

MODULE-4

Module - 4

1. Solution of Higher order differential Equation:-

Solution of Homogeneous differential Equation:-

If the given equation is of the form

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + a_n y = 0 \rightarrow (1)$$

where $a_0, a_1, a_2, a_3, \dots$, constants.

Working procedure:-

1. Let $D = \frac{d}{dx}$, $\rightarrow$ eq-(1) becomes

$$a_0 D^n y + a_1 D^{n-1} y + a_2 D^{n-2} y + \dots + a_n y = 0$$
$$(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_n) y = 0$$

2. Auxially equation $D \rightarrow m$

$$a_0 m^n + a_1 m^{n-1} + a_2 m^{n-2} + \dots + a_n = 0$$

Solving the above equation we get different types of roots.

Depending on nature of the roots we write solution for given differential equation.

We solve general quadratic equation in the form of

$$am^2 + bm + C = 0$$

SI
No.

Nature of the root

Solution of D.E

1. Real & distinct say m_1 & m_2

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$$

2. Roots are equal
say $m_1 = m_2 = m$

$$y = (C_1 + C_2 x) e^{mx}$$

3. Complex pairs of root $a \pm ib$

$$y = e^{ax} (C_1 \cos bx + C_2 \sin bx)$$

1. Solve $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 0$

Sol:-

put $\frac{d}{dx} = D$

$$D^2y - 4Dy + 3y = 0$$

$$(D^2 - 4D + 3)y = 0$$

Auxiliary equation is given by $D \rightarrow m$

$$m^2 - 4m + 3 = 0, m = 1, 3$$

$m_1 = 1, m_2 = 3$ roots are real and distinct

$\therefore$ Solution is given by

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$$

$$\Rightarrow y = C_1 e^x + C_2 e^{3x}$$

2. Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$

put $\frac{d}{dx} = D$

$$D^2y - 2Dy + y = 0$$

$$(D^2 - 2D + 1)y = 0$$

$$m^2 - 2m + 1 = 0$$

$$a = 1, b = -2, c = 1, m = 1, 1,$$

roots are equal

$\therefore$ Solution is given by

$$y = (C_1 + C_2 x)e^x$$

3. Solve $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$

$$Dy^2 - 4Dy + 4y = 0 \rightarrow (D^2 - 4D + 4)y = 0$$

A.E is given by $D \rightarrow m$

$$m^2 - 4m + 4 = 0$$

$$a = 1, b = -4, c = 4$$

$$m = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(4)}}{2}$$

$$= \frac{4 \pm \sqrt{16 - 16}}{2} = \frac{4 \pm \sqrt{0}}{2} = \frac{4 \pm 0}{2}$$

$$m = \frac{4 \pm i0\sqrt{0}}{2}$$

$$m = 2 \pm i\sqrt{0}$$

Roots are complex pairs

$$a = 2, b = \sqrt{0}$$

$$y = e^{ax} (C_1 \cos bx + C_2 \sin bx)$$

$$y = e^{2x} (C_1 \cos \sqrt{0}x + C_2 \sin \sqrt{0}x)$$

$$4. (D^3 + 6D^2 + 11D + 6)y = 0$$

Sol:- Auxiliary equation is given by

$$m^3 + 6m^2 + 11m + 6 = 0$$

Synthetic division

$$\begin{array}{r|rrrr} & 1 & 6 & 11 & 6 \\ -1 & 0 & -1 & -5 & -6 \\ \hline & 1 & 5 & 6 & 0 \end{array}$$

$$m^2 + 5m + 6 = 0$$

$$m^2 + 3m + 2m + 6 = 0$$

$$m(m+3) + 2(m+3) = 0$$

$$(m+3)(m+2) = 0$$

$$m+3 = 0 \quad m+2 = 0$$

$$m = -3 \quad m = -2$$

$\therefore$ Roots are real and distinct

$$m_1 = -3, m_2 = -2, m_3 = -1$$

Complete solution is by

$$y = C_1 e^{-3x} + C_2 e^{-2x} + C_3 e^{-x}$$

$$5. \text{ Solve } (D^5 - D^4 - D + 1)y = 0$$

Sol:- A.E is given by

$$m^5 - m^4 - m + 1 = 0$$

$$m^4(m-1) - (m-1) = 0$$

$$(m-1)(m^4 - 1) = 0$$

$$m-1 = 0 \quad m^4 - 1 = 0$$

$$\boxed{m = 1}$$

$$(m^2)^2 - (1^2)^2 = 0$$

$$(m^2 + 1)(m^2 - 1) = 0$$

$$m^2 + 1 = 0 \quad (or) \quad m^2 - 1 = 0$$

$$m^2 = -1 \quad m^2 = 1$$

$$m = \pm \sqrt{-1} \quad \boxed{m = \pm 1}$$

$$\boxed{m = \pm i} \quad \therefore \text{Roots are } 1, 1, -1, \pm i$$

C.S is given by

$$y = (C_1 + C_2 x)e^{mx} + C_3 e^{m_1 x} + e^{ax} (C_4 \cos bx + C_5 \sin bx)$$

$$y = (C_1 + C_2 x)e^x + C_3 e^{-x} + e^0 (C_4 \cos x + C_5 \sin x)$$

6. $y''' + y = 0$

Sol:- $\frac{d^3 y}{dx^3} + y = 0$

Put $\frac{d}{dx} = D$

$$D^3 y + y = 0$$

$$(D^3 + 1)y = 0$$

$$(m^3 + 1) = 0$$

$$a = 1, b = 0, c = 0, d = 1$$

Roots are $m = -1, a \pm ib = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

$$a = \frac{1}{2}, b = \frac{\sqrt{3}}{2}$$

C.S is given by

$$y = C_1 e^{-x} + e^{\frac{x}{2}} \left(C_2 \cos \frac{\sqrt{3}}{2} x + C_3 \sin \frac{\sqrt{3}}{2} x \right)$$

Non-Homogeneous linear differential equation:-

If the given differential equation is the form

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + a_n y = 0 \rightarrow (1)$$

is called Non-Homogeneous linear D.E.

$$(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_n) y = f(x)$$

Complete solution = Complementary function + Particular Integral

$$C.S = C.F + P.I$$

$$y = y_c + y_p$$

i> Complementary function (solution of LHS)

ii> Particular Integral (solution of RHS)

Solution of LHS i.e y_c is nothing but finding solution of homogeneous differential equation and we follow the same procedure.

While finding solution of RHS, we need to study different cases.

Case-i> $y_p = \frac{e^{ax+b}}{f(D)}$

Replace $D \rightarrow a$ if $f(a) \neq 0$

if $f(a) = 0$

Multiply by 'x' to numerator &
Differentiate denominator once then

Replace $D \rightarrow a$ if $f'(a) \neq 0$

$$1. \frac{d^2y}{dx^2} - 6 \frac{dy}{dx} + 9y = e^x$$

Sol: Consider, $\frac{d^2y}{dx^2} - 6 \frac{dy}{dx} + 9y = e^x \rightarrow (1)$

$$D^2y - 6Dy + 9y = e^x$$

$$(D^2 - 6D + 9)y = e^x$$

A.E is $m^2 - 6m + 9 = 0$

$$(m-3)^2 = 0$$

$$m = 3, 3$$

$\therefore$ Roots are real and repeated.

$$y_c = (C_1 + C_2x)e^{3x}$$

To find P.I $y_p = \frac{f(x)}{f(D)} = \frac{e^{ax+b}}{f(D)}$

$$y_p = \frac{e^x}{D^2 - 6D + 9}$$

$$D \rightarrow a = 1$$

$$y_p = \frac{e^x}{1^2 - 6 + 9} = \frac{e^x}{4}$$

$\therefore$ Complete solution is

$$y = y_c + y_p$$

$$y = (C_1 + C_2x)e^{3x} + \frac{e^x}{4}$$

$$2. \frac{d^2y}{dx^2} - 6 \frac{dy}{dx} + 9y = e^{3x}$$

Sol:- $D^2y - 6Dy + 9y = e^{3x}$

$$(D^2 - 6D + 9)y = e^{3x}$$

Replace $D \rightarrow m$

$$m^2 - 6m + 9 = 0 \Rightarrow (m-3)^2 = 0$$

$$m = 3, 3$$

$\therefore$ Roots are real and repeated.

$$y_c = (C_1 + C_2 x) e^{3x}$$

To find $y_p = \frac{e^{ax+b}}{f(x)}$

$$y_p = \frac{e^{3x}}{D^2 - 6D + 9}$$

$D \rightarrow a = 3$

$$y_p = \frac{e^{3x}}{9 - 18 + 9} = \frac{e^{3x}}{0} \quad \text{Since, } f(a) = f(3) = 0$$

Multiply x in numerator & diff. denominator once.

$$y_p = x \cdot \frac{e^{3x}}{2D - 6}$$

$$D \rightarrow a = 3$$

$$x \cdot \frac{6e^{3x}}{6 - 6} = \frac{e^{3x}}{0} \quad f'(a) = f'(3) = 0$$

$\therefore$ Multiply x in num & diff. deno. once

$$y_p = x^2 \cdot \frac{e^{3x}}{2 - 0} \Rightarrow y_p = \frac{x^2 \cdot e^{3x}}{2}$$

$\therefore$ C.S is given by

$$y = (C_1 + C_2 x) e^{3x} + \frac{x^2 \cdot e^{3x}}{2}$$

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3. Solve $(D^2 - D - 2)y = 5^x + \cosh x + \log x$ (Inverse differential operator method)

Sol: $\rightarrow 5^x = e^{\log 5^x} = e^{x \log 5}$

$$\rightarrow \cosh x = \frac{e^x + e^{-x}}{2}$$

$$\rightarrow \log x = \log x \cdot 1 = \log x \cdot e^{0x}$$

$$(D^2 - D - 2)y = e^{x \log 5} + \frac{e^x + e^{-x}}{2} + \log x \cdot e^{0x}$$

Replace $D \rightarrow m$

$$m^2 - m - 2 = 0$$

$$m = 2, -1$$

$$y_c = C_1 e^{2x} + C_2 e^{-x}$$

$$y_p = \frac{e^{x \log 5} + \frac{e^x + e^{-x}}{2} + \log x \cdot e^{0x}}{D^2 - D - 2}$$

$$= \frac{e^{x \log 5}}{D^2 - D - 2} + \frac{1}{2} \left[\frac{e^x}{D^2 - D - 2} + \frac{e^{-x}}{D^2 - D - 2} \right] + \frac{\log x \cdot e^{0x}}{D^2 - D - 2}$$

$$= \frac{e^{x \log 5}}{(\log 5)^2 - \log 5 - 2} + \frac{1}{2} \left[\frac{e^x}{1 - 1 - 2} + \frac{e^{-x}}{1 + 1 - 2} \right] + \frac{\log x \cdot e^{0x}}{0 - 0 - 2}$$

$$= \frac{e^{x \log 5}}{(\log 5)^2 - \log 5 - 2} + \frac{1}{2} \left[\frac{e^x}{-2} + \frac{x \cdot e^{-x}}{2 \cdot 0 - 1} \right] + \frac{\log x \cdot e^{0x}}{-2}$$

$$= \frac{e^{x \log 5}}{(\log 5)^2 - \log 5 - 2} + \frac{1}{2} \left[\frac{e^x}{-2} + \frac{x e^{-x}}{-3} \right] + \frac{\log x}{-2}$$

$$y_p = \frac{5^x}{(\log 5)^2 - \log 5 - 2} + \frac{1}{2} \left[\frac{e^x}{-2} - \frac{x e^{-x}}{3} \right] + \frac{\log x}{-2}$$

$$y = y_c + y_p$$

$$y = C_1 e^{2x} + C_2 e^{-x} + \frac{5^x}{(\log 5)^2 - \log 5 - 2} + \frac{1}{2} \left[\frac{e^x}{-2} - \frac{e^{-x}}{3} \right] + \frac{\log 2}{-2}$$

Case - ii If $f(x) = \sin(ax+b)$ or $\cos(ax+b)$

$$y_p = \frac{\sin(ax+b)}{f(D)}$$

$$D^2 \rightarrow -a^2 \quad \text{if } (-a^2) \neq 0$$

$$\text{if } (-a^2) = 0$$

then Multiply by x to numerator & differentiate denominator.

1. Solve $y'' - 4y = \cos 2x$

Sol:- Consider :- $y'' - 4y = \cos 2x$

$$y = y_c + y_p$$

$$\frac{d^2 y}{dx^2} - 4y = \cos 2x$$

$$D = \frac{d}{dx}$$

$$D^2 y - 4y = \cos 2x$$

$$(D^2 - 4)y = \cos 2x$$

$$\text{A.E is } m^2 - 4 = 0$$

$$m^2 = 4 \quad m = \pm 2$$

$$y_c = C_1 e^{-2x} + C_2 e^{2x}$$

$$\text{To find } y_p \quad y_p = \frac{\cos(ax+b)}{f(D)}$$

$$y_p = \frac{\cos 2x}{D^2 - 4}$$

$$y_1 = \frac{\cos 2x}{-4-4} = -\frac{\cos 2x}{8}$$

∴ C.S is

$$y = C_1 e^{-2x} + C_2 e^{2x} - \frac{\cos 2x}{8}$$

Q. Solve $(D^2+1)y = \cos x$

Sol:- $(D^2+1)y = \cos x$

$$D^2 \rightarrow m$$

$$m^2+1=0$$

$$m^2 = -1, \boxed{m = \pm i} \quad \begin{matrix} R.P = 0 \\ I.P = 1 \end{matrix}$$

$$y_c = e^{0x} (C_1 \cos x + C_2 \sin x)$$

To find y_p

$$y_p = \frac{\cos(ax+b)}{f(D)} \quad \begin{matrix} f(D) \\ D^2 \rightarrow -a^2 = -1 \end{matrix}$$

$$= \frac{\cos x}{D^2+1} = \frac{\cos x}{-1+1} = \frac{\cos x}{0}$$

Multiply x to num. & diff. deno.

$$y_p = \frac{x \cdot \cos x}{2D}$$

$$y_p = \frac{x}{2} \cdot \int \cos x \cdot dx$$

$$y_p = \frac{x}{2} \cdot \sin x$$

multiply & $\div D$ in RHS

$$= \frac{x}{2} \cdot \frac{D(\cos x)}{D^2}$$

$$= \frac{x}{2} \cdot \frac{-\sin x}{-1}$$

$$y_p = \frac{x}{2} \cdot \sin x$$

$$y = y_c + y_p$$

$$y = C_1 \cos x + C_2 \sin x + \frac{x}{8} \cdot \sin x$$

$$3. (D^2 - 4D + 3)y = \sin 3x \cdot \cos 2x$$

$$\text{Sol:- } (D^2 - 4D + 3)y = \sin 3x \cos 2x$$

$$D \rightarrow m$$

$$m^2 - 4m + 3 = 0$$

$$m^2 - 3m - m + 3 = 0$$

$$m(m-3) - (m-3) = 0$$

$$(m-3) = 0 \quad (m-1) = 0$$

$$m = 3 \quad m = 1$$

$$y_c = C_1 e^{3x} + C_2 e^x$$

To find y_p

$$y_p = \frac{\sin 3x \cdot \cos 2x}{(D^2 - 4D + 3)}$$

$$\sin A \cdot \cos B = \frac{1}{2} (\sin(A+B) + \sin(A-B))$$

$$y_p = \frac{1}{2} \left[\frac{\sin 5x + \sin x}{D^2 - 4D + 3} \right]$$

$$= \frac{1}{2} \left[\frac{\sin 5x}{D^2 - 4D + 3} + \frac{\sin x}{D^2 - 4D + 3} \right]$$

$$D^2 \rightarrow -\alpha^2 = -25 \quad , \quad D^2 \rightarrow -\alpha^2 = -1$$

$$= \frac{1}{2} \left[\frac{\sin 5x}{-25 - 4D + 3} + \frac{\sin x}{-1 - 4D + 3} \right]$$

$$= \frac{1}{2} \left[\frac{\sin 5x}{-4D - 22} + \frac{\sin x}{-4D + 2} \right]$$

$$= \frac{1}{2} \left[\frac{\sin 5x}{-2(2D + 11)} + \frac{\sin x}{-2(2D - 1)} \right]$$

$$8. \frac{d^2 y}{dx^2} - 4y = \sin^3 x$$

Sol:- $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$$\sin^3\theta = \frac{3\sin\theta - \sin 3\theta}{4}$$

$$D^2 y - 4y = \frac{3\sin x - \sin 3x}{4}$$

$$(D^2 - 4)y = \frac{3\sin x - \sin 3x}{4}$$

$$D \rightarrow m$$

$$m^2 - 4 = 0 \quad \boxed{m = \pm 2}$$

$$y_c = C_1 e^{2x} + C_2 e^{-2x}$$

To find y_p

$$y_p = \frac{\sin(ax+b)}{f(D)}$$

$$= \frac{\frac{3\sin x - \sin 3x}{4}}{D^2 - 4}$$

$$= \frac{1}{4} \left[\frac{3\sin x}{D^2 - 4} - \frac{\sin 3x}{D^2 - 4} \right]$$

$D^2 \rightarrow -a^2 = -1 \quad D^2 \rightarrow -a^2 = -9$

$$= \frac{1}{4} \left[\frac{3\sin x}{-5} + \frac{\sin 3x}{13} \right]$$

$$y_p = \frac{1}{4} \left[\frac{3\sin x}{-5} + \frac{\sin 3x}{13} \right]$$

$$y = y_c + y_p$$

$$y = C_1 e^{2x} + C_2 e^{-2x} + \frac{1}{4} \left[\frac{3\sin x}{-5} + \frac{\sin 3x}{13} \right]$$

$$y = C_1 e^{2x} + C_2 e^{-2x} + \frac{1}{4} \left[\frac{\sin 3x}{13} - \frac{3\sin x}{5} \right]$$

Multiply & $\div$ $2D-11$

Multiply & $\div$ $2D+1$

$$= -\frac{1}{4} \left[\frac{\sin 5x (2D-11)}{(2D+11)(2D-11)} + \frac{\sin x (2D+1)}{(2D-1)(2D+1)} \right]$$

$$= -\frac{1}{4} \left[\frac{\sin 5x (2D-11)}{4D^2 - 11^2} + \frac{\sin x (2D+1)}{4D^2 - 1} \right]$$

$$D^2 \rightarrow -a^2 = -25$$

$$D^2 \rightarrow -a^2 = -1$$

$$= -\frac{1}{4} \left[\frac{\sin 5x (2D-11)}{4(-25) - 121} + \frac{\sin x (2D+1)}{-4-1} \right]$$

$$= -\frac{1}{4} \left[\frac{\sin 5x (2D-11)}{-221} + \frac{\sin x (2D+1)}{-5} \right]$$

$$= \frac{1}{4} \left[\frac{2 \cos 5x \cdot (5) - 11 \sin 5x}{221} + \frac{2 \cos x + \sin x}{5} \right]$$

$$= \frac{1}{4} \left[\frac{10 \cos 5x - 11 \sin 5x}{221} + \frac{2 \cos x + \sin x}{5} \right]$$

$$y = y_c + y_p$$

$$y = C_1 e^x + C_2 e^{3x} + \frac{1}{4} \left[\frac{10 \cos 5x - 11 \sin 5x}{221} + \frac{2 \cos x + \sin x}{5} \right]$$

Q. Solve $(D^2+1)y = \sin^3 x \cdot \cos^3 x$

Sol:- $(D^2+1)y = \sin^3 x \cdot \cos^3 x$

A.E is given by

$$m^2 + 1 = 0$$

$$m = \pm i \quad \begin{matrix} R.P = 0 \\ I.P = 1 \end{matrix}$$

$$y_c = e^{0x} (C_1 \cos x + C_2 \sin x)$$

$$y_p = \frac{\sin^3 x \cdot \cos^3 x}{D^2+1}$$

$$\sin^3 x \cdot \cos^3 x = (\sin x \cdot \cos x)^3$$

$$= \left(\frac{\sin 2x}{2} \right)^3$$

$$= \frac{\sin^3 2x}{4}$$

$$\sin^3 2x = \frac{1 - \cos 4x}{2}$$

$$\sin^3 2x = \frac{1 - \cos 4x}{2 \times 4}$$

$$y_p = \frac{1 - \cos 4x}{8(D^2+1)}$$

$$= \frac{1}{8} \left[\frac{1 \times e^{0x}}{D^2+1} - \frac{\cos 4x}{D^2+1} \right]$$

$$D^2 \rightarrow -a^2 = 0 \quad D^2 \rightarrow a^2 = -16$$

$$= \frac{1}{8} \left[\frac{1}{0+1} - \frac{\cos 4x}{-15} \right]$$

$$y = e^{0x} (C_1 \cos x + C_2 \sin x) + \frac{1}{8} \left[\frac{1}{1} + \frac{\cos 4x}{15} \right]$$

Case - iii :- Particular integral of the form $\frac{p(x)}{f(D)}$

where $p(x)$ is polynomial in x

Divide numerator using denominator writing $p(x)$ in descending order & $f(D)$ in ascending order.

Continue this division until we get remainder '0', the quotient so obtain is particular integral of the given problem.

1. Solve $(D^2 + 3D + 2)y = 1 + 3x + x^2$

Sol:- $(D^2 + 3D + 2)y = 1 + 3x + x^2$

$$D \rightarrow m$$

$$m^2 + 3m + 2 = 0$$

$$m = -1, -2$$

$$y_c = C_1 e^{-x} + C_2 e^{-2x}$$

To find y_p

$$y_p = \frac{1 + 3x + x^2}{D^2 + 3D + 2}$$

$$\begin{array}{r} (2 + 3D + D^2) \overline{x^2 + 3x + 1} \left(\frac{x^2}{2} \right) \\ \underline{x^2 + 3x + 1} \\ 0 \end{array}$$

$$y_p = \frac{x^2}{2}$$

$$y = C_1 e^{-x} + C_2 e^{-2x} + \frac{x^2}{2}$$

$$\begin{aligned} & \frac{x^2}{2} (2 + 3D + D^2) \\ &= 2 \cdot \frac{x^2}{2} + \frac{3}{2} 2x + \frac{1}{2} 2(1) \\ &= x^2 + 3x + 1 \end{aligned}$$

2. Solve $\frac{d^2y}{dx^2} + 4y = x^2$

Sol: $\frac{d^2y}{dx^2} + 4y = x^2$

$$D^2y + 4y = x^2$$

$$(D^2 + 4)y = x^2$$

A.E is $m^2 + 4 = 0$

$$m = \pm 2i$$

$$y_c = C_1 e^{2ix} + C_2 e^{-2ix}$$

To find y_p

$$y_p = \frac{x^2}{D^2 + 4}$$

$$\begin{array}{r} (4 + D^2) \frac{x^2}{x^2 + \frac{1}{4}} \left(\frac{x^2}{4} - \frac{1}{8} \right) \\ \hline - \frac{1}{2} \\ - \frac{1}{2} \\ \hline 0 \end{array}$$

$$\frac{x^2}{4} (4 + D^2)$$

$$4 \frac{x^2}{4} + \frac{1}{2}$$

$$y_p = \frac{x^2}{4} - \frac{1}{8}$$

$$y = C_1 e^{2ix} + C_2 e^{-2ix} + \frac{x^2}{4} - \frac{1}{8}$$

3. Solve $(D^3 + 8)y = x^4 + 2x + 1$

Sol:- $(D^3 + 8)y = x^4 + 2x + 1$

A.E is

$$m^3 + 8 = 0, m = -2, 1 \pm \sqrt{3}i$$

$$y_c = C_1 e^{-2x} + e^x (C_2 \cos \sqrt{3}x + C_3 \sin \sqrt{3}x)$$

$$\begin{array}{r}
 (8 + D^3) \cancel{x^4} + 9x + 1 \left(\frac{x^4}{8} - \frac{x}{8} + \frac{1}{8} \right) \\
 \hline
 x^4 + 3x \\
 \hline
 -x + 1 \\
 \hline
 -x + 0 \\
 \hline
 1 \\
 \hline
 1 \\
 \hline
 0
 \end{array}$$

$$y = C_1 e^{-2x} + e^x (C_2 \cos \sqrt{3}x + C_3 \sin \sqrt{3}x) + \frac{x^4}{8} - \frac{x}{8} + \frac{1}{8}$$

4. Solve $y'' + 2y' - y = e^{-2x} + x$

Sol:- $y'' + 2y' - y = e^{-2x} + x$

$$(D^2 y + 2Dy - y) = e^{-2x} + x$$

$$(D^2 + 2D - 1)y = e^{-2x} + x$$

A.E is given by

$$m^2 + 2m - 1 = 0$$

$$m = -1 \pm \sqrt{2}$$

$$y_c = C_1 e^{-1+\sqrt{2}} + C_2 e^{-1-\sqrt{2}}$$

$$y_p = \frac{e^{-2x} + x}{D^2 + 2D - 1}$$

$$= \frac{e^{-2x}}{D^2 + 2D - 1} + \frac{x}{D^2 + 2D - 1}$$

$$y_p = (y_p)_1 + (y_p)_2$$

$$(y_p)_1 = \frac{e^{-2x}}{D^2 + 2D - 1}$$

$$D \rightarrow a = -2$$

$$= \frac{e^{-2x}}{(-2)^2 + 2(-2) - 1}$$

$$= \frac{e^{-2x}}{4 - 4 - 1}$$

$$= \frac{e^{-2x}}{-1}$$

$$\begin{array}{r} -1 + 2D + D^2 \bigg) \cancel{(-x+2)} \\ \underline{x-2} \\ 2 \\ -2 \\ \hline 0 \end{array}$$

$$y_1 = C_1 e^{(-1+\sqrt{2})x} + C_2 e^{(-1-\sqrt{2})x} - e^{-2x} - x + 2$$

5. Solve $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = x^2 + e^x$

Sol:- $D^2y - 3Dy + 2y = x^2 + e^x$

$$(D^2 - 3D + 2)y = x^2 + e^x$$

$$D \rightarrow m$$

$$m^2 - 3m + 2 = 0$$

$$m = 1, 2$$

$$y_c = C_1 e^x + C_2 e^{2x}$$

$$y_p = \frac{x^2 + e^x}{D^2 - 3D + 2}$$

$$\frac{x^2}{D^2 - 3D + 2} + \frac{e^x}{D^2 - 3D + 2}$$

$$\Rightarrow y_p = (y_p)_1 + (y_p)_2$$

$$(y_p)_1 = \frac{x^2}{D^2 - 3D + 2}$$

$$\frac{(2 - 3D + D^2) \cancel{x^2} \left(\frac{x^2}{2} + \frac{3}{2}x + \frac{1}{4} \right)}{\cancel{x^2} - 3x + 1}$$

$$\frac{\cancel{3x} - 1}{\underset{(-)}{3x} - \underset{(+)}{9/2} + 0}$$

$$\frac{7}{2}$$

$$\frac{\underset{(-)}{1}}{\underset{(+)}{2}}$$

$$\frac{0}{0}$$

$$\Rightarrow \frac{x^2}{2} (2 - 3D + D^2)$$

$$2 \cdot \frac{x^2}{2} - \frac{3}{2} 2x + \frac{1}{2} 2(1)$$

$$= x^2 - 3x + 1$$

$$\Rightarrow \frac{3}{2}x (2 - 3D + D^2)$$

$$= 2 \cdot \frac{3x}{2} - \frac{9}{2}(1) + \frac{3}{2}(0)$$

$$\Rightarrow \frac{1}{4} (2 - 3D + D^2)$$

$$= 2 \cdot \frac{1}{4} - 0 + 0$$

$$(y_p)_2 = \frac{e^x}{D^2 - 3D + 2}$$

$$D \rightarrow 1$$

$$= \frac{e^x}{1 - 3 + 2} = \frac{e^x}{0}$$

$x^p y \propto$ in nume & diff. deno

$$= \frac{x \cdot e^x}{2D - 3}$$

$$D \rightarrow (1)$$

$$= \frac{x \cdot e^x}{2 - 3} = \frac{x \cdot e^x}{-1}$$

$$y = -x \cdot e^x + \frac{x^2}{2} + \frac{3x}{2} + \frac{1}{4} + C_1 e^x + C_2 e^{2x}$$

$$6. \frac{d^2 y}{dx^2} + 4y = x^2 + \cos 2x + e^{-x}$$

Sol:- $D^2 y + 4y = x^2 + \cos 2x + e^{-x}$

$$(D^2 + 4)y = x^2 + \cos 2x + e^{-x}$$

$$D \rightarrow m \quad A.E$$

$$m^2 + 4 = 0, \quad m = \pm 2i$$

$$y_c = e^{0x} (C_1 \cos 2x - C_2 \sin 2x)$$

$$y_p = (y_p)_1 + (y_p)_2 + (y_p)_3$$

$$y_p = \frac{x^2}{D^2 + 4} + \frac{\cos 2x}{D^2 + 4} + \frac{e^{-x}}{D^2 + 4}$$

$$(y_p)_1 = \frac{x^2}{D^2 + 4}$$

$$\begin{array}{r} (4 + D^2) x^2 \left(\frac{x^2}{4} - \frac{1}{8} \right) \\ \hline \frac{x^2 + \frac{1}{2}}{(-) \quad (-) \quad 2} \\ \hline -\frac{1}{2} \\ \hline -\frac{1}{2} \\ \hline 0 \end{array}$$

$$\Rightarrow \frac{x^2}{4} (4 + D^2)$$

$$4 \cdot \frac{x^2}{4} + \frac{1}{4} 2(1)$$

$$\Rightarrow -\frac{1}{8} (4 + D^2)$$

$$= 4 \cdot -\frac{1}{8} + 0$$

$$(y_p)_2 = \frac{\cos 2x}{D^2 + 4}$$

$$D^2 \rightarrow -a^2 = -4$$

$$\frac{\cos 2x}{-4 + 4} = \frac{\cos 2x}{0}$$

$$\frac{\cos 2x}{0} = \frac{x \cos 2x}{2} \quad \frac{x \cos 2x}{4}$$

$$\begin{aligned}
 (y_p)_3 &= \frac{2^{-x}}{D^2+4} \\
 &= \frac{e^{\log 2^{-x}}}{D^2+4} = \frac{e^{-x \log 2}}{D^2+4} \xrightarrow{D \rightarrow a} \\
 &= \frac{e^{-x \log 2}}{(\log 2)^2+4}
 \end{aligned}$$

$$y_p = \frac{x^2}{4} - \frac{1}{8} + \frac{x \cos 2x}{4} + \frac{e^{-x \log 2}}{(\log 2)^2+4} + e^{0x}(C_1 \cos 2x - C_2 \sin 2x)$$

$$\Rightarrow \frac{d^2 x}{dt^2} - 6 \frac{dx}{dt} + 25x = e^{2t} + \sin t + t$$

Sol:- Let $\frac{d}{dt} = D$

$$D^2 x - 6Dx + 25x = e^{2t} + \sin t + t$$

$$(D^2 - 6D + 25)x = e^{2t} + \sin t + t$$

A.E is given by

$$m^2 - 6m + 25 = 0$$

$$m = 3 \pm 4i$$

$$x_{fc} = e^{3t} (C_1 \cos 4t + C_2 \sin 4t)$$

$$\begin{aligned}
 x_p &= \frac{e^{2t} + \sin t + t}{D^2 - 6D + 25} \\
 &= \left[\frac{e^{2t}}{D^2 - 6D + 25} + \frac{\sin t}{D^2 - 6D + 25} + \frac{t}{D^2 - 6D + 25} \right]
 \end{aligned}$$

$$x_p = (x_p)_1 + (x_p)_2 + (x_p)_3$$

$$= \frac{e^{2t}}{D^2 - 6D + 25}$$

$$(x_p)_1 = \frac{e^{at}}{D^2 - 6D + 25} \quad D \rightarrow a$$

$$(x_p)_1 = \frac{e^{2t}}{(2)^2 - 6(2) + 25} = \frac{e^{2t}}{4 - 12 + 25} = \frac{e^{2t}}{17}$$

$$(x_p)_2 = \frac{\sin t}{D^2 - 6D + 25}$$

$$D^2 \rightarrow -a^2 = -1$$

$$= \frac{\sin t}{(-1) - 6D + 25} = \frac{\sin t}{24 - 6D}$$

$$= \frac{\sin t (24 + 6D)}{(24 - 6D)(24 + 6D)} = \frac{\sin t (24 + 6D)}{(24)^2 - (6D)^2}$$

$$= \frac{\sin t (24 + 6D)}{576 - 36D^2}$$

$$= \frac{1}{36} \left[\frac{\sin t (24 + 6D)}{16 - D^2} \right]$$

$$= \frac{1}{36} \left[\frac{-\sin t (24 + 6D)}{D^2 - 16} \right]$$

$$= -\frac{1}{36} \left[\frac{24 \sin t + 6 \cos t}{D^2 - 16} \right]$$

$$= -\frac{1}{36} \left[\frac{4 \sin t + \cos t}{D^2 - 16} \right]$$

$$= -\frac{1}{6} \left[\frac{4 \sin t + \cos t}{D^2 - 16} \right]$$

$$D^2 \rightarrow -a^2 = -1$$

$$= -\frac{1}{6} \left[\frac{4 \sin t + \cos t}{-1 - 16} \right]$$

$$= \frac{e^{2t}}{(2)^2 - 6D + 25} = \frac{e^{2t}}{4 - 6D + 25} = \frac{e^{2t}}{29 - 6(2)} = \frac{e^{2t}}{17D}$$

$$(x p)_2 = \frac{\sin t}{D^2 - 6D + 25} \quad D^2 \rightarrow -a^2 = -1 = \frac{\sin t}{(-1)^2 - 6(-1)D + 25} = \frac{\sin t}{-6D + 26}$$

$$= \frac{-\sin t}{6(D-4)}$$

Xly & ÷ by (D-4)

$$\therefore \frac{-\sin t (D+4)}{6(D+4)(D-4)} = -\frac{1}{6} \left(\frac{D(\sin t) + 4\sin t}{D^2 - 16} \right) = -\frac{1}{6} \left(\frac{\cos t + 4\sin t}{D^2 - 16} \right) = -\frac{1}{6} \left(\frac{\cos t + 4\sin t}{(-1)^2 - 16} \right)$$

$$(x p)_3 = \frac{t}{D^2 - 6D + 25}$$

$$(25 - 6D + D^2)t \quad \left(\frac{t}{25} + \frac{6}{625} \right)$$

$$\frac{t - \frac{6(1)}{25} + 0}{\frac{6}{25} - \frac{6}{25}\sqrt{0}}$$

$$8) \quad y'' + 4y' - 12y = e^{2x} - 3\sin 2x$$

Sol:- $D^2 + 4D - 12y = e^{2x} - 3\sin 2x$

$D \rightarrow m$ A.E is given

$$(m^2 + 4m - 12) = 0$$

$$m = 2, -6$$

$$y_c = C_1 e^{2x} + C_2 e^{-6x}$$

$$y_p = \frac{e^{2x} - 3\sin 2x}{(D^2 + 4D - 12)}$$

$$y_p = \frac{e^{2x}}{D^2 + 4D - 12} - \frac{3\sin 2x}{D^2 + 4D - 12}$$

$$(y_p)_1 = \frac{e^{2x}}{D^2 + 4D - 12}$$

$$D \rightarrow a$$

$$= \frac{e^{2x}}{(2)^2 + 4(2) - 12} = \frac{e^{2x}}{4 + 8 - 12} = \frac{e^{2x}}{0}$$

$$= \frac{x e^{2x}}{2D + 4}$$

$$(y_p)_1 = \frac{x \cdot e^{2x}}{8}$$

$$(y_p)_2 = \frac{3 \sin 2x}{D^2 + 4D - 12}$$

$$D^2 \rightarrow -a^2 = -4$$

$$= \frac{3 \sin 2x}{-4 + 4D - 12} = \frac{3 \sin 2x}{4D - 16} = \frac{3}{4} \left[\frac{\sin 2x}{D - 4} \right]$$

$$= \frac{3}{4} \left[\frac{\sin 2x (D + 4)}{(D - 4)(D + 4)} \right]$$

$$= \frac{3}{4} \left[\frac{\sin 2x (D + 4)}{D^2 - 16} \right]$$

$$= \frac{3}{4} \left[\frac{2 \cos 2x + 4 \sin 2x}{-4 - 16} \right]$$

$$= \frac{3}{4} \left[\frac{2 \cos 2x + 4 \sin 2x}{-20} \right]$$

$$= \frac{3 \cdot 2}{4 \cdot 2} \left[\frac{\cos 2x + 2 \sin 2x}{-20} \right]$$

$$(y_p)_2 = \frac{3}{2} \left[\frac{\cos 2x + 2 \sin 2x}{-20} \right]$$

$$y = (y_p)_1 + (y_p)_2 + y_c$$

$$y = C_1 e^{3x} + C_2 e^{-6x} - \frac{3}{8} \left[\frac{\cos 3x + 3 \sin 3x}{30} \right] + \frac{x \cdot e^{3x}}{8}$$

9. $(D^3 - D)y = 3e^x + 4\cos x$

Sol:- $(D^3 - D)y = 3e^x + 4\cos x$

$D \rightarrow m$ A.E is

$$m^3 - m = 0$$

$$m = \pm 1, 0$$

$$y_c = C_1 e^x + C_2 e^{-x} + C_3 e^{0x}$$

$$y_p = \frac{3e^x}{D^3 - D} + \frac{4\cos x}{D^3 - D}$$

$$y_p = (y_p)_1 + (y_p)_2$$

$$(y_p)_1 = \frac{3e^x}{D^3 - D} \quad D \rightarrow \alpha$$

$$= \frac{3e^x}{1-1} = \frac{3e^x}{0}$$

$x^1 y$ & x in nume & diff deno.

$$= \frac{x \cdot 3e^x}{3D^2 - D}$$

$$= \frac{x \cdot 3e^x}{3-1} = \frac{3x e^x}{2}$$

$$(y_p)_1 = x \cdot e^x$$

$$(y_p)_2 = \frac{4 \cos x}{D^3 - D}$$

$$= \frac{4 \cos x}{D(D^2 - 1)} = \frac{4 \cos x}{D(-2)}$$

$$= -\frac{2 \cos x}{D}$$

$$= -2 \int \cos x$$

$$(y_p)_2 = -2 \sin x$$

$$y = (y_c) + y_p$$

$$= C_1 e^x + C_2 e^{-x} + C_3 e^{0x} + x \cdot e^x + -2 \sin x$$

$$10) \frac{d^2 y}{dx^2} - 4y = \cosh(2x-1) + 3^x$$

$$\text{sol: } (D^2 - 4)y = \cosh(2x-1) + 3^x$$

$$(D^2 - 4)y = 0$$

$$m^2 - 4 = 0$$

$$\therefore m = \pm 2$$

$$y_c = C_1 e^{-2x} + C_2 e^{2x}$$

$$y_p = \frac{\cosh(2x-1) + 3^x}{D^2 - 4}$$

$$= \frac{e^{2x-1} + e^{-(2x-1)}}{2} + \frac{e^{x \log 3}}{D^2 - 4}$$

$$\cosh = \frac{e^{ax} + e^{-ax}}{2}$$

$$= \frac{1}{2} \left[\frac{e^{2x-1}}{D^2 - 4} + \frac{e^{-2x+1}}{D^2 - 4} \right]$$

$$0 \rightarrow a \rightarrow 2$$

$$0 \rightarrow a \rightarrow -2$$

$$+ \frac{e^{x \log 3}}{D^2 - 4} \quad D \rightarrow a = \log 3$$

$$= \frac{1}{2} \left[\frac{e^{2x-1}}{0} + \frac{e^{-2x-1}}{0} \right] + \frac{e^{x \log 3}}{(\log 3)^2 - 4}$$

$$= \frac{1}{2} \left[\frac{x \cdot e^{2x-1}}{20} + \frac{x \cdot e^{-2x-1}}{20} \right] + \frac{e^{x \log 3}}{(\log 3)^2 - 4}$$

$$y_p = \frac{1}{2} \left[\frac{x \cdot e^{2x-1}}{4} + \frac{x \cdot e^{-2x-1}}{-4} \right] + \frac{e^{x \log 3}}{(\log 3)^2 - 4}$$

$$y = y_c + y_p$$

$$y = c_1 e^{-2x} + c_2 e^{2x} + \frac{1}{2} \left[\frac{x \cdot e^{2x-1}}{4} + \frac{x \cdot e^{-2x-1}}{-4} \right] + \frac{e^{x \log 3}}{(\log 3)^2 - 4}$$

11) Solve $(D^2 + D)y = x^2 + 2x + 4$.

Sol: Consider $(D^2 + D)y = x^2 + 2x + 4$.

$$(D^2 + D)y = 0$$

$$\text{A.E } m^2 + m = 0$$

$$\therefore m = -1, m = 0$$

$$y_c = c_1 e^0 + c_2 e^{-1x}$$

$$\therefore y_c = c_1 e^0 + c_2 e^{-x}$$

$$y_p = \frac{x^2 + 2x + 4}{D^2 + D}$$

$$D(D+1) \frac{x^2 + 2x + 4}{D^2 + D} = \frac{x^2 + 2x + 4}{D}$$

$$\begin{array}{r} x^2 + 2x + 4 \\ \underline{4} \\ -4 \\ \underline{-4} \\ 0 \end{array}$$

$$(D+D^2) \left(\frac{x^3}{3} \right)$$

$$= D \cdot \frac{x^3}{3} + D^2 \cdot \frac{x^3}{3}$$

$$= \frac{1}{3} \cdot 3x^2 + \frac{1}{3} \cdot 3(2x)$$

$$= x^2 + 2x$$

$$(D+D^2)(4x)$$

$$= D(4x) + D^2(4x) = 4 + 0$$

$$y = y_c + y_p$$

$$= c_1 e^0 + c_2 e^{-x} + \frac{x^3}{3} + 4x$$

12) $(D^3 + 3D^2 + D)y = x^2 + x$

Sol $(D^3 + 3D^2 + D)y = x^2 + x$

$$(D^3 + 3D^2 + D)y = 0$$

$$m^3 + 3m^2 + m = 0$$

$$\therefore m = 0 \quad m = \frac{-3 \pm \sqrt{5}}{2}$$

$$y_c = c_1 e^{0 \cdot x} + c_2 e^{\frac{-3 + \sqrt{5}}{2}} + c_3 e^{\frac{-3 - \sqrt{5}}{2}}$$

$$y_p = \frac{x^2 + x}{D^3 + 3D^2 + D}$$

$$(D + 3D^2 + D^3) \frac{x^2 + x}{D^3 + 3D^2 + D} = \left(\frac{x^3}{3} - \frac{5x^2}{2} + 13x \right)$$

$$\begin{array}{r} x^2 + 6x + 2 \\ (-) \quad (-) \quad (-) \\ \hline -5x - 2 \\ -5x + 15 \\ \hline 13 \\ 13 + 0 \\ \hline 0 \end{array}$$

$$y_p = \frac{x^3}{3} - \frac{5x^2}{2} + 13x$$

$$y = y_c + y_p$$

$$y = c_1 e^{0x} + c_2 e^{-3 + \sqrt{5}/2} + c_3 e^{-3 - \sqrt{5}/2} + \frac{x^3}{3} - \frac{5x^2}{2} + 13x$$

$$\frac{x^3}{3} (D + 3D^2 + D^3)$$

$$= \frac{1}{3} \cdot 3x^2 + 3 \cdot \frac{1}{3} (6x) + \frac{6}{3}$$

$$= x^2 + 6x + \frac{6}{3}$$

$$= x^2 + 6x + 2$$

$$-\frac{5x^2}{2} (D + 3D^2 + D^3)$$

$$= -\frac{5}{2} \cdot 2x + 3 \left(-\frac{5}{2} \right) (-2)$$

Method of variation of parameter:-

Consider second order linear differential equation,

$$a_0 \frac{d^2 y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = \phi(x) \rightarrow (1)$$

Working procedure:-

1. Write auxiliary equation, then according to nature of the root
2. Complete solution is assumed as $y = Ay_1 + By_2$ using complementary function (y_c) where we need to find the values of A & B where

$$A = \int \frac{-y_2(x)\phi(x)}{\omega} dx \quad B = \int \frac{y_1(x)\phi(x)}{\omega} dx$$

where, ω is $\omega = y_1 y_2' - y_2 y_1'$

It is called Wronskian

1. Solve $\frac{d^2 y}{dx^2} + y = \sec x \cdot \tan x$ by using variation of parameter method.

Sol:- $\frac{d^2 y}{dx^2} + y = \sec x \cdot \tan x$

$$D = \frac{d}{dx}, \quad \phi(x) = \sec x \cdot \tan x$$

$$D^2 y + y = \sec x \cdot \tan x \Rightarrow (D^2 + 1)y = \sec x \cdot \tan x$$

A.E is given by

$$m^2 + 1 = 0$$

$$y_c = e^{\alpha x} (C_1 \cos x + C_2 \sin x)$$

$$y_c = C_1 \cos x + C_2 \sin x$$

Assume C.S as $y = Ay_1 + By_2$

$$y = A \cos x + B \sin x \rightarrow (1)$$

$$y_1 = \cos x \quad y_2 = \sin x$$

$$y_1' = -\sin x \quad y_2' = \cos x$$

$$\omega = y_1 y_2' - y_2 y_1'$$

$$\omega = \cos x (\cos x) - \sin x (-\sin x)$$

$$\omega = \cos^2 x + \sin^2 x$$

$$\boxed{\omega = 1}$$

$$A = \int \frac{-y_2 \phi(x)}{\omega} dx$$

$$A = \int \frac{-\sin x \cdot \sec x \cdot \tan x}{1} dx$$

$$A = - \int \sin x \cdot \frac{1}{\cos x} \cdot \tan x \cdot dx$$

$$A = - \int \tan^2 x \cdot dx$$

$$= - \int (\sec^2 x - 1) dx$$

$$= - \int \sec^2 x \cdot dx + \int dx$$

$$\boxed{A = -\tan x + x + K_1}$$

$$B = \int \frac{y_1(x) \phi(x)}{\omega} dx$$

$$= \int \frac{\cos x \cdot \sec x \tan x \, dx}{1}$$

$$= \int \tan x \, dx$$

$$B = \log \sec x + k_2$$

$\therefore$ (1) becomes

$$y = (-\tan x + x + k_1) \cos x + (\log \sec x + k_2) \sin x$$

2. $y'' + y = \tan x$

Sol:- $D^2 y + y = \tan x$

$$(D^2 + 1)y = \tan x$$

A.E is given by $m^2 + 1 = 0$

$$\boxed{m = \pm i}$$

$$y_c = e^{0x} (C_1 \cos x + C_2 \sin x)$$

$$y_c = C_1 \cos x + C_2 \sin x$$

Assume C.S as $y = Ay_1 + By_2$

$$y = A \cos x + B \sin x \rightarrow (1)$$

$$y_1 = \cos x$$

$$y_2 = \sin x$$

$$y_1' = -\sin x$$

$$y_2' = \cos x$$

$$w = y_1 y_2' - y_2 y_1'$$

$$w = \cos x (\cos x) - \sin x (-\sin x)$$

$$w = \cos^2 x + \sin^2 x$$

$$A = \int \frac{-y_2(x) \phi(x)}{\omega} dx$$

$$= \int \frac{-\sin x \cdot \tan x}{1} dx$$

$$= - \int \sin x \cdot \tan x dx$$

$$= - \int \frac{\sin^2 x}{\cos x} dx = \int \frac{1 - \cos^2 x}{\cos x} dx$$

$$= - \int \frac{1}{\cos x} dx + \int \cos x dx$$

$$= - \int \sec x dx + \int \cos x dx$$

$$= -\log(\sec x + \tan x) + \sin x + K_1$$

$$B = \int \frac{y_1(x) \phi(x)}{\omega} dx$$

$$= \int \frac{\cos x \cdot \tan x}{1} dx = \int \cos x \cdot \frac{\sin x}{\cos x} dx$$

$$= \int \sin x dx$$

$$B = -\cos x + K_2$$

$$y = (-\log(\sec x + \tan x) + K_1) \cos x + (-\cos x + K_2) \sin x$$

$$3. y'' - 2y' + 2y = e^x \tan x$$

Sol:- $D^2y - 2Dy + 2y = e^x \tan x$

$$(D^2 - 2D + 2)y = e^x \tan x$$

A.E is given by $m^2 - 2m + 2 = 0$

$$m = 1 \pm i \quad \text{R.P} = 1, \text{I.P} = 1$$

$$y_c = e^x (C_1 \cos x + C_2 \sin x) \Rightarrow C_1 e^x \cos x + C_2 e^x \sin x$$

Assume C.S as $Ay_1 + By_2$

$$y_1 = e^x \cos x \quad y_2 = e^x \sin x$$

$$y_1' = e^x(-\sin x) + \cos x \cdot e^x \quad y_2' = e^x \cos x + \sin x \cdot e^x$$

$$y_1' = \cos x \cdot e^x - \sin x \cdot e^x$$

$$\omega = y_1 y_2' - y_2 y_1'$$

$$= e^x \cos x (\cos x \cdot e^x + \sin x \cdot e^x) - e^x \sin x (\cos x \cdot e^x - \sin x \cdot e^x)$$

$$= e^{2x} (\cos^2 x + \cancel{\cos x \sin x} - \cancel{\sin x \cos x} + \sin^2 x)$$

$$\omega = e^{2x} (1)$$

$$\boxed{\omega = e^{2x}}$$

$$A = \int \frac{-y_2(x) \phi(x)}{\omega} dx$$

$$= \int \frac{-e^x \sin x \cdot e^x \tan x}{e^{2x}} dx = - \int \frac{e^{2x} \sin x \tan x}{e^{2x}} dx$$

$$= - \int \frac{\sin^2 x}{\cos x} dx = - \int \frac{1 - \cos^2 x}{\cos x} dx$$

$$= - \int \frac{1}{\cos x} dx + \int \cos x \cdot dx$$

$$= - \int \sec x \cdot dx + \int \cos x \cdot dx$$

$$A = -\log(\sec x + \tan x) + \sin x + k_1$$

$$B = \int \frac{y_1(x) \phi(x)}{\omega} dx = \int \frac{e^x \cos x \cdot e^x \tan x}{e^{2x}} dx$$

$$= \int \frac{e^{2x} \cdot \cos x \cdot \sin x}{e^{2x} \cos x} dx$$

$$B = \int \sin x \cdot dx = -\cos x + k_2$$

$$y = (-\log(\sec x + \tan x) + \sin x + k_1) \cos x + (-\cos x + k_2) \sin x$$

$$4. \quad y'' - 6y' + 9y = \frac{e^{3x}}{x^2}$$

$$\text{Sol:- } (D^2 - 6D + 9)y = \frac{e^{3x}}{x^2}$$

A.E is given by

$$m^2 - 6m + 9 = 0$$

$$(m-3)^2 = 0$$

$$m = 3, 3$$

$$y_c = (C_1 + C_2 x) e^{3x} \Rightarrow y_c = C_1 e^{3x} + C_2 e^{3x} \cdot x$$

C.S is given by $y = Ay_1 + By_2$

$$y = A e^{3x} + B e^{3x} \cdot x$$

$$y_1 = e^{3x}$$

$$y_2 = x e^{3x}$$

$$y_1' = 3e^{3x}$$

$$y_2' = x \cdot 3e^{3x} + e^{3x}$$

$$\omega = y_1 y_2' - y_2 y_1'$$

$$\omega = e^{3x} (3x \cdot e^{3x} + e^{3x}) - x e^{3x} (3e^{3x})$$

$$= e^{3x} \cdot e^{3x} (3x + 1 - 3x)$$

$$\boxed{\omega = e^{6x}}$$

$$5. y'' + 2y' + y = e^{-x} \log x$$

Sol: $D^2 y + 2Dy + y = e^{-x} \log x$

$$(D^2 + 2D + 1)y = e^{-x} \log x$$

A.E is given by $m^2 + 2m + 1 = 0$

$$m = -1, -1$$

$$y_c = (C_1 + C_2 x) e^{-x} \Rightarrow e^{-x} C_1 + C_2 e^{-x} x$$

C.S is $y = Ay_1 + By_2$

$$y = A e^{-x} + B e^{-x}$$

$$y_1 = e^{-x}$$

$$y_2 = e^{-x} \cdot x$$

$$y_1' = -e^{-x}$$

$$y_2' = -e^{-x}(x) + e^{-x}$$

$$\omega = y_1 y_2' - y_2 y_1'$$

$$= e^{-x}(-x e^{-x} + e^{-x}) - x e^{-x}(-e^{-x})$$

$$= e^{-2x}(-x + 1 + x)$$

$$\boxed{\omega = e^{-2x}}$$

$$A = \int \frac{-y_2(x) \phi(x)}{\omega} dx = \int \frac{-x \cdot e^{-x} \cdot e^{-x}}{e^{-2x}} \cdot d \log x \cdot dx$$

$$= - \int x \cdot \log x \cdot dx$$

$$= - \left[\log x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{1}{x} \cdot dx \right] + K_1$$

$$A = - \log x \cdot \frac{x^2}{2} - \frac{x^2}{4} + K_1$$

$$B = \int \frac{y_1(x) \phi(x)}{\omega} dx$$

$$= \int \frac{e^{-x} \cdot e^{-x} \cdot \log x}{e^{-2x}} dx = \int_1^x (1) \cdot \log x \cdot dx$$

$$= \log x \cdot x - \int x \cdot \frac{1}{x} \cdot dx + k_2$$

$$B = x \cdot \log x - x + k_2$$

$$y = \left(-\log x \cdot \frac{x^2}{2} - \frac{x^2}{4} + k_1 \right) e^{-x} + (x \cdot \log x - x + k_2) e^{-x} \cdot x$$

$$6. (D^2 + 3D + 2)y = e^x$$

$$\text{Sol:- } (D^2 + 3D + 2)y = e^x$$

$$\text{A.E is given by } m^2 + 3m + 2 = 0$$

$$m = -1, -2$$

$$y_c = C_1 e^{-x} + C_2 e^{-2x}$$

$$\text{C.S is given by } y = Ay_1 + By_2$$

$$y_1 = e^{-x}$$

$$y_2 = e^{-2x}$$

$$y_1' = -e^{-x}$$

$$y_2' = -2e^{-2x}$$

$$\omega = y_1 y_2' - y_2 y_1'$$

$$= e^{-x}(-2e^{-2x}) - e^{-2x}(-e^{-x})$$

$$= -2e^{-3x} + e^{-3x} = e^{-3x}(-2+1)$$

$$\boxed{\omega = -e^{-3x}}$$

$$A = \int \frac{-y_2(x) \phi(x)}{\omega} dx$$

$$A = \int \frac{e^{-2x} \cdot e^{e^x}}{e^{3x}} dx + k_1$$

$$= \int e^{-2x} \cdot e^{3x} \cdot e^{e^x} dx = \int e^x \cdot e^{e^x} dx$$

$$\text{put } e^x = z$$

$$e^x \cdot dx = dz$$

$$A = \int e^z dz + k_1$$

$$A = e^{e^x} + k_1$$

$$B = \int \frac{e^{-x} \cdot e^{e^x}}{-e^{3x}} dx + k_2$$

$$= - \int e^{-x} \cdot e^{3x} \cdot e^{e^x} dx + k_2$$

$$= - \int e^{2x} \cdot e^{e^x} dx + k_2 = - \int e^x \cdot e^{e^x} \cdot e^x dx + k_2$$

$$\text{put } e^x = z$$

$$e^x \cdot dx = dz$$

$$= - \int z \cdot e^z \cdot dz$$

$$= - \left[z \int e^z \cdot dz - \int e^z \frac{d(z)}{dz} dz \right] + k_2$$

$$B = - [z \cdot e^z - e^z] + k_2$$

$$B = - [e^x e^{e^x} - e^{e^x}] + k_2$$

$$\therefore y = (e^{e^x} + k_1) e^{-x} + (-e^x e^{e^x} + e^{e^x} + k_2) e^{-2x}$$

$$\Rightarrow (D^2+1)y = \operatorname{cosec} x \cdot \cot x$$

Sol: $(D^2+1)y = \operatorname{cosec} x \cdot \cot x$

A.E is given by $m^2+1=0$

$$\boxed{m = \pm i}$$

$$y_c = e^{0x} (C_1 \cos x + C_2 \sin x)$$

$$y_c = C_1 \cos x + C_2 \sin x$$

$$y = Ay_1 + By_2$$

$$y_1 = \cos x$$

$$y_2 = \sin x$$

$$y_1' = -\sin x$$

$$y_2' = \cos x$$

$$W = y_1 y_2' - y_2 y_1'$$

$$= \cos x (\cos x) - \sin x (-\sin x)$$

$$= \cos^2 x + \sin^2 x$$

$$\boxed{W = 1}$$

$$A = \int \frac{-y_2(x) \phi(x)}{W} dx$$

$$= - \int \frac{\sin x \cdot \operatorname{cosec} x \cdot \cot x}{1} dx$$

$$= - \int \cancel{\sin x} \cdot \frac{1}{\cancel{\sin x}} \cdot \cot x \cdot dx$$

$$= - \int \cot x \cdot dx$$

$$A = -\log \sin x + K_1$$

$$B = \int \frac{\cos x \cdot \operatorname{cosec} x \cdot \cot x}{1} dx$$

$$= \int \cos x \cdot \frac{1}{\sin x} \cdot \frac{\cos x}{\sin x} dx$$

$$= \int \frac{\cos^2 x}{\sin^2 x} dx$$

$$= \int \frac{1}{\sin^2 x} dx - \int 1 \cdot dx$$

$$= \int \operatorname{cosec}^2 x \cdot dx - \int dx$$

$$B = -\cot x - x + K_2$$

$$y = (-\log \sin x + K_1) \cos x + (-\cot x - x + K_2) \sin x$$

Differential Equation reducible to D.E constant co-efficient:

Here, we study the D.E involving the variable co-efficient. This can be reduced to equation with constant co-efficient by specific substitution.

Here we study two types of equations.

a) Legendre's linear second order D.E:-

If the given D.E is of the form

$$(ax+b)^2 \frac{d^2 y}{dx^2} + (ax+b) \frac{dy}{dx} + y = \phi(x) \rightarrow (1)$$

$(ax+b)^2 D^2 y + (ax+b) Dy + y = \phi(x)$. Then the D.E is called Legendre's linear second order D.E which contains variable where $\phi(x)$ is function of x

$$\text{put } \log(ax+b) = t$$

$$(ax+b) Dy = a Dy$$

$$(ax+b)^2 D^2 y = a^2 D(D-1)y$$

with this substitution eq-(1) reduces to differential equation with constant variable and is known method to solve it.

by Cauchy's linear second order D.E

if the given D.E is of the form

$$x^2 y'' + xy' + y = \phi(x)$$

$$\text{put } \log x = t \rightarrow x = e^t$$

$$xy' = Dy$$

$$x^2 y'' = D(D-1)y$$

1. Solve $(3x+2)^3 y'' + 3(3x+2)y' - 36y = 8x^3 + 4x + 1$

Sol:- put $\log(3x+2) = t$ $\Rightarrow 3x+2 = e^t$
 $(3x+2)y' = 3Dy$ $x = \frac{e^t - 2}{3}$
 $(3x+2)^3 y'' = 9D(D-1)y$

$\therefore (1)$ becomes

$$9D(D-1)y + 3(3Dy) - 36y = 8\left(\frac{e^t - 2}{3}\right)^3 + 4\left(\frac{e^t - 2}{3}\right) + 1$$

$$9(D^2 - D + D - 4)y = \frac{8}{9}(e^{3t} + 4 - 4e^t) + \frac{4}{3}(e^t - 2) + 1$$

$$9(D^2 - 4)y = \frac{1}{9} \left[\frac{8}{9}e^{3t} + \frac{32}{9} - \frac{32e^t}{9} + \frac{4e^t}{3} - \frac{8}{3} + 1 \right]$$

$$(D^2 - 4)y = \frac{1}{9} \left[\frac{8}{9}e^{3t} - \frac{32e^t}{9} + \frac{4e^t}{3} + \frac{32}{9} - \frac{5}{3} \right]$$

$$(D^2 - 4)y = \frac{1}{9} \left[\frac{8}{9}e^{3t} - \frac{20}{9}e^t + \frac{17}{9} \right]$$

A.E is $m^2 - 4 = 0$, $m = \pm 2$

$$y_c = C_1 e^{2t} + C_2 e^{-2t}$$

$$y_p = \frac{1}{9} \left[\frac{8}{9}e^{3t} - \frac{20}{9}e^t + \frac{17}{9} \right]$$

$$y_p = \frac{1}{9} \left[\frac{8}{9} \frac{e^{3t}}{(D^2 - 4)} - \frac{20}{9} \frac{e^t}{(D^2 - 4)} + \frac{17}{9} \frac{1 \times e^{0t}}{(D^2 - 4)} \right]$$

$$y_p = \frac{1}{9} \left[\frac{8}{9} \frac{e^{3t}}{0} - \frac{20}{9} \frac{e^t}{(-3)} + \frac{17}{9} \frac{1}{(-4)} \right]$$

$$y_p = \frac{1}{9} \left[\frac{8}{9} + \frac{e^{3t}}{27} + \frac{20e^t}{27} - \frac{17}{36} \right]$$

$$y_p = \frac{1}{9} \left[\frac{8}{9} + \frac{e^{3t}}{27} + \frac{20e^t}{27} - \frac{17}{36} \right]$$

$$y = y_c + y_p$$

$$y = C_1 e^{3t} + C_2 e^{-3t} + \frac{1}{9} \left[\frac{2}{9} t e^{3t} + \frac{20}{27} e^t - \frac{17}{36} \right]$$

$$y = C_1 (3x+2)^3 + C_2 (3x+2)^{-3} + \frac{1}{9} \left[\frac{2}{9} \log(3x+2) (3x+2)^{-3} + \frac{20}{27} (3x+2) - \frac{17}{36} \right]$$

2. Solve $(x+2)y'' - (x+2)y' + y = 3x+4 \rightarrow (1)$

put $\log(x+2) = t$

$$x+2 = e^t$$

$$(x+2)y' = Dy$$

$$x = e^t - 2$$

$$(x+2)^2 y'' = D(D-1)y$$

$\therefore (1)$ becomes

$$D(D-1)y - Dy + y = 3x+4$$

$$(D^2 - D)y - Dy + y = 3(e^t - 2) + 4$$

$$D^2 y - Dy - Dy + y = 3e^t - 6 + 4$$

$$D^2 y - 2Dy + y = 3e^t - 2$$

$$(D^2 - 2D + 1)y = 3e^t - 2$$

A.E is $m^2 - 2m + 1 = 0$

$$(m-1)^2 = 0$$

$$m = 1, 1$$

$$y_c = (C_1 + C_2 t) e^t$$

$$y_p = \frac{3x+4}{(D^2 - 2D + 1)}$$

$$y_p = \frac{3(e^t - 2) + 4}{D^2 - 2D + 1}$$

$$y_p = \frac{3e^t - 2}{D^2 - 2D + 1}$$

$$y_p = \frac{3e^t}{D^2 - 2D + 1} - \frac{2}{D^2 - 2D + 1}$$

Since 1 is root of A.E, Replacing $D \rightarrow \alpha$
deno. of 1st term becomes 0 two times.

$$y_p = \frac{3t^2 \cdot e^t}{2} - \frac{2e^{0t}}{0 - 0 + 1}$$

$$y_p = \frac{3t^2 e^t}{2} - 2$$

$$y = y_c - y_p$$

$$y = (C_1 + C_2 t)e^t + \frac{3t^2 e^t}{2} - 2$$

$$y = (C_1 + C_2 \log(x+2)) + \frac{3}{2} (\log(x+2))^2 (x+2) - 2$$

3. Solve $x^2 y'' - 3xy' + 4y = (1+x)^2$

Sol: put $\log x = t$, $x = e^t$

$$xy' = Dy$$

$$x^2 y'' = D(D-1)y$$

$$D(D-1)y - 3Dy + 4y = (1+e^t)^2$$

$$(D^2 - D - 3D + 4)y = 1 + e^{2t} + 2e^t$$

$$(D^2 - 4D + 4)y = 1 + e^{2t} + 2e^t$$

A.E is $m^2 - 4m + 4 = 0$

$$(m-2)^2 = 0$$

$$m = 2, 2$$

$$y_c = (C_1 + C_2 t) e^{2t}$$

$$P.I = y_p = \frac{1 + e^{2t} + 2e^t}{D^2 - 4D + 4}$$

$$y_p = \frac{1 \times e^{0t}}{D^2 - 4D + 4} + \frac{e^{2t}}{D^2 - 4D + 4} + 2 \cdot \frac{e^t}{D^2 - 4D + 4}$$

$D \rightarrow a=0 \quad D \rightarrow a=2 \quad D \rightarrow a=1$

$$y_p = \frac{1}{0-0+4} + \frac{e^{2t}}{4-8+4} + 2 \cdot \frac{e^t}{1-4+4}$$

$$= \frac{1}{4} + \frac{t \cdot e^{2t}}{2D-4} + 2e^t$$

$$D \rightarrow 2$$

$$y_p = \frac{1}{4} + \frac{t^2 e^{2t}}{2} + 2e^t$$

$$y = y_c + y_p$$

$$y = (C_1 + C_2 t) e^{2t} + \frac{1}{4} + \frac{t^2 e^{2t}}{2} + 2e^t$$

$$y = (C_1 + C_2 t) e^{2t} + \frac{1}{4} + \frac{t^2 x^2}{2} + 2x$$

4) Solve $x^2 y'' - xy' + 4y = \cos(\log x)$

Sol: put $\log x = t$ $x = e^t$

$$xy' = Dy$$

$$x^2 y'' = D(D-1)y$$

$$D(D-1)y - Dy + 4y = \cos(t)$$

$$(D^2 - D - D + 4)y = \cos t$$

$$(D^2 - 2D + 4)y = \cos t$$

$$\text{A.E is } m^2 - 2m + 4 = 0$$

$$m = 1 \pm \sqrt{3}i$$

$$y_c = e^t (C_1 \cos \sqrt{3}t + C_2 \sin \sqrt{3}t)$$

$$y_p = \frac{\cos t}{D^2 - 2D + 4}$$

$$D^2 \rightarrow -a^2 = -1$$

$$= \frac{\cos t}{-1 - 2D + 4} = \frac{\cos t}{-2D + 3} = -\frac{\cos t (2D + 3)}{(2D - 3)(2D + 3)}$$

$$= -\left[\frac{2(-\sin t) + 3\cos t}{4D^2 - 9} \right] \quad D^2 \rightarrow -a^2 \rightarrow -1$$

$$y_p = \frac{2 \sin t}{-13} + \frac{(-3\cos t)}{-13}$$

$$y = e^t (C_1 \cos \sqrt{3}t + C_2 \sin \sqrt{3}t) - \frac{2 \sin t}{13} + \frac{3 \cos t}{13}$$

$$y = x(C_1 \cos(\sqrt{3} \log x) + C_2 \sin(\sqrt{3} \log x)) - \frac{2}{13} \sin(\log x) + \frac{3}{13} \cos(\log x)$$

$$5. \quad x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + y = \sin(\log x^2)$$

Sol: $x^2 y'' + x y' + y = \sin(\log x^2)$

put $\log x = t$, $x = e^t$

$$x y' = Dy$$

$$x^2 y'' = D(D-1)y$$

$$D(D-1)y + Dy + y = \sin 2t$$

$$(D^2 - D + D + 1)y = \sin 2t$$

$$(D^2 + 1)y = \sin 2t$$

A.E is $m^2 + 1 = 0$

$$y_c = e^{0t} (C_1 \cos t + C_2 \sin t) \quad m = \pm i$$

$$y_p = \frac{\sin 2t}{D^2 + 1}$$

$$D^2 \rightarrow -\alpha^2 = -4$$

$$y_p = \frac{\sin 2t}{-4+1} = \frac{\sin 2t}{-3}$$

$$y_p = \frac{\sin 2t}{-3}$$

$$y = y_c + y_p$$

$$= e^{0t} (C_1 \cos t + C_2 \sin t) - \frac{\sin 2t}{3}$$

$$y = C_1 \cos(\log x) + C_2 \sin(\log x) - \frac{\sin(2 \log x)}{3}$$

$$6. x^2 y'' + xy' + 9y = 3x^2 + \sin(3 \log x)$$

Sol: put $\log x = t$, $e^t = x$

$$xy' = Dy$$

$$x^2 y'' = D(D-1)y$$

$$D(D-1)y + Dy + 9y = 3e^{3t} + \sin(3t)$$

$$(D^2 - D + D + 9)y = 3e^{3t} + \sin 3t$$

$$(D^2 - 9)y = 3e^{3t} + \sin 3t$$

A.E is $m^2 - 9 = 0$

$$m = \pm 3$$

$$y_c = (C_1 e^{3t} + C_2 e^{-3t})$$

$$P.I = y_p = \frac{3e^{3t} + \sin 3t}{D^2 - 9}$$

$$y_p = \frac{3e^{3t}}{D^2 - 9} + \frac{\sin 3t}{D^2 - 9}$$

$$= \frac{3e^{3t}}{-4-9} + \frac{\sin 3t}{9-9}$$

$$= \frac{3e^{3t}}{-5} + \frac{t \sin 3t \times (2D+9)}{(2D-9) \times (2D+9)}$$

$$= \frac{3e^{3t}}{-5} + \frac{t \sin 3t (2D+9)}{4D^2 - 81} \quad D^2 - 9$$

$$y_p = \frac{3e^{3t}}{-5} + \frac{t \cdot 2(3) \cos 3t + 9t \sin 3t}{4(-9) - 81}$$

$$y_p = \frac{3e^{3t}}{-5} + \left[\frac{6t \cos 3t + 9t \sin 3t}{-117} \right]$$

$$y_c = C_1 x^3 + C_2 x^{-3} + \frac{6 \log x \cos(3 \log x) + 9 \log x \sin(3 \log x)}{117}$$