

Module-1

Linear Algebra

VECTOR SPACE:

Rank of a matrix: The number of non-zero rows in row echelon form, and is denoted by $\rho(A)$

1. Find the Basis using row space

$$A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$$

$$\text{Sol : } R_2 \rightarrow -2R_1 + R_2, \quad R_3 \rightarrow -R_1 + R_3$$

Apply Row Echelon form (row elementary transformation)

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 1 & 1 & 3 \end{bmatrix}$$

$$R_3 \rightarrow R_2 + R_3$$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_2 \rightarrow -R_2$$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore \rho(A) = 2$, Linearly dependent.

$$\text{Basis} = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 1 & 3 \end{bmatrix}$$

1. 2. Find the Basis using row space

$$\begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix}$$

$$\text{Sol : } A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix}$$

$$: R_2 \rightarrow R_2 - 2R_1, \quad R_3 \rightarrow R_3 - R_1$$

$$A \sim \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & -14 & 16 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - 2R_2$$

$$A \sim \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A \sim \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & 0 & 0 \end{bmatrix}$$

$$: R_2 \rightarrow -R_2/7$$

$$A \sim \begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & -8/7 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \rho(A) = 2$$

3. Find the row space to find the Basis of the set of vectors $(0,1,-3,-1), (1,0,1,1), (3,1,0,2), (1,1,-2,0)$.

Sol

$$A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

$$: R_1 \leftrightarrow R_2$$

$$A \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - 3R_1, \quad R_4 \rightarrow R_4 - R_1$$

$$A \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - R_2, \quad R_4 \rightarrow R_4 - R_2$$

$$A \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore \rho(A) = 2$ and Linearly dependent.

$$\text{Basis} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \end{bmatrix}$$

5. Find the Basis and rank of matrix to the following

$$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

Sol : $R_1 \leftrightarrow R_2$

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

$$R_4 \rightarrow R_4 - (R_1 + R_2 + R_3)$$

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - 3R_2, \quad R_2 \rightarrow R_2 - 2R_1$$

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \rho(A) = 3 \quad \text{Basis} = \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 0 & 3 & 2 \end{bmatrix}$$

Find the Basis using Column space for the set of vectors $(1,1,0), (2,3,-2), (-1,-4,6)$

Solution:

The given set of vectors should be written as column matrix

$$\begin{bmatrix} 1 & 2 & -1 \\ 1 & 3 & -4 \\ 0 & -2 & 6 \end{bmatrix}$$

Using Row Reduced Echelon form

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -3 \\ 0 & -2 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{Basis are } \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$$

PROBLEMS on Linear combination.

1. Is $(6,5,8)$ is Linear combination of the set of vectors $(1,1,3), (1,-1,2), (3,1,1)$ Test for consistency and solve:

Solution: -

The set of vectors can be written using column matrix

$$\text{Let } [A:B] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & -1 & 2 & 5 \\ 3 & 1 & 1 & 8 \end{bmatrix}$$

$$: R_2 \rightarrow R_2 - R_1, R_3 \rightarrow R_3 - 3R_1$$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -2 & 1 & -1 \\ 0 & -2 & -2 & -10 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - R_2$$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -2 & 1 & -1 \\ 0 & 0 & -3 & -9 \end{bmatrix}$$

From this we have,

$\rho(A)=3$ and $\rho(A:B)=3$

The given system of equations is consistent.

Rewriting the above matrix in equation format we get,

$$-3z = -9; \quad -2y + z = -1; \quad x + y + z = 6$$

$$\mathbf{z} = 3 \quad -2y + 3 = -1 \quad x + 2 + 3 = 6$$

$$-2y = -4 \quad x = 6 - 5$$

$$\mathbf{Y} = 2 \quad \mathbf{x} = 1$$

Solving the above equations, we get,

$(x, y, z) = (1, 2, 3)$ is the required Unique Solution.

$$\begin{bmatrix} 6 \\ 5 \\ 8 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 6 \\ 5 \\ 8 \end{bmatrix} \text{ is linear combination of } \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

2. $(-3, -2, 7)$ is Linear span of the set of vectors $(1, 3, 2), (1, 1, 4), (1, -2, 7)$.

Solution:

The above set of vectors written using column matrix, so that it can be written as

$$\begin{bmatrix} 1 & 1 & 1: & 3 \\ 3 & 1 & -2: & 2 \\ 2 & 4 & 7: & 7 \end{bmatrix}$$

Using Row reduced Echelon form, we get

$$: R_2 \rightarrow R_2 - 3R_1, \quad R_3 \rightarrow R_3 - 2R_1$$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1: & 3 \\ 0 & -2 & -5: & -7 \\ 0 & 2 & 5: & 1 \end{bmatrix}$$

$$R_3 \rightarrow R_3 + R_2$$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1: & 3 \\ 0 & -2 & -5: & -7 \\ 0 & 0 & 0: & -6 \end{bmatrix}$$

We do not get the solution as left hand side of the matrix in the last row are zero.

$$\begin{bmatrix} -3 \\ -2 \\ 7 \end{bmatrix} \text{ is not linear span of } \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 7 \end{bmatrix}$$

3. Show that (12,33,20) is Linear combination of the set of vectors (2,4,8), (1,11,-3),(4,-1,2)

Solution: -

$$\text{Let } \begin{bmatrix} 2 & 1 & 4: & 12 \\ 4 & 11 & -1: & 33 \\ 8 & -3 & 2: & 20 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 2R_1, \quad R_3 \rightarrow R_3 - 4R_1$$

$$[A:B] \sim \begin{bmatrix} 2 & 1 & 4: & 12 \\ 0 & 9 & -9: & 9 \\ 0 & -7 & -14: & -28 \end{bmatrix}$$

$$R_2 \rightarrow \frac{1}{9}R_2 \quad R_3 \rightarrow -\frac{1}{7}R_3$$

$$[A:B] \sim \begin{bmatrix} 2 & 1 & 4: & 12 \\ 0 & 1 & -1: & 1 \\ 0 & 1 & 2: & 4 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - R_2$$

$$[A:B] \sim \begin{bmatrix} 2 & 1 & 4: & 12 \\ 0 & 1 & -1: & 1 \\ 0 & 0 & 3: & 3 \end{bmatrix}$$

The system of equations as follows:

$$2x_1 + x_2 + 4x_3 = 12 \quad ; \quad x_2 - x_3 = 1 \quad ; \quad 3x_3 = 3$$

Solving the above equations, we get,

$(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = (\mathbf{3}, 2, 1)$ is the required solution.

$(12, 33, 20)$ is Linear combination of the set of vectors $(2, 4, 8)$, $(1, 11, -3)$, $(4, -1, 2)$