R = \{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (d, d), (c, a), (c, b), (c, c), (x, x), (y, y), (z, z)\}$$

Since R is reflexive, symmetric and transitive, R is an equivalence relation.

Equivalence classes:

$[a] = \{x \mid x R a\} = \{a, b, c\}$	$[d] = \{x \mid x R d\} = \{d\}$
$[b] = \{x \mid x R b\} = \{a, b, c\}$	$[x] = \{x \mid x R x\} = \{x\}$
$[c] = \{x \mid x R c\} = \{a, b, c\}$	$[y] = \{x \mid x R y\} = \{y\}$
	$[z] = \{x \mid x R z\} = \{z\}$

Partition of A: Partition of A = $\{\{a, b, c\}, \{d\}, \{x\}, \{y\}, \{z\}\}$

9. If R is a relation defined on $A = \{1, 2, 3, 4\}$ by

$R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 3), (3, 3), (4, 4)\}$. Determine the partition induced by the equivalence relation.

Equivalence classes are

$[1] = \{x \mid x R 1\} = \{1, 2\}$	$[3] = \{x \mid x R 3\} = \{3, 4\}$
$[2] = \{x \mid x R 2\} = \{1, 2\}$	$[4] = \{x \mid x R 4\} = \{3, 4\}$

Partition induced by the given equivalence relation is $\{\{1, 2\}, \{3, 4\}\}$

10. Let $A = \{a, b, c, d, e, f, g\}$ and consider the partition $P = \{\{a, c, d\}, \{b\}, \{e, g\}, \{f\}\}$.

Determine the corresponding equivalence relation R .

$$R = \{(a, a), (a, c), (a, d), (c, a), (c, c), (c, d), (d, a), (d, c), (d, d), (b, b), (e, e), (e, g), (g, e), (g, g), (f, f)\}$$

11. Let R be an equivalence relation on A and $a, b \in A$. Then prove the following:

(i) $a \in [a]$ (ii) aRb iff $[a] = [b]$ (iii) If $[a] \cap [b] \neq \phi$ then $[a] = [b]$

Proof:

(i) R is reflexive

$$\Rightarrow (a, a) \in R \Rightarrow a \in \{a\}$$

(ii) **To prove:** $aRb \Rightarrow [a] = [b]$

Let $(a, b) \in R$

$x \in [a]$ $\Rightarrow (x, a) \in R$ $\Rightarrow (x, a), (a, b) \in R$ $\Rightarrow (x, b) \in R$ $\Rightarrow x \in [b]$ $\therefore [a] \subseteq [b].$	$x \in [b]$ $\Rightarrow (x, b) \in R$ $\Rightarrow (x, b), (a, b) \in R$ $\Rightarrow (x, b), (b, a) \in R$ $\Rightarrow (x, a) \in R$ $\Rightarrow x \in [a]$ $\therefore [b] \subseteq [a].$
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Therefore, $[a] = [b]$.

To prove: $[a] = [b] \Rightarrow aRb$

$$[a] = [b]$$

$$a \in [a] \Rightarrow a \in [b]$$

$$\Rightarrow (a, b) \in R$$

$$\Rightarrow a R b$$

(iii) $[a] \cap [b] \neq \phi$

$$\Rightarrow \text{Let } x \in [a] \cap [b]$$

$$\Rightarrow x \in [a] \text{ and } x \in [b]$$

$$\Rightarrow (x, a), (x, b) \in R \text{ for some } x \in R \text{ } (\because \text{By definition})$$

$$\Rightarrow (a, x), (x, b) \in R \text{ for some } x \in R \text{ } (\because R \text{ is symmetric})$$

$$\Rightarrow (a, b) \in R \text{ } (\because R \text{ is transitive})$$

$$\Rightarrow [a] = [b] \text{ } (\because \text{By (ii)})$$

3.6 Partial orders and Hasse diagram

Partial orders:

A relation R on a set A is said to be a partial order on A if R reflexive, antisymmetric and transitive. A set with a partial order R is called a partially ordered set or a poset. It is denoted by (A, R) . (e-g) $(\mathbb{Z}, \leq)$ is a poset.

Note: A digraph of a partial order has no cycle of length greater than one.

Hasse diagram:

The digraph of a partial order with the following conditions is called Hasse diagram.

- ❖ All vertices have a cycle of length one. (Since it is known, need not produce loop)
- ❖ If there is an edge from a to b and from b to c then there is an edge from a to c . (Since it is known, need not produce an edge from a to c)
- ❖ All edges point upward (Since it is known, need not put arrows in the edges)

1. Let $A = \{1, 2, 3, 4\}$ and $R = \{(1, 1), (1, 2), (2, 2), (2, 4), (1, 3), (3, 3), (3, 4), (1, 4), (4, 4)\}$.

Verify that R is a partial order on A . Write down the Hasse diagram for R .

$$(1, 1), (2, 2), (3, 3), (4, 4) \in R.$$

Therefore, $(x, x) \in R, \forall x \in A$.

$$(1, 3), (3, 4) \in R \Rightarrow (1, 4) \in R$$

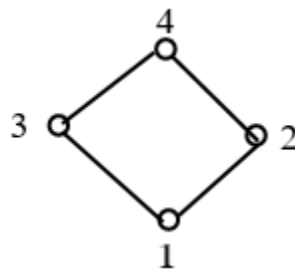
$$(x, y), (y, z) \in R \Rightarrow (x, z) \in R, \text{ for any } x, y, z \in A$$

R does not contain (x, y) and (y, x) with $x \neq y$.

Therefore, R is reflexive, anti-symmetric and transitive.

Therefore, R is a partial order on A .

Hasse diagram:



2. If R is a relation on a set $A = \{1, 2, 3, 4\}$ defined on xRy iff $x|y$. Prove that (A, R) is a poset. Draw its Hasse diagram.

$$R = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$$

$$(1, 1), (2, 2), (3, 3), (4, 4) \in R.$$

Therefore, $(x, x) \in R, \forall x \in A$.

$$(1, 2), (2, 4) \in R \Rightarrow (1, 4) \in R$$

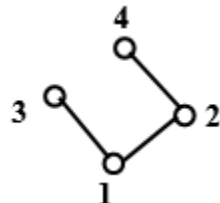
$$(x, y), (y, z) \in R \Rightarrow (x, z) \in R, \text{ for any } x, y, z \in A$$

R does not contain (x, y) and (y, x) with $x \neq y$.

Therefore, R is reflexive, anti-symmetric and transitive.

Therefore, R is a partial order on A .

Hasse diagram:



3. Let $A = \{1, 2, 3, 4, 6, 12\}$. On A define relation R by aRb iff $a|b$. Prove that R is a partial order on A . Draw the Hasse diagram for this relation.

$$R = \left\{ (1, 1), (1, 2), (1, 3), (1, 4), (1, 6), (1, 12), (2, 2), (2, 4), (2, 6), (2, 12), (3, 3), (3, 6), (3, 12), (4, 4), (4, 12), (6, 6), (6, 12), (12, 12) \right\}$$

$$(1, 1), (2, 2), (3, 3), (4, 4), (6, 6), (12, 12) \in R.$$

Therefore, $(x, x) \in R, \forall x \in A$.

$$x|y, y|z \in R \Rightarrow x|z$$

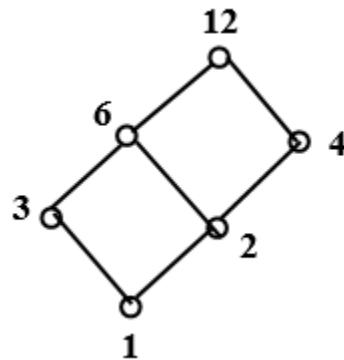
$$(x, y), (y, z) \in R \Rightarrow (x, z) \in R, \text{ for any } x, y, z \in A$$

R does not contain (x, y) and (y, x) with $x \neq y$.

Therefore, R is reflexive, anti-symmetric and transitive.

Therefore, R is a partial order on A .

Haase diagram:



4. Let $S = \{1, 2, 3\}$ and $P(S)$ the power set of S . On $P(S)$ define the relation R by xRy iff $X \subseteq Y$. Show that the relation is a partial order on $P(S)$. Draw its Hasse diagram.

$$P(S) = R = \{\{\}, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

$$= \{S_1, S_2, S_3, S_4, S_5, S_6, S_7, S_8\}$$

$$(i) \quad S_i \subseteq S_j, \forall 1 \leq i, j \leq 8$$

$$(S_i, S_i) \in R, \forall 1 \leq i \leq 8$$

$$(ii) \quad S_i \subseteq S_j, S_j \subseteq S_k \Rightarrow S_i \subseteq S_k, \forall 1 \leq i, j, k \leq 8$$

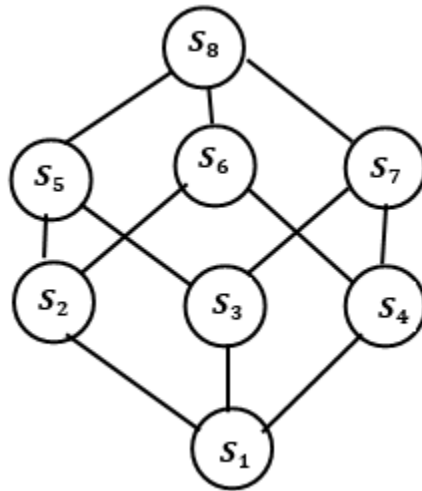
$$(S_i, S_j), (S_j, S_k) \in R \Rightarrow (S_i, S_k) \in R, \text{ for any } 1 \leq i, j, k \leq 8$$

$$(iii) \quad R \text{ does not contain } S_i \subseteq S_j \text{ and } S_j \subseteq S_i \text{ with } S_i \neq S_j$$

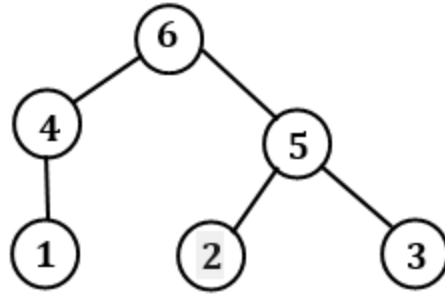
$$R \text{ does not contain } (S_i, S_j) \text{ and } (S_j, S_i) \text{ with } S_i \neq S_j$$

Therefore, R is reflexive, anti-symmetric and transitive.

Therefore, R is partial order relation.



5. The Hasse diagram of the partial order R on a set $A = \{1, 2, 3, 4, 5, 6\}$ is as given below:

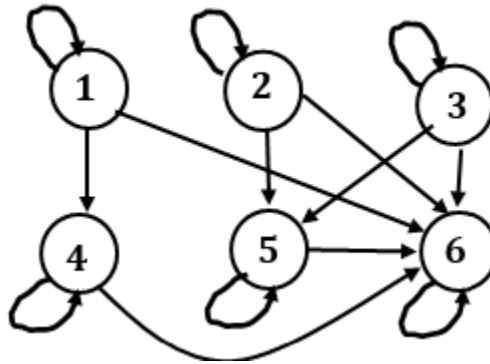


Write down R as a subset of $A \times A$. Construct its digraph.

$R =$

$\{(1, 1), (1, 4), (1, 6), (2, 2), (2, 5), (2, 6), (3, 3), (3, 5), (3, 6), (4, 4), (4, 6), (5, 5), (5, 6), (6, 6)\}$

Digraph:

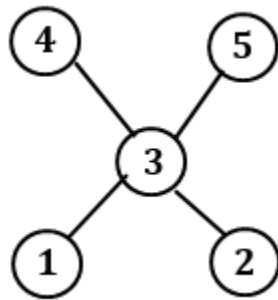


6. Draw the Hasse diagram of the relation R on $A = \{1, 2, 3, 4, 5\}$ whose matrix is as

given below: $M_R = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

$$R = \{(1, 1), (1, 3), (1, 4), (1, 5), (2, 2), (2, 3), (2, 4), (2, 5), (3, 3), (3, 4), (3, 5), (4, 4), (5, 5)\}$$

Hasse diagram:

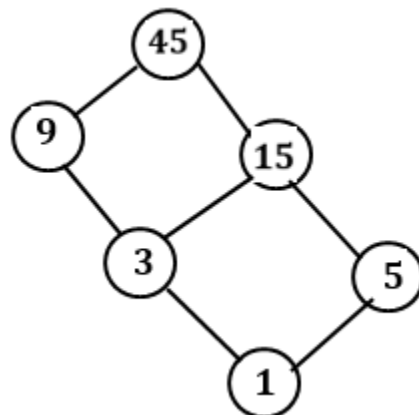


7. Write down the Hasse diagram for the positive divisors of 45.

$$A = \text{Set of all divisors of } 45 = \{1, 3, 5, 9, 15, 45\}$$

$$R = \{(1, 1), (1, 3), (1, 5), (1, 15), (1, 45), (3, 3), (3, 15), (3, 45), (9, 9), (9, 45), (15, 15), (15, 45), (45, 45)\}$$

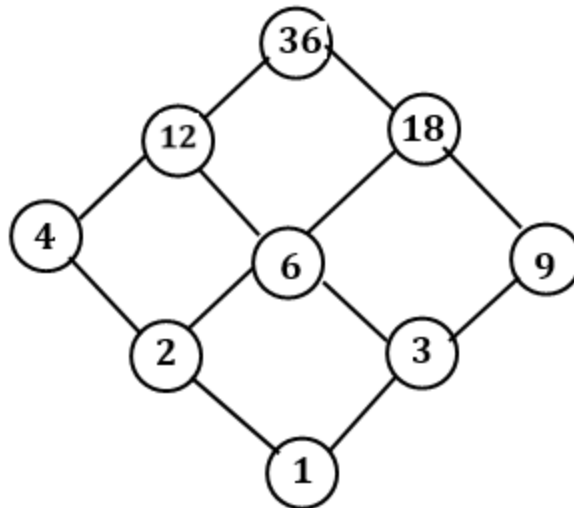
Hasse diagram:



8. Draw the Hasse diagram to represent the positive divisors of 36.

$A = \text{Set of all divisors of } 36 = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$

Hasse diagram:



3.7 Functions

Definition:

Let A and B be any two non empty sets. Then a function f from A to B is a relation such that for each $a \in A$, there is a unique $b \in B$.

Note:

- ❖ If $(a, b) \in f$ then $b = f(a)$ is the image a and a is a pre-image of b .
- ❖ If $f: A \rightarrow B$ is a function then A -Domain of f , B -Codomain of f , $f(A)$ -Range of f .

Example:

- ❖ $\{(1, a), (2, a), (2, c), (3, b)\}$ is not a function.
- ❖ $\{(x, z), (y, x), (y, y)\}$ is not a function.
- ❖ $\{(p, q), (q, q), (r, q)\}$ is a function.

Problems:

1. Let $A = \{1, 2, 3, 4\}$. Determine whether or not the following relations on A are functions: (i) $f = \{(2, 3), (1, 4), (2, 1), (3, 2), (4, 4)\}$ (ii) $g = \{(3, 1), (4, 2), (1, 1)\}$ (iii) $h = \{(2, 1), (3, 4), (1, 4), (4, 4)\}$.
 - (i) No. 2 has two images.
 - (ii) No. 2 doesn't have image.
 - (iii) Yes. Each element has a unique image.
2. Let a function $f: R \rightarrow R$ be defined by $f(x) = x^2 + 1$. Determine the images of the following subsets of R : $A = \{2, 3\}$, $B = \{-2, 0, 3\}$, $C = (0, 1)$, $D = [-6, 3]$.

$$f(A) = \{2^2 + 1, 3^2 + 1\} = \{5, 10\},$$

$$f(B) = \{(-2)^2 + 1, 0^2 + 1, 3^2 + 1\} = \{5, 1, 10\}$$

$$f(C) = \{x^2 + 1 \mid 0 < x < 1\}$$

$$f(D) = \{x^2 + 1 \mid -6 \leq x \leq 3\}$$

3. Let $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{6, 7, 8, 9, 10\}$. If a function $f: A \rightarrow B$ is defined by $\{(1, 7), (2, 7), (3, 8), (4, 6), (5, 9), (6, 9)\}$ determine $f^{-1}(6)$ and $f^{-1}(9)$. If $B_1 = \{7, 8\}$ and $B_2 = \{8, 9, 10\}$, find $f^{-1}(B_1)$ and $f^{-1}(B_2)$.

$$f^{-1}(6) = \text{Pre image of } 6 = \{4\}$$

$$f^{-1}(9) = \text{Set of pre images of } 9 = \{5, 6\}$$

$$f^{-1}(B_1) = \text{Set of pre images of } 7 \text{ and } 9 = \{1, 2, 3\}$$

$$f^{-1}(B_2) = \text{Set of pre images of } 8, 9 \text{ and } 10 = \{3, 5, 6\}$$

4. Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{3, 4, 5\}$. A function $f: A \rightarrow B$ is given by $f = \{(1, 3), (2, 3), (3, 4), (4, 5), (5, 4)\}$. Find $f^{-1}(3)$, $f^{-1}(4)$, $f^{-1}(B_1)$, $f^{-1}(B_2)$. Where $B_1 = \{3, 4\}$, $B_2 = \{4, 5\}$.

$$f^{-1}(3) = \text{Pre image of } 3 = \{1, 2\}$$

$$f^{-1}(4) = \text{Set of pre images of } 4 = \{3, 5\}$$

$$f^{-1}(B_1) = \text{Set of pre images of } 3 \text{ and } 4 = \{1, 2, 3, 5\}$$

$$f^{-1}(B_2) = \text{Set of pre images of } 4 \text{ and } 5 = \{3, 4, 5\}$$

5. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = \begin{cases} 3x - 5, & \text{for } x > 0 \\ -3x + 1, & \text{for } x \leq 0 \end{cases}$.

Determine $f(0)$, $f(-1)$, $f\left(\frac{5}{3}\right)$, $f\left(-\frac{5}{3}\right)$, $f(2)$

$$f(0) = -3(0) + 1 = 1$$

$$f(-1) = -3(-1) + 1 = 4$$

$$f\left(\frac{5}{3}\right) = 3\left(\frac{5}{3}\right) - 5 = 0$$

$$f\left(-\frac{5}{3}\right) = -3\left(-\frac{5}{3}\right) + 1 = 6$$

$$f(2) = 3(2) - 5 = 1$$

6. Let $f: R \rightarrow R$ be defined by $f(x) = \begin{cases} 3x - 5, & \text{for } x > 0 \\ -3x + 1, & \text{for } x \leq 0 \end{cases}$.

Find $f^{-1}(0), f^{-1}(1), f^{-1}(-1), f^{-1}(3), f^{-1}(-3), f^{-1}(-6)$

$f(x) = 3x - 5, x > 0$ $\Rightarrow 0 = 3x - 5, x > 0$ $\Rightarrow x = 5/3$	$f(x) = -3x + 1, x \leq 0$ $\Rightarrow 0 = -3x + 1, x \leq 0$ $\Rightarrow x = 1/3, \text{impossible.}$
--	--

Therefore, $f^{-1}(0) = \frac{5}{3}$

$f(x) = 3x - 5, x > 0$ $\Rightarrow 1 = 3x - 5, x > 0 \Rightarrow$ $\Rightarrow x = 2$	$f(x) = -3x + 1, x \leq 0$ $\Rightarrow 1 = -3x + 1, x \leq 0$ $\Rightarrow x = 0$
--	--

Therefore, $f^{-1}(1) = \{2, 0\}$

$f(x) = 3x - 5, x > 0$ $\Rightarrow -1 = 3x - 5, x > 0$ $\Rightarrow x = 4/3$	$f(x) = -3x + 1, x \leq 0$ $\Rightarrow -1 = -3x + 1, x \leq 0$ $\Rightarrow x = 2/3$
---	---

Therefore, $f^{-1}(-1) = \{4/3\}$

$f(x) = 3x - 5, x > 0$ $\Rightarrow 3 = 3x - 5, x > 0$ $\Rightarrow x = \frac{8}{3}$	$f(x) = -3x + 1, x \leq 0$ $\Rightarrow 3 = -3x + 1, x \leq 0$ $\Rightarrow x = -\frac{2}{3}, \text{impossible.}$
--	---

Therefore, $f^{-1}(3) = \left\{\frac{8}{3}, -\frac{2}{3}\right\}$

$f(x) = 3x - 5, x > 0$ $\Rightarrow -3 = 3x - 5, x > 0$ $\Rightarrow x = \frac{2}{3}$	$f(x) = -3x + 1, x \leq 0$ $\Rightarrow -3 = -3x + 1, x \leq 0$ $\Rightarrow x = \frac{4}{3}, \text{impossible.}$
---	---

Therefore, $f^{-1}(-3) = \left\{\frac{2}{3}\right\}$

$f(x) = 3x - 5, x > 0$ $\Rightarrow -6 = 3x - 5, x > 0$ $\Rightarrow x = -\frac{1}{3}, \text{impossible.}$	$f(x) = -3x + 1, x \leq 0$ $\Rightarrow -6 = -3x + 1, x \leq 0$ $\Rightarrow x = \frac{7}{3}, \text{impossible.}$
--	---

Therefore, $f^{-1}(-6) = \phi$

7. If $f: Z \rightarrow R$ is defined by $f(x) = x^2 + 5$ find $f^{-1}(\{6\})$, $f^{-1}([6, 7])$, $f^{-1}([6, 10])$, $f^{-1}([-4, 5])$, $f^{-1}([5, \infty))$.

$x = f^{-1}(6)$ $\Rightarrow f(x) = 6$ $\Rightarrow x^2 + 5 = 6$ $\Rightarrow x^2 = 1$ $\Rightarrow x = \pm 1$ $\Rightarrow x = \{-1, 1\}$	$x = f^{-1}([6, 7])$ $\Rightarrow f(x) = [6, 7]$ $\Rightarrow 6 \leq f(x) \leq 7$ $\Rightarrow 6 \leq x^2 + 5 \leq 7$ $\Rightarrow 1 \leq x^2 \leq 2$ $\Rightarrow x = \{-1, 1\}$	$x = f^{-1}([6, 10])$ $\Rightarrow f(x) = [6, 10]$ $\Rightarrow 6 \leq f(x) \leq 10$ $\Rightarrow 6 \leq x^2 + 5 \leq 10$ $\Rightarrow 1 \leq x^2 \leq 4$ $\Rightarrow x = \{-2, -1, 1, 2\}$
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$x = f^{-1}([-4, 5])$ $\Rightarrow f(x) = [-4, 5]$ $\Rightarrow -4 \leq f(x) \leq 5$ $\Rightarrow -4 \leq x^2 + 5 \leq 5$ $\Rightarrow -9 \leq x^2 \leq 0$ $\Rightarrow x = \{0\}$	$x = f^{-1}([5, \infty))$ $\Rightarrow f(x) = [5, \infty)$ $\Rightarrow -5 \leq f(x) < \infty$ $\Rightarrow -5 \leq x^2 + 5 < \infty$ $\Rightarrow -25 \leq x^2 < \infty$ $\Rightarrow x = Z$
---	--

8. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = \begin{cases} 3x - 5, & \text{for } x > 0 \\ -3x + 1, & \text{for } x \leq 0 \end{cases}$.

Find $f^{-1}([-5, 5])$ and $f^{-1}([-6, 5])$

$-5 \leq f(x) \leq 5$ $\Rightarrow -5 \leq 3x - 5 \leq 5, x > 0$ $\Rightarrow 0 \leq 3x \leq 10, x > 0$ $\Rightarrow 0 < x \leq 10/3$ $\Rightarrow x \in \left(0, \frac{10}{3}\right]$	$-5 \leq f(x) \leq 5$ $\Rightarrow -5 \leq -3x + 1 \leq 5, x \leq 0$ $\Rightarrow -6 \leq -3x \leq 4, x \leq 0$ $\Rightarrow -4 \leq 3x \leq 6, x \leq 0$ $\Rightarrow -\frac{4}{3} \leq x \leq 2, x \leq 0$ $\Rightarrow x \in [-4/3, 0]$
--	---

Therefore, $f^{-1}([-5, 5]) = \left[-\frac{4}{3}, \frac{10}{3}\right]$

$-6 \leq f(x) \leq 5$ $\Rightarrow -6 \leq 3x - 5 \leq 5, x > 0$ $\Rightarrow -1 \leq 3x \leq 10, x > 0$ $\Rightarrow -\frac{1}{3} \leq x \leq \frac{10}{3}, x > 0$ $\Rightarrow x \in \left(0, \frac{10}{3}\right]$	$-5 \leq f(x) \leq 5$ $\Rightarrow -6 \leq -3x + 1 \leq 5, x \leq 0$ $\Rightarrow -7 \leq -3x \leq 4, x \leq 0$ $\Rightarrow -4 \leq 3x \leq 7, x \leq 0$ $\Rightarrow -\frac{4}{3} \leq x \leq \frac{7}{3}, x \leq 0$ $\Rightarrow x \in [-4/3, 0]$
--	---

Therefore, $f^{-1}([-5, 5]) = \left[-\frac{4}{3}, \frac{10}{3}\right]$

9. Let $f: X \rightarrow Y$ be a function, A be an arbitrary non empty subset of X and C, D be arbitrary non empty subsets of Y. Then prove that $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$

(ii) $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$ (iii) $f^{-1}(\bar{C}) = \overline{f^{-1}(C)}$.

$$\begin{aligned}
 x \in f^{-1}(C \cup D) &\Leftrightarrow f(x) \in C \cup D \\
 &\Leftrightarrow f(x) \in C \text{ or } f(x) \in D \\
 &\Leftrightarrow x \in f^{-1}(C) \text{ or } x \in f^{-1}(D) \\
 &\Leftrightarrow x \in f^{-1}(C) \cup f^{-1}(D)
 \end{aligned}$$

Therefore, $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$

10. Let $f: X \rightarrow Y$ be a function, A be a arbitrary non empty subset of X and C, D be arbitrary non empty subsets of Y . Then prove that $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$.

$$\begin{aligned} x \in f^{-1}(C \cap D) &\Leftrightarrow f(x) \in C \cap D \\ &\Leftrightarrow f(x) \in C \text{ and } f(x) \in D \\ &\Leftrightarrow x \in f^{-1}(C) \text{ and } x \in f^{-1}(D) \\ &\Leftrightarrow x \in f^{-1}(C) \cap f^{-1}(D) \end{aligned}$$

Therefore, $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$

11. Let $f: X \rightarrow Y$ be a function, A be a arbitrary non empty subset of X and C, D be arbitrary non empty subsets of Y . Then prove that $f^{-1}(\bar{C}) = \overline{f^{-1}(C)}$.

$$\begin{aligned} x \in f^{-1}(\bar{C}) &\Leftrightarrow f(x) \in \bar{C} \\ &\Leftrightarrow f(x) \notin C \\ &\Leftrightarrow x \notin f^{-1}(C) \\ &\Leftrightarrow x \in \overline{f^{-1}(C)} \end{aligned}$$

Therefore, $f^{-1}(\bar{C}) = \overline{f^{-1}(C)}$.

12. If A and B are finite sets with $|A| = m$ and $|B| = n$. Find how many functions are possible from A to B .

Let $A = \{a_1, a_2, \dots, a_m\}, B = \{b_1, b_2, \dots, b_n\}, |A| = m, |B| = n$

The function f is of the form, $f = \{(a_1, x), (a_2, x), \dots, (a_m, x)\}$

x has n chances in (a_1, x) , x has n chances in (a_2, x) and so on.

Total number of functions from A to $B = n \times n \times \dots \times n = n^m$

13. If there are 2187 functions from A to B and $|B| = 3$. What is $|A|$?

If $|A| = m$, $|B| = n$ then the total number of functions from A to B = n^m

By data, the total number of functions from A to B = 2187

$$n^m = 2187$$

By data, $n = 3$.

$$3^m = 2187$$

$$m \log 3 = \log 2187$$

$$m = \frac{\log 2187}{\log 3} = 7$$

14. If $A = \{0, \pm 1, \pm 2\}$ and $f: A \rightarrow R$ is defined by $f(x) = x^2 - x + 1$, $x \in R$.
find the range of f .

$$f(0) = 0^2 - 0 + 1 = 1$$

$$f(-1) = (-1)^2 - (-1) + 1 = 3$$

$$f(1) = 1^2 - 1 + 1 = 1$$

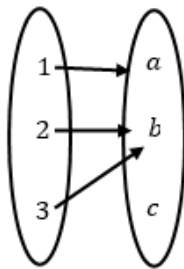
$$f(-2) = (-2)^2 - (-2) + 1 = 7$$

$$f(2) = 2^2 - 2 + 1 = 3$$

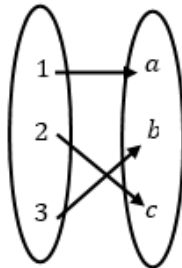
Therefore, $x = \{1, 3, 7\}$

3.8 1-1 and onto function

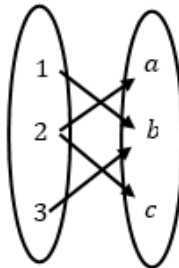
- ❖ **1-1 function and onto function or injective function:** Different elements of A have different images in B under f . That is, $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$, for $a_1, a_2 \in A$.
- ❖ **Onto function or surjective function:** Every element of B has a pre image in A under f . That is, $f(A) = B$.
- ❖ **1-1 correspondence or bijective function:** f is 1-1 and onto. That is, Every element of A has a unique image in B and every element of B has a unique element in A.
- ❖ **Identity function:** $f: A \rightarrow A$ such that $f(a) = a$, for every $a \in A$. It is denoted by I_A .
- ❖ **Constant function:** $f: A \rightarrow A$ such that $f(a) = c$, for every $a \in A$.
- ❖ **Constant function:** $f: A \rightarrow A$ such that $f(a) = c$, for every $a \in A$.



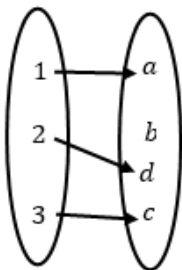
Neither 1-1 nor onto



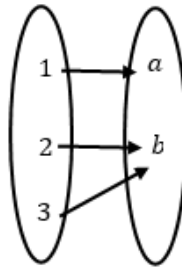
1-1 and onto



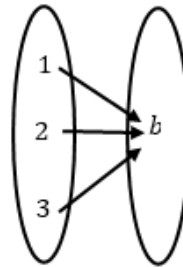
Not a function



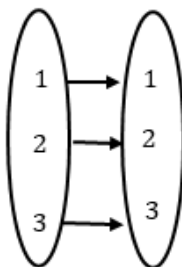
1-1 but not onto



Onto but not 1-1



Constant function



Identity function

Problems:

1. Find the nature of the following functions defined on $A = \{1, 2, 3\}$,

(i) $f = \{(1, 1), (2, 2), (3, 3)\}$

(ii) $g = \{(1, 2), (2, 2), (3, 2)\}$

(iii) $h = \{(1, 2), (2, 2), (3, 1)\}$

(iv) $p = \{(1, 2), (2, 3), (3, 1)\}$

(i) $a = f(a), \forall a \in A$. Therefore, f is a identity function.

(ii) $g(a) = 2, \forall a \in A$. Therefore, g is a constant function.

(iii) $h(1) = h(2) = 2$. 3 has no pre-image. Therefore, h is neither 1-1 nor onto.

(iv) $p(1) = 2, p(2) = 3, p(3) = 1$. All are distinct images. Range and co-domain are same. Therefore, p is 1-1 and onto.

2. In each of the following cases determine f is 1-1 or onto or both or neither:

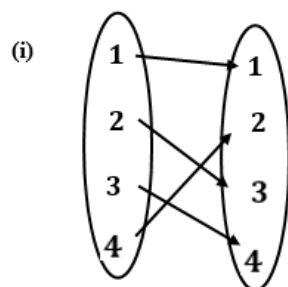
(i) $A = B = \{1, 2, 3, 4\}$ and $f = \{(1, 1), (2, 3), (3, 4), (4, 2)\}$

(ii) $A = \{a, b, c\}, B = \{1, 2, 3, 4\}, f = \{(a, 1), (b, 1), (c, 3)\}$

(iii) $A = \{1, 2, 3\}, B = \{1, 2, 3, 4, 5\}, f = \{(1, 1), (2, 3), (3, 4)\}$

(iv) $A = \{1, 2, 3\}, B = \{1, 2, 3, 4, 5\}, f = \{(1, 1), (2, 3), (3, 3)\}$

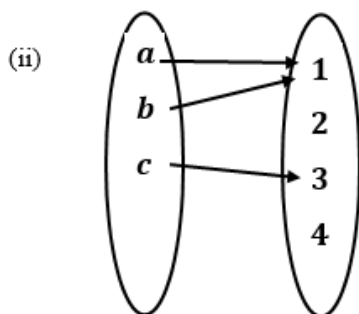
(v) $A = \{1, 2, 3, 4\}, B = \{a, b, c, d\}, f = \{(1, a), (2, a), (3, d), (4, c)\}$



Different elements of A have different elements in B.

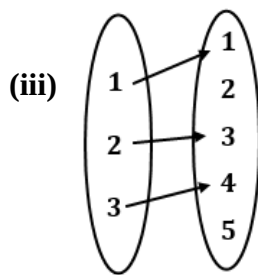
Also $f(A) = B$. Therefore, it is 1-1 correspondence

(or) Bijective function.



$f(a) = f(b) = 1$ and $f(A) = \{1, 3\} \neq B$.

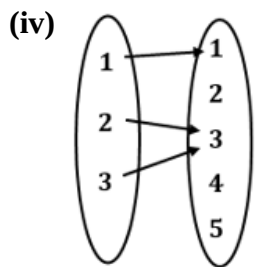
Therefore, this function is neither 1-1 nor onto.



Different elements of A have different elements in B.

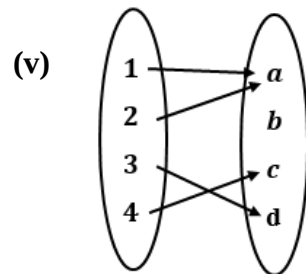
But $f(A) = \{1, 3, 4\} \neq B$.

Therefore, this function is 1-1 but not onto.



$f(2) = f(3) = 3$ and $f(A) = \{1, 3\} \neq B$.

Therefore, this function is 1-1 but not onto.



$f(1) = f(2) = a$ and $f(A) = \{a, c, d\} \neq B$.

Therefore, this function is 1-1 but not onto.

3. The functions $f: R \rightarrow R$ and $g: R \rightarrow R$ are defined by $f(x) = 3x + 7$ for all $x \in R$ and $g(x) = x(x^3 - 1)$ for all $x \in R$. Verify f is 1-1 but g is not.

$$\begin{aligned}f(x_1) = f(x_2) &\Rightarrow 3x_1 + 7 = 3x_2 + 7 \\&\Rightarrow 3x_1 = 3x_2 \\&\Rightarrow x_1 = x_2, \forall x_1, x_2 \in R\end{aligned}$$

Therefore, f is 1-1.

$$g(0) = g(1) = 0, \text{ but } 0 \neq 1.$$

Therefore, g is not 1-1.

4. The function $f: Z \rightarrow Z$ be defined by $f(a) = a + 1$ for $a \in Z$. Find whether 1-1 or onto.

$$\begin{aligned}f(a) = f(b) &\Rightarrow a + 1 = b + 1 \\&\Rightarrow a = b, \forall a, b \in Z\end{aligned}$$

Therefore, f is 1-1.

For any $b \in Z$, we can find a pre image $b - 1$ as $f(b - 1) = b - 1 + 1 = b$.

Therefore, f is onto.

5. The function $f: Z \times Z \rightarrow Z$ is defined by $f(x, y) = 2x + 3y$. P.T f is onto but not 1-1.

To prove: f is 1-1

$$n = 4n - 3n = 2(2n) + 3(-n), \forall n \in Z.$$

Therefore, for any $n \in Z$, we can find a pre-image $(2n, -n)$ such that

$$f(2n, -n) = 2(2n) + 3(-n) = n. \text{ Therefore, } f \text{ is onto.}$$

To prove: f is not onto

$$f(0, 2) = 2(0) + 3(2) = 6$$

$$f(3, 0) = 2(3) + 3(0) = 6$$

$$f(0, 2) = f(3, 0) = 6. \text{ But } (2, 0) \neq (3, 0)$$

Therefore, f is onto.

6. If $f: R \rightarrow R$ in each of the following cases determine f is 1-1 or onto or both:

(i) $f(x) = 2x - 3$ (ii) $f(x) = x^2 + x$ (iii) $f(x) = e^x$ (iv) $f(x) = \sin x$

(i) Check whether f is 1-1.

$$f(x_1) = f(x_2) \Rightarrow 2x_1 - 3 = 2x_2 - 3$$

$$\Rightarrow 2x_1 = 2x_2$$

$$\Rightarrow x_1 = x_2, \forall x_1, x_2 \in R$$

Therefore, f is 1-1.

Check whether f is onto.

$$y = 2x - 3 \Rightarrow x = \frac{y+3}{2}$$

For any $y \in R$, there exist $x \in R$ such that $f(x) = f\left(\frac{y+3}{2}\right) = 2\left(\frac{y+3}{2}\right) - 3 = y$.

Therefore, f is onto.

(ii) Check whether f is 1-1.

$$f(-1) = (-1)^2 - 1 = 0$$

$$f(0) = 0^2 + 0 = 0$$

$$f(-1) = f(0), \text{ But } -1 \neq 0$$

Therefore, f is not 1-1.

Check whether f is onto.

If $f(x) = -2$, then $x^2 + x + 2 = 0$ has complex roots.

Therefore, f is not onto.

7. If $f: R \rightarrow R$ in each of the following cases determine f is 1-1 or onto or both:

(i) $f(x) = e^x$ (ii) $f(x) = \sin x$

(i) Check whether f is 1-1 .

$$f(x_1) = f(x_2) \Rightarrow e^{x_1} = e^{x_2}$$

$$\Rightarrow x_1 = x_2, \forall x_1, x_2 \in R$$

Therefore, f is 1-1.

Check whether f is onto.

$$y = e^x \Rightarrow x = \log y$$

For any $y \in R$, there exist $x \in R$ such that $f(x) = f(\log y) = e^{\log y} = y$.

Therefore, f is onto.

(ii) Check whether f is 1-1

$$f(0) = \sin 0 = 0$$

$$f(\pi) = \sin \pi = 0$$

$$f(0) = f(\pi), \text{ But } 0 \neq \pi$$

Therefore, f is not 1-1.

Check whether f is onto.

Range = $[-1, 1] \subset R$. Range is a subset of co-domain.

Therefore, f is not onto.

8. Let $f: X \rightarrow Y$ be a function and A and B be arbitrary non empty subsets of X . then prove the following: (i) If $A \subseteq B$ then $f(A) \subseteq f(B)$ (ii) $f(A \cup B) = f(A) \cup f(B)$.

$$\begin{aligned} \text{(i)} \quad y \in f(A) &\Rightarrow y = f(x) \text{ for some } x \in A \\ &\Rightarrow y = f(x) \text{ for some } x \in B \quad (\because A \subseteq B) \\ &\Rightarrow y \in f(B) \\ \text{Therefore, } f(A) &\subseteq f(B) \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad y \in f(A \cup B) &\Rightarrow y = f(x) \text{ for some } x \in A \cup B \\ &\Rightarrow y = f(x) \text{ for } x \in A \text{ or } x \in B \\ &\Rightarrow y \in f(A) \text{ or } y \in f(B) \\ &\Rightarrow y \in f(A) \cup f(B) \end{aligned}$$

$$\text{Therefore, } f(A \cup B) \subseteq f(A) \cup f(B) \quad \text{-----} \quad (1)$$

$$A \subseteq A \cup B, B \subseteq A \cup B$$

$$\Rightarrow f(A) \subseteq f(A \cup B), f(B) \subseteq f(A \cup B)$$

$$\Rightarrow f(A) \cup f(B) \subseteq f(A \cup B) \quad \text{-----} \quad (2)$$

9. Let A and B be finite sets and f be a function from A to B . Then prove the following: (i) If f is 1-1, $|A| \leq |B|$ (ii) If f is onto, $|B| \leq |A|$ (iii) If f is bijection, $|A| = |B|$.

(i) Let $|A| = m, |B| = n$ and $A = \{a_1, a_2, \dots, a_m\}$.
 $a_1 \neq a_2 \neq \dots \neq a_m \Rightarrow f(a_1) \neq f(a_2) \neq \dots \neq f(a_m)$
 $\Rightarrow$ All images in B are distinct.
 $\Rightarrow B$ has at least m elements.
 $\Rightarrow |B| \geq m$
 $\Rightarrow |B| \geq |A|$
 $\Rightarrow |A| \leq |B|$

(ii) f is onto $\Rightarrow$ For each $b \in B$, there exists $a \in A$ such that $f(a) = b$.
 $\Rightarrow$ As f is a function, all pre-images in A are distinct.
 $\Rightarrow A$ has at least n elements.
 $\Rightarrow |A| \geq n$
 $\Rightarrow |A| \geq |B|$
 $\Rightarrow |B| \leq |A|$

(iii) f is a bijection $\Rightarrow f$ is 1-1 and onto
 $\Rightarrow |A| \leq |B|$ and $|B| \leq |A|$
 $\Rightarrow |A| = |B|$

10. Let A and B be finite sets with $|A| = m$ and $|B| = n$ How many 1-1 functions are possible from A to B?

Let $A = \{a_1, a_2, \dots, a_m\}$ and $B = \{b_1, b_2, \dots, b_n\}$

f is 1-1 $\Rightarrow |A| \leq |B| \Rightarrow m \leq n$.

$f: A \rightarrow B$ is of the form $f = \{(a_1, x), (a_2, x), \dots, (a_m, x)\}$

There are n choices for x in (a_1, x) .

There are $n - 1$ choices for x in (a_2, x) .

....

There are $n - (m - 1)$ choices for x in (a_m, x) .

The total number of choices for $x = n(n - 1)(n - 2) \dots (n - (m - 1))$

$$= \frac{n!}{(n-m)!}$$

Therefore, number of 1-1 functions from A to B = $\frac{n!}{(n-m)!}$

11. If there are 60 1-1 functions from A to B and $|A| = 3$, what is $|B|$?

By data, $m = |A| = 3$.

Number of 1-1 functions from A to B = $\frac{n!}{(n-m)!}$

$$60 = \frac{n!}{(n-3)!}$$

$$60 = n(n - 1)(n - 2)$$

Therefore, $n = 5$.

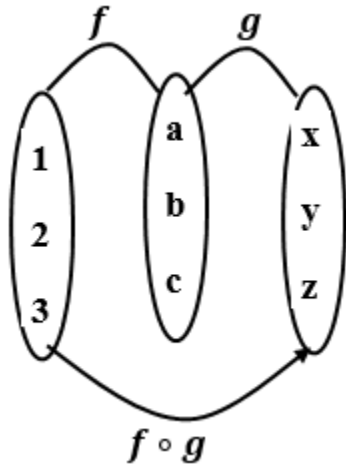
3.9 composition of functions

Introduction:

- ❖ Consider three non empty sets A , B and C and functions $f: A \rightarrow B$ and $g: B \rightarrow C$.

The composition of these two functions is defined as

$g \circ f: A \rightarrow C$ with $(g \circ f)(a) = g[f(a)], \forall a \in A$



Problems:

1. Let $A = \{1, 2, 3, 4\}$, $B = \{a, b, c\}$ and $C = \{w, x, y, z\}$ with $f: A \rightarrow B$ and $g: B \rightarrow C$ given by $f = \{(1, a), (2, a), (3, b), (4, c)\}$ and $g = \{(a, x), (b, y), (c, z)\}$ find $g \circ f$

$$(g \circ f)(1) = g[f(1)] = g(a) = x$$

$$(g \circ f)(2) = g[f(2)] = g(a) = x$$

$$(g \circ f)(3) = g[f(3)] = g(b) = y$$

$$(g \circ f)(4) = g[f(4)] = g(c) = z$$

$$\text{Therefore, } g \circ f = \{(1, x), (2, x), (3, y), (4, z)\}$$

2. If $A = \{1, 2, 3, 4\}$ and $f = \{(1, 2), (2, 2), (3, 1), (4, 3)\}$ find f^2 and f^3 .

To find: f^2

$$f^2(1) = (f \circ f)(1) = f[f(1)] = f(2) = 2$$

$$f^2(2) = (f \circ f)(2) = f[f(2)] = f(2) = 2$$

$$f^2(3) = (f \circ f)(3) = f[f(3)] = f(1) = 2$$

$$f^2(4) = (f \circ f)(4) = f[f(4)] = f(3) = 1$$

$$\text{Therefore, } f^2 = \{(1, 2), (2, 2), (3, 2), (4, 1)\}$$

To find: f^3

$$f^3(1) = (f \circ f^2)(1) = f[f^2(1)] = f(2) = 2$$

$$f^3(2) = (f \circ f^2)(2) = f[f^2(2)] = f(2) = 2$$

$$f^3(3) = (f \circ f^2)(3) = f[f^2(3)] = f(2) = 2$$

$$f^3(4) = (f \circ f^2)(4) = f[f^2(4)] = f(1) = 2$$

$$\text{Therefore, } f^3 = \{(1, 2), (2, 2), (3, 2), (4, 2)\}$$

3. Consider the functions f and g defined by $f(x) = x^3$ and $g(x) = x^2 + 1, \forall x \in R$.

Find $g \circ f, f \circ g, f^2$ and g^2 .

$$(g \circ f)(x) = g[f(x)] = g(x^3) = (x^3)^2 + 1 = x^6 + 1$$

$$(f \circ g)(x) = f[g(x)] = f(x^2 + 1) = (x^2 + 1)^3$$

$$f^2(x) = (f \circ f)(x) = f[f(x)] = f(x^3) = (x^3)^3 = x^9$$

$$g^2(x) = (g \circ g)(x) = g[g(x)] = g(x^2 + 1) = (x^2 + 1)^2 + 1$$

4. Consider the functions f and g defined by $f(x) = ax + b$ and $g(x) = 1 - x + x^2$. If $(g \circ f)(x) = 9x^2 - 9x + 3$ determine a and b .

$$9x^2 - 9x + 3 = (g \circ f)(x) = g[f(x)] = g(ax + b)$$

$$= 1 - (ax + b) + (ax + b)^2$$

$$= a^2x^2 + (2ab - a)x + (1 - b + b^2)$$

Equating coefficients on both the sides,

$$a^2 = 9, 2ab - a = -9, 1 - b + b^2 = 3$$

$$a = \pm 3, a(2b - 1) = -9, b(b - 1) = 2$$

$$\text{If } a = 3, 2b - 1 = -3, b = -1$$

$$\text{If } a = -3, 2b - 1 = 3, b = 2$$

$$\text{Therefore, } a = 3, b = -1 \text{ (or) } a = -3, b = 2.$$

5. Let f, g and h be three functions from Z to Z defined by $f(x) = x - 1, g(x) = 2x$

and $h(x) = \begin{cases} 7, & \text{if } x \text{ is even} \\ 4, & \text{if } x \text{ is odd} \end{cases}$ Find $fog, gof, goh, hog, fo(goh), (fog)oh$.

$$(f \circ g)(x) = f[g(x)] = f(2x) = 2x - 1$$

$$(g \circ f)(x) = g[f(x)] = g(x - 1) = 2(x - 1)$$

$$(g \circ h)(x) = g[h(x)] = 2[h(x)] = \begin{cases} 2(7), & \text{if } x \text{ is even} \\ 2(4), & \text{if } x \text{ is odd} \end{cases} = \begin{cases} 14, & \text{if } x \text{ is even} \\ 8, & \text{if } x \text{ is odd} \end{cases}$$

$$(h \circ g)(x) = h[g(x)] = h(2x) = 7$$

$$\begin{aligned} [f \circ (g \circ h)](x) &= f[(g \circ h)(x)] \\ &= \begin{cases} f(14), & \text{if } x \text{ is even} \\ f(8), & \text{if } x \text{ is odd} \end{cases} \\ &= \begin{cases} 14 - 1, & \text{if } x \text{ is even} \\ 8 - 1, & \text{if } x \text{ is odd} \end{cases} \\ &= \begin{cases} 13, & \text{if } x \text{ is even} \\ 7, & \text{if } x \text{ is odd} \end{cases} \end{aligned}$$

$$\begin{aligned} [(f \circ g) \circ h](x) &= (f \circ g)[h(x)] \\ &= \begin{cases} (f \circ g)(7), & \text{if } x \text{ is even} \\ (f \circ g)(4), & \text{if } x \text{ is odd} \end{cases} \\ &= \begin{cases} f[g(7)], & \text{if } x \text{ is even} \\ f[g(4)], & \text{if } x \text{ is odd} \end{cases} \\ &= \begin{cases} f[2(7)], & \text{if } x \text{ is even} \\ f[2(4)], & \text{if } x \text{ is odd} \end{cases} \\ &= \begin{cases} f(14), & \text{if } x \text{ is even} \\ f(8), & \text{if } x \text{ is odd} \end{cases} \\ &= \begin{cases} 14 - 1, & \text{if } x \text{ is even} \\ 8 - 1, & \text{if } x \text{ is odd} \end{cases} \\ &= \begin{cases} 13, & \text{if } x \text{ is even} \\ 7, & \text{if } x \text{ is odd} \end{cases} \end{aligned}$$

Basic results:

(i) $f \circ (g \circ h) = (f \circ g) \circ h$

(ii) $f \circ g \neq g \circ f$

6. Let $f: A \rightarrow B$ and $g: B \rightarrow C$ be any two functions. Prove the following:

(i) If f and g are 1-1, so is $g \circ f$. (ii) If $g \circ f$ is 1-1 then f is 1-1.

(i) Since f is 1-1, $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ ----- (1)

Since g is 1-1, $g(y_1) = g(y_2) \Rightarrow y_1 = y_2$ ----- (2)

$$\begin{aligned}(g \circ f)(x_1) &= (g \circ f)(x_2) \Rightarrow g[f(x_1)] = g[f(x_2)] \\ &\Rightarrow f(x_1) = f(x_2) \quad [\because \text{by (2)}] \\ &\Rightarrow x_1 = x_2 \quad [\because \text{by (1)}]\end{aligned}$$

Therefore, $g \circ f$ is 1-1.

(ii) Since $g \circ f$ is 1-1, $(g \circ f)(x_1) = (g \circ f)(x_2) \Rightarrow x_1 = x_2$ ----- (3)

$$\begin{aligned}f(x_1) &= f(x_2) \Rightarrow g[f(x_1)] = g[f(x_2)] \\ &\Rightarrow (g \circ f)(x_1) = (g \circ f)(x_2) \\ &\Rightarrow x_1 = x_2 \quad [\because \text{by (2)}]\end{aligned}$$

Therefore, f is 1-1.

7. Let $f: A \rightarrow B$ and $g: B \rightarrow C$ be any two functions. Prove the following:

(i) If f and g are onto, so is $g \circ f$. (ii) If $g \circ f$ is onto, then g is onto.

(i) Since f is onto, for any $b \in B$, there exists $a \in A$ such that $f(a) = b$.

Since g is onto, for any $c \in C$, there exists $b \in B$ such that $g(b) = c$.

For any $c \in C$, there exists $a \in A$ such that

$$(g \circ f)(a) = g[f(a)] = g(b) = c.$$

Therefore, $g \circ f$ is onto.

(ii) Since $g \circ f$ is onto, for any $c \in C$, there exists $a \in A$ such that $(g \circ f)(a) = c$.

For any $c \in C$, there exists $b \in B$ such that

$$g[b] = g[f(a)] = (g \circ f)(a) = c.$$

Therefore, $g \circ f$ is onto.

8. Let $f: A \rightarrow B, g: B \rightarrow C, h: C \rightarrow D$ be three functions. Then $(hog)of = ho(gof)$

$$[(hog)of](a) = (h \circ g)[f(a)] = (h \circ g)(b) = h[g(b)] = h(c) = d, \forall a \in A$$

$$[ho(gof)](a) = h[(g \circ f)(a)] = h\{g[f(a)]\} = h[g(b)] = h(c) = d, \forall a \in A$$

Therefore, $(hog)of = ho(gof)$

9. Let f, g and h be three functions from Z to Z defined by $f(x) = x - 1, g(x) = 3x$

and $h(x) = \begin{cases} 0, & \text{if } x \text{ is even} \\ 1, & \text{if } x \text{ is odd} \end{cases}$ Verify that $fo(goh) = (fog)oh$.

$$\begin{aligned} [fo(goh)](x) &= f\{(g \circ h)(x)\} \\ &= f\{g[h(x)]\} \\ &= \begin{cases} f[g(0)], & \text{if } x \text{ is even} \\ f[g(1)], & \text{if } x \text{ is odd} \end{cases} \\ &= \begin{cases} f[3(0)], & \text{if } x \text{ is even} \\ f[3(1)], & \text{if } x \text{ is odd} \end{cases} \\ &= \begin{cases} f(0), & \text{if } x \text{ is even} \\ f(3), & \text{if } x \text{ is odd} \end{cases} \\ &= \begin{cases} 0 - 1, & \text{if } x \text{ is even} \\ 3 - 1, & \text{if } x \text{ is odd} \end{cases} \\ &= \begin{cases} -1, & \text{if } x \text{ is even} \\ 2, & \text{if } x \text{ is odd} \end{cases} \end{aligned}$$

$$\begin{aligned} [(f \circ g) \circ h](x) &= (f \circ g)[h(x)] \\ &= \begin{cases} (f \circ g)(0), & \text{if } x \text{ is even} \\ (f \circ g)(1), & \text{if } x \text{ is odd} \end{cases} \\ &= \begin{cases} f[g(0)], & \text{if } x \text{ is even} \\ f[g(1)], & \text{if } x \text{ is odd} \end{cases} \\ &= \begin{cases} f[3(0)], & \text{if } x \text{ is even} \\ f[3(1)], & \text{if } x \text{ is odd} \end{cases} \\ &= \begin{cases} f(0), & \text{if } x \text{ is even} \\ f(3), & \text{if } x \text{ is odd} \end{cases} \\ &= \begin{cases} 0 - 1, & \text{if } x \text{ is even} \\ 3 - 1, & \text{if } x \text{ is odd} \end{cases} \\ &= \begin{cases} -1, & \text{if } x \text{ is even} \\ 2, & \text{if } x \text{ is odd} \end{cases} \end{aligned}$$

10. Let f and g be functions from $\mathbb{R}$ to $\mathbb{R}$ defined by $f(x) = x^2$ and $g(x) = x + 5$. Prove that $g \circ f \neq f \circ g$.

$$(g \circ f)(x) = g[f(x)] = g(x^2) = x^2 + 5, \forall x \in \mathbb{R}$$

$$(f \circ g)(x) = f[g(x)] = f(x + 5) = (x + 5)^2, \forall x \in \mathbb{R}$$

Therefore, $g \circ f \neq f \circ g$

11. Let f and g be functions from $\mathbb{R}$ to $\mathbb{R}$ defined by $f(x) = x^2$ and $g(x) = x + 5$, $h(x) = \sqrt{x^2 + 2}$. Verify that $f \circ (g \circ h) = (f \circ g) \circ h$.

$$\begin{aligned} [f \circ (g \circ h)](x) &= f\{(g \circ h)(x)\} \\ &= f\{g[h(x)]\} \\ &= f\left\{g\left(\sqrt{x^2 + 2}\right)\right\} \\ &= f\left\{\sqrt{x^2 + 2} + 5\right\} \\ &= \left(\sqrt{x^2 + 2} + 5\right)^2 \end{aligned}$$

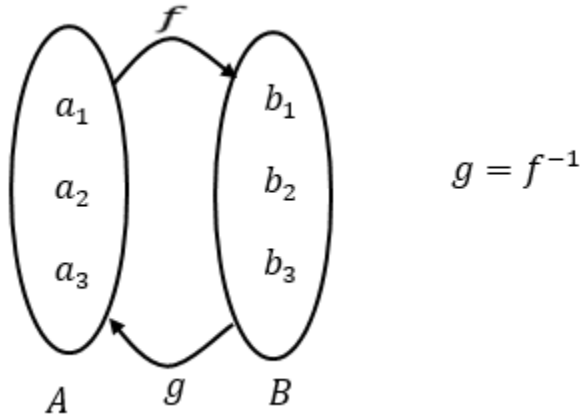
$$\begin{aligned} [(f \circ g) \circ h](x) &= (f \circ g)[h(x)] \\ &= (f \circ g)\left[\sqrt{x^2 + 2}\right] \\ &= f\left\{g\left(\sqrt{x^2 + 2}\right)\right\} \\ &= f\left\{\sqrt{x^2 + 2} + 5\right\} \\ &= \left(\sqrt{x^2 + 2} + 5\right)^2 \end{aligned}$$

$$[f \circ (g \circ h)](x) = [(f \circ g) \circ h](x), \text{ for any } x \in \mathbb{R}$$

Therefore, $f \circ (g \circ h) = (f \circ g) \circ h$.

3.10 Invertible function

A function $f: A \rightarrow B$ is said to be invertible if there exists a function $g: B \rightarrow A$ such that $g \circ f = I_A$ and $f \circ g = I_B$. Where I_A - identity function on A, I_B - Identity function on B.



- Let $A = \{1, 2, 3, 4\}$ and f, g be functions from A to A given by $f = \{(1, 4), (2, 1), (3, 2), (4, 3)\}$ and $g = \{(1, 2), (2, 3), (3, 4), (4, 1)\}$. Prove that g and f inverses of each other.

$$(g \circ f)(1) = g[f(1)] = g(4) = 1 = I_A(1)$$

$$(g \circ f)(2) = g[f(2)] = g(1) = 2 = I_A(2)$$

$$(g \circ f)(3) = g[f(3)] = g(2) = 3 = I_A(3)$$

$$(g \circ f)(4) = g[f(4)] = g(3) = 4 = I_A(4)$$

$$(g \circ f)(x) = I_A(x), \forall x \in A$$

Therefore, $g \circ f = I_A$ and hence g is the inverse of f .

$$(f \circ g)(1) = f[g(1)] = f(2) = 1 = I_A(1)$$

$$(f \circ g)(2) = f[g(2)] = f(3) = 2 = I_A(2)$$

$$(f \circ g)(3) = f[g(3)] = f(4) = 3 = I_A(3)$$

$$(f \circ g)(4) = f[g(4)] = f(1) = 4 = I_A(4)$$

$$(f \circ g)(x) = I_A(x), \forall x \in A$$

Therefore, $f \circ g = I_A$ and hence f is the inverse of g .

2. Consider the function $f: R \rightarrow R$ defined by $f(x) = 2x + 5$. Let a function $g: R \rightarrow R$ be defined by $g(x) = \frac{1}{2}(x - 5)$. Prove that g is an inverse of f .

$$(g \circ f)(x) = g[f(x)] = g(2x + 5) = \frac{1}{2}(2x + 5 - 5) = x = I_R(x), \quad \forall x \in R.$$

$$(f \circ g)(x) = f[g(x)] = f\left(\frac{x - 5}{2}\right) = 2\left(\frac{x - 5}{2}\right) + 5 = x = I_R(x), \quad \forall x \in R.$$

Therefore, $g \circ f = f \circ g = I_R$ and hence g is an inverse of h .

3. Prove that if a function $f: A \rightarrow B$ is invertible then it has a unique inverse.

Let g be an inverse of f such that $g \circ f = I_A$ and $f \circ g = I_B$

Let h be another inverse of f such that $h \circ f = I_A$ and $f \circ h = I_B$

Then $h = h \circ I_B = h \circ (f \circ g) = (h \circ f) \circ g = I_A \circ g = g$.

Therefore, f has a unique inverse.

4. Prove that If $f(a) = b$ then $f^{-1}(b) = a$, provided f is invertible.

Let f be invertible and g be the inverse of f . Then $g \circ f = I_A$

Let $f(a) = b$.

Then $a = I_A(a) = (g \circ f)(a) = g[f(a)] = g(b) = f^{-1}(b)$

Therefore, $f^{-1}(b) = a$

5. A function $f: A \rightarrow B$ is invertible if and only if it is 1-1 and onto.

IF PART:

Given: f is invertible.

To prove: f is 1-1 and onto.

f is invertible.

$\Rightarrow$ There exist a unique function $g: B \rightarrow A$ such that $g \circ f = I_A, f \circ g = I_B$.

$$f(a_1) = f(a_2) \Rightarrow g[f(a_1)] = g[f(a_2)]$$

$$\Rightarrow (g \circ f)(a_1) = (g \circ f)(a_2)$$

$$\Rightarrow I_A(a_1) = I_A(a_2)$$

$$\Rightarrow a_1 = a_2 \text{ for any } a_1, a_2 \in A$$

Therefore, f is 1-1.

For any b in B , there exists $a \in A$ such that $b = I_B(b) = (f \circ g)(b) = f[g(b)] = f(a)$.

Therefore, f is onto.

ONLY IF PART:

Given: f is 1-1 and onto.

To prove: f is invertible.

f is 1-1 and onto.

$\Rightarrow$ For any b in B , there exists a unique $a \in A$ such that $b = f(a)$.

Define $g: B \rightarrow A$ by $g(b) = a$.

Then $(g \circ f)(a) = g[f(a)] = g(b) = a = I_A(a)$, for any $a \in A$

$(f \circ g)(b) = f[g(b)] = f(a) = b = I_B(b)$, for any $b \in B$

Therefore, $g \circ f = I_A$ and $f \circ g = I_B$ and hence f is invertible.

Note:

f is invertible $\Leftrightarrow f$ is 1-1 and onto.

6. Consider the function $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = x^2, \forall x \in \mathbb{R}$. Is f invertible?

$f(a) = f(-a) = a^2$. But $a \neq -a$.

Therefore, f is not 1-1.

Therefore, f is not invertible.

Note:

If $f: A \rightarrow B$ is an invertible function then $f \circ f^{-1} = I_B$ and $f^{-1} \circ f = I_A$.

If $g: B \rightarrow C$ is an invertible function then $g \circ g^{-1} = I_C$ and $g^{-1} \circ g = I_B$.

7. If $f: A \rightarrow B$ and $g: B \rightarrow C$ are invertible functions then $g \circ f: A \rightarrow C$ is an invertible function and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

f and g are invertible functions

$\Rightarrow f$ and g are 1-1 and onto.

$\Rightarrow g \circ f$ is 1-1 and onto.

$\Rightarrow g \circ f$ is invertible.

$f: A \rightarrow B$ and $g: B \rightarrow C$ are invertible functions.

$\Rightarrow f^{-1}: B \rightarrow A$ and $g^{-1}: C \rightarrow B$ are invertible.

$\Rightarrow f^{-1} \circ g^{-1}: C \rightarrow A$ is invertible.

$\Rightarrow h: C \rightarrow A$ is invertible. Where $h = f^{-1} \circ g^{-1}$.

Therefore,

$$(g \circ f) \circ h = (g \circ f) \circ (f^{-1} \circ g^{-1}) = g \circ (f \circ f^{-1}) \circ g^{-1} = g \circ I_B \circ g^{-1} = g \circ g^{-1} = I_C$$

$$h \circ (g \circ f) = (f^{-1} \circ g^{-1}) \circ (g \circ f) = f^{-1} \circ (g^{-1} \circ g) \circ f = f^{-1} \circ I_B \circ f = f^{-1} \circ f = I_A$$

Therefore,

h is the inverse of $g \circ f$.

$$\Rightarrow (g \circ f)^{-1} = h$$

$$\Rightarrow (g \circ f)^{-1} = f^{-1} \circ g^{-1}$$

8. Let $A = B = \mathbb{R}$. The set of all real numbers and the functions $f: A \rightarrow B$ and $g: B \rightarrow A$ be defined by $f(x) = 2x^3 - 1, \forall x \in A$ and $g(y) = \left\{\frac{1}{2}(y + 1)\right\}^{\frac{1}{3}}, \forall y \in B$. Show that g and f are inverse to each other.

$$(g \circ f)(x) = g[f(x)] = g(2x^3 - 1) = \left[\frac{1}{2}(2x^3 - 1 + 1)\right]^{\frac{1}{3}} = x = I_A(x), \text{ for any } x \in A$$

$$(f \circ g)(y) = f[g(y)] = f\left[\frac{1}{2}(y + 1)\right]^{\frac{1}{3}} = 2\left[\frac{1}{2}(y + 1)\right] - 1 = y = I_B(y), \text{ for any } y \in B$$

Therefore, $g \circ f = I_A$ and $f \circ g = I_B$ and hence f and g are inverse to each other.

9. Let $A = B = C = \mathbb{R}$. Let $f: A \rightarrow B$ and $g: B \rightarrow C$ be defined by

$$f(a) = 2a + 1, \forall a \in A \text{ and } g(b) = \frac{1}{3}b, \forall b \in B.$$

Compute $g \circ f$. Show that $g \circ f$ is invertible. Also find $(g \circ f)^{-1}$.

$$(i) \quad (g \circ f)(a) = g[f(a)] = g(2a + 1) = \frac{1}{3}(2a + 1).$$

$$(ii) \quad f(a_1) = f(a_2) \Rightarrow 2a_1 + 1 = 2a_2 + 1 \Rightarrow 2a_1 = 2a_2 \Rightarrow a_1 = a_2$$

$$\text{For any } b \in B, \text{ there exists } a \in A \text{ such that } 2a + 1 = b \Rightarrow a = \frac{b-1}{2}$$

Therefore, f is 1-1 and onto, hence f is invertible.

$$g(b_1) = g(b_2) \Rightarrow \frac{1}{3}b_1 = \frac{1}{3}b_2 \Rightarrow b_1 = b_2$$

$$\text{For any } c \in C, \text{ there exists } b \in B \text{ such that } \frac{1}{3}b = c \Rightarrow b = 3c.$$

Therefore, g is 1-1 and onto, hence g is invertible.

Since f and g are invertible, $g \circ f$ is also invertible.

$$(iii) \quad (g \circ f)^{-1}(c) = (f^{-1} \circ g^{-1})(c) = f^{-1}[g^{-1}(c)] = f^{-1}(3c) = \frac{3c-1}{2}.$$

10. For the functions f and g from $\mathbb{R}$ to $\mathbb{R}$ defined by $f(x) = 2x$ and $g(x) = 3x - 2$.

Verify that $(gof)^{-1} = f^{-1}og^{-1}$.

$f(x) = 2x$ $\Rightarrow y = 2x$ $\Rightarrow \frac{y}{2} = x$ $\Rightarrow \frac{y}{2} = f^{-1}(y)$ Therefore, $f^{-1}(x) = \frac{x}{2}$	$g(x) = 3x - 2$ $\Rightarrow z = 3x - 2$ $\Rightarrow \frac{z+2}{3} = x$ $\Rightarrow \frac{z+2}{3} = g^{-1}(z)$ Therefore, $g^{-1}(x) = \frac{x+2}{3}$
---	---

$$(gof)(x) = g[f(x)] = g(2x) = 3(2x) - 2 = 6x - 2$$

$$\Rightarrow p = 6x - 2 \quad [\text{Assume } (gof)(x) = p]$$

$$\Rightarrow \frac{p+2}{6} = x$$

$$\Rightarrow \frac{p+2}{6} = (gof)^{-1}(p)$$

$$\text{Therefore, } (gof)^{-1}(x) = \frac{x+2}{6} \quad \text{----- (1)}$$

$$(f^{-1}og^{-1})(x) = f^{-1}[g^{-1}(x)] = f^{-1}\left(\frac{x+2}{3}\right) = \frac{x+2}{6} \quad \text{----- (2)}$$

From (1), (2) it is clear that $(gof)^{-1}(x) = (f^{-1}og^{-1})(x)$, for any $x \in \mathbb{R}$.

Therefore, it is verified that $(gof)^{-1} = f^{-1}og^{-1}$.

3.11 The pigeonhole principle

- ❖ If m Pigeons occupies n pigeonholes and $m > n$ then atleast one pigeonhole has two or more pigeons in it. In general, there are at least $p + 1$ pigeons in it. Where $p = \left\lfloor \frac{m-1}{n} \right\rfloor$.

Problems:

1. **If five colours are used to paint 26 doors, show that at least 6 doors will have the same colours.**

m = Number of Pigeons = Number of doors = 26

n = Number of Pigeonholes = Number of colours = 5

$$p + 1 = \left\lfloor \frac{m-1}{n} \right\rfloor + 1 = \left\lfloor \frac{26-1}{5} \right\rfloor + 1 = 5 + 1 = 6$$

By Pigeonhole principle, there are at least $p + 1 = 6$ doors will have the same colours.

2. **If $n + 1$ numbers from 1 to $2n$ are selected then show that at least two of them will have their sum equal to $2n + 1$.**

Consider $A_1 = \{1, 2n\}, A_2 = \{2, 2n - 1\}, \dots, A_{n-1} = \{n - 1, n + 2\}$ and $A_n = \{n, n + 1\}$

There are n sets containing exactly two elements with their sum $2n + 1$.

m = Number of Pigeons = Numbers from 1 to $2n = n + 1$

n = Number of Pigeonholes = Number of sets = n

$$p + 1 = \left\lfloor \frac{m-1}{n} \right\rfloor + 1 = \left\lfloor \frac{n+1-1}{n} \right\rfloor + 1 = 1 + 1 = 2$$

By Pigeonhole principle,

There are at least $p + 1 = 2$ of them will have their sum equal to $2n + 1$.

3. **Show that if any seven numbers from 1 to 12 are chosen then at least two of them will add to 13.**

Consider $A_1 = \{1, 12\}, A_2 = \{2, 11\}, A_3 = \{3, 10\}, A_4 = \{4, 9\}, A_5 = \{5, 8\}, A_6 = \{6, 7\}$

There are 6 sets containing exactly two elements with their sum 13.

m = Number of Pigeons = Numbers selected = 7

n = Number of Pigeonholes = Number of sets = 6

$$p + 1 = \left\lfloor \frac{m-1}{n} \right\rfloor + 1 = \left\lfloor \frac{7-1}{6} \right\rfloor + 1 = 1 + 1 = 2$$

By Pigeonhole principle, there are at least $p + 1 = 2$ of them will add to 13.

4. **Show that if any 14 integers are selected from the set $S = \{1, 2, 3, \dots, 25\}$ there are at least two integers whose sum is 26.**

Consider

$$A_1 = \{1, 25\}, A_2 = \{2, 24\}, A_3 = \{3, 23\}, A_4 = \{4, 22\}, A_5 = \{5, 21\}, A_6 = \{6, 20\}, \\ A_7 = \{7, 19\}, A_8 = \{8, 18\}, A_9 = \{9, 17\}, A_{10} = \{10, 16\}, A_{11} = \{11, 15\}, A_{12} = \{12, 14\}$$

There are 12 sets containing exactly two elements with their sum 26.

$$m = \text{Number of Pigeons} = \text{Numbers selected} = 14$$

$$n = \text{Number of Pigeonholes} = \text{Number of sets} = 12$$

$$p + 1 = \left\lfloor \frac{m-1}{n} \right\rfloor + 1 = \left\lfloor \frac{14-1}{12} \right\rfloor + 1 = 1 + 1 = 2$$

By Pigeonhole principle, there are at least $p + 1 = 2$ integers whose sum is 26.

5. **Shirts numbered consecutively from 1 to 20 are worn by 20 students of a class. When any three of these students are chosen to a debating team from the class, the sum of their numbers is used as the code number of the team. Show that if any 8 of the 20 are selected then from these 8, we may form at least two different teams having the same code number.**

$$\text{Total number of teams with 3 students} = \binom{8}{3} = 56$$

$$\text{Smallest possible code to a team} = 1 + 2 + 3 = 6$$

$$\text{highest possible code to a team} = 18 + 19 + 20 = 57$$

We can generate codes varying from 6 to 57 and there are 52 codes.

$$m = \text{Number of Pigeons} = \text{Number of teams} = 56$$

$$n = \text{Number of Pigeonholes} = \text{Number of codes} = 52$$

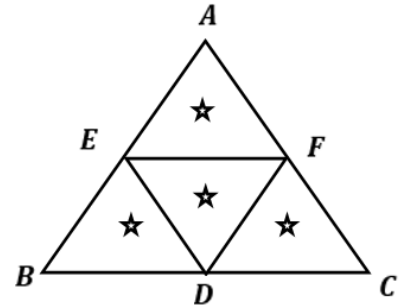
$$p + 1 = \left\lfloor \frac{m-1}{n} \right\rfloor + 1 = \left\lfloor \frac{56-1}{52} \right\rfloor + 1 = 1 + 1 = 2$$

By Pigeonhole principle, there are at least $p + 1 = 2$ different teams having the same code number.

6. Let ABC be an equilateral triangle with $AB = 1\text{ cm}$. Show that if we select 5 points in the interior of the triangle, there must be at least two points whose distance is less than $\frac{1}{2}\text{ cm}$.

Consider the triangle ABC with $AB = BC = CA = 1\text{ cm}$.

Consider another triangle DEF formed by joining mid points of BC , AB and AC .



ΔABC is partitioned into 4 small equivalent triangles each of which have sides $\frac{1}{2}\text{ cm}$.

$m = \text{Number of Pigeons} = \text{Number of points} = 5$

$n = \text{Number of Pigeonholes} = \text{Number of partitions} = 4$

$$p + 1 = \left\lfloor \frac{m-1}{n} \right\rfloor + 1 = \left\lfloor \frac{5-1}{4} \right\rfloor + 1 = 1 + 1 = 2$$

By Pigeonhole principle,

There are at least $p + 1 = 2$ points whose distance is less than $\frac{1}{2}\text{ cm}$.

7. How many persons must be chosen in order that at least 5 of them will have birthdays in the same calendar month.

$m = \text{Number of Pigeons} = \text{Number of persons} = m$

$n = \text{Number of Pigeonholes} = \text{Number of calendar months} = 12$

By Pigeonhole principle,

$$p + 1 = \left\lfloor \frac{m-1}{n} \right\rfloor + 1 = \left\lfloor \frac{m-1}{12} \right\rfloor + 1 = 5$$

$$\left\lfloor \frac{m-1}{12} \right\rfloor = 4$$

Least possible value of m is 49.

49 persons must be chosen in order that at least 5 of them will have birthdays in the same calendar month.