

MODULE 3

Differential Equation of First Order and First Degree

An equation which contains one dependent variable and its derivatives with respect to one or more variables is called differential equations.

There are 2 types :-

1. Ordinary Differential Equations [ODE]

2. Partial Differential Equations [PDE]

1. Ordinary Differential Equations [ODE]

An equation which contains one dependent variables and its derivatives with respect to one independent variables is called Ordinary Differential Equations [ODE].

Ex:-

$$\Rightarrow x^2 \frac{d^2 y}{dx^2} - 2 \left[\frac{dy}{dx} \right]^2 + y = 0$$

$$\Rightarrow x^2 \frac{d^2 y}{dx^2} - 2xy \frac{dy}{dx} + 3y = 0$$

2. Partial Differential Equations [PDE]

An equation which contains one dependent variable and its derivatives with respect to two or more independent variables is called PDE.

Ex:-

$$\Rightarrow x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = z$$

$$\Rightarrow x^2 \frac{\partial^2 z}{\partial x^2} - 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial x} - 3 \frac{\partial z}{\partial y} + z = 0$$

Solution of ODE of 1st Order and 1st Degree

Step 1:- Write the given differential equation to the standard form.

$$M(x, y) dx + N(x, y) dy = 0 \quad \text{--- (1)}$$

and identify M and N.

Step 2:- Find $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$, if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then eqn (1) is called an exact differential equation and follows the solution as,

$$\int M(x, y) dx + \int (\text{The terms which do not contain } x \text{ in } N) dy = C$$

Step 03 :- If $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, the given D.E can be called as non-exact D.E

Step 04 :- Non-exact D.E may have a solutions by reading reducing to exact form as given below,

- * Find $f(x)$ by $\frac{1}{N} \left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right]$ and find the integrating factor $IF = e^{\int f(x) dx}$.
- * Find $g(y)$ through $\frac{1}{M} \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right] = g(y)$ and find the integrating factor $IF = e^{\int g(y) dy}$
- * Multiply the integrating factor to the given D.E on both sides and verify the exactness.

Problems :-

1. Solve $\frac{dy}{dx} + \frac{2x+3y-1}{3x+4y-2} = 0$

Sol :- $(3x+4y-2)dy + (2x+3y-1)dx = 0$
 $\Rightarrow \frac{(3x+4y-2)dy + (2x+3y-1)dx}{(3x+4y-2)dx} = 0$

$\Rightarrow (3x+4y-2)dy + (2x+3y-1)dx = 0$

$\Rightarrow (2x+3y-1)dx + (3x+4y-2)dy = 0$ ——— (1)

$Mdx + Ndy = 0$

$\therefore M = 2x+3y-1$, $N = 3x+4y-2$

$$\frac{\partial M}{\partial y} = 3, \quad \frac{\partial N}{\partial x} = 3$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Eqn (1) is an exact D.E

$\therefore$ The solution is ;

$$\int (2x + 3y - 1) dx + \int (4y - 2) dy = C$$

$$\Rightarrow 2 \int x dx + 3y \int 1 dx - \int 1 dx + 4 \int y dy - 2 \int 1 dy = C$$

$$\Rightarrow \frac{2x^2}{2} + 3yx - x + \frac{4y^2}{2} - 2y = C$$

$$\Rightarrow x^2 + 3yx - x + 2y^2 - 2y = C$$

2. Solve $(2x + y + 1) dx + (x + 2y + 1) dy = 0$

$$\Rightarrow (2x + y + 1) dx + (x + 2y + 1) dy = 0 \quad \text{--- (1)}$$

$$M dx + N dy = 0$$

$$M = 2x + y + 1$$

$$N = x + 2y + 1$$

$$\frac{\partial M}{\partial y} = 1$$

$$\frac{\partial N}{\partial x} = 1$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Eqn (1) is an exact D.E

$\therefore$ The solution is :

$$\int (2x + y + 1) dx + \int (x + 2y + 1) dy = 0$$

$$\Rightarrow 2 \int x dx + y \int 1 dx + \int 1 dx + 2 \int y dy + \int 1 dy = 0$$

$$\Rightarrow \frac{2x^2}{2} + yx + x + \frac{2y^2}{2} + y = C$$

$$\Rightarrow x^2 + yx + y + x = C$$

3. Solve $(5x^4 + 3x^2y^2 - 2xy^3)dx + (2x^3y - 3x^2y^2 - 5y^4)dy = 0$

$$\Rightarrow (5x^4 + 3x^2y^2 - 2xy^3)dx + (2x^3y - 3x^2y^2 - 5y^4)dy = 0$$

$$Mdx + Ndy = 0$$

$$M = 5x^4 + 3x^2y^2 - 2xy^3, \quad N = 2x^3y - 3x^2y^2 - 5y^4$$

$$\frac{\partial M}{\partial y} = 6x^2y - 6xy^2$$

$$\frac{\partial N}{\partial x} = 6x^2 - 6xy^2$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Eqn (1) is an EDE

$\therefore$ The soln is;

$$\int (5x^4 + 3x^2y^2 - 2xy^3)dx + \int (-5y^4)dy = 0$$

$$\Rightarrow 5 \int x^4 dx + 3y^2 \int x^2 dx - 2y^3 \int x dx - 5 \int y^4 dy = C$$

$$\Rightarrow \cancel{5} \frac{x^5}{\cancel{5}} + 3y^2 \frac{x^3}{3} - 2y^3 \frac{x^2}{2} - \cancel{5} \frac{y^5}{\cancel{5}} = C$$

$$\Rightarrow x^5 + x^3y^2 - x^2y^3 - y^5 = C$$

4. Solve, $\frac{dy}{dx} + \frac{y \cos x + \sin y + y}{\sin x + x \cos y + x} = 0$

$$\Rightarrow (\sin x + x \cos y + x) dy + (y \cos x + \sin y + y) dx = 0$$

$$\Rightarrow (y \cos x + \sin y + y) dx + (\sin x + x \cos y + x) dy = 0 \quad \text{--- (1)}$$

$$Mdx + Ndy = 0$$

$$M = y \cos x + \sin y + y, \quad N = \sin x + x \cos y + x$$

$$\frac{\partial M}{\partial y} = \cos x + \cos y + 1$$

$$\frac{\partial N}{\partial x} = \cos x + \cos y + 1$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Eqn (1) is an EDE

∴ The solution is :-

$$\int (y \cos x + \sin y + y) dx + \int 0 \cdot dy = C$$

$$\Rightarrow y \sin x + \sin y \int 1 dx + y \int 1 dx = C$$

$$\Rightarrow y \sin x + \sin y x + xy = C$$

5. Solve $(1 + e^{x/y}) dx + e^{x/y} (1 - x/y) dy = 0$

$$\Rightarrow (1 + e^{x/y}) dx + e^{x/y} (1 - x/y) dy = 0 \quad \text{--- (1)}$$

$$M \cdot dx + N \cdot dy = 0$$

$$M = 1 + e^{x/y}$$

$$N = e^{x/y} (1 - x/y)$$

$$\frac{\partial M}{\partial y} = e^{x/y} - \frac{x}{y^2}$$

$$= -\frac{x}{y^2} e^{x/y}$$

$$\frac{\partial N}{\partial x} = e^{x/y} \left[-\frac{1}{y}\right] + \left[1 - \frac{x}{y}\right] e^{x/y} \cdot \frac{1}{y}$$

$$= -\frac{e^{x/y}}{y} + \frac{e^{x/y}}{y} - \frac{x}{y^2} e^{x/y}$$

$$= -\frac{x}{y^2} e^{x/y}$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Eqn (1) is an exact d.e

$$\therefore \int (1 + e^{x/y}) dx + \int 0 \cdot dy = C$$

$$\Rightarrow x + \frac{e^{x/y}}{1/y} = C$$

$$\Rightarrow x + y e^{x/y} = C$$

6. Solve $[4x^3y^2 + y\cos(xy)]dx + [2x^4y + x\cos(xy)]dy$

$$\Rightarrow [4x^3y^2 + y\cos(xy)]dx + [2x^4y + x\cos(xy)]dy = 0 \quad \text{--- (1)}$$

$$M \cdot dx + N \cdot dy = 0$$

$$M = 4x^3y^2 + y\cos(xy)$$

$$\Rightarrow \frac{\partial M}{\partial y} = 8x^3y + y(-\sin xy)x + \cos(xy)(1)$$

$$= 8x^3y + xy(\sin xy) + \cos xy$$

$$= 8x^3y - xy\sin xy + \cos xy$$

$$N = 2x^4y + x\cos(xy)$$

$$\Rightarrow \frac{\partial N}{\partial x} = 8x^3y + x(-\sin xy)y + \cos(xy)(1)$$

$$= 8x^3y - xy\sin xy + \cos xy$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Eqn (1) is an EDE

$\therefore$ The solution is,

$$\int [4x^3y^2 + y\cos(xy)]dx + \int 0 dy = c$$

$$\Rightarrow \frac{4x^4y^3}{4} + \frac{y\sin(xy)}{y} = c$$

$$\Rightarrow x^4y^3 + \sin(xy) = c$$

7. Solve $[y(1 + \frac{1}{x}) + \cos y]dx + [x + \log x - x\sin y]dy = 0$

$$\Rightarrow [y(1 + \frac{1}{x}) + \cos y]dx + [x + \log x - x\sin y]dy = 0$$

$$M dx + N dy = 0$$

$$M = y\left(1 + \frac{1}{x}\right) + \cos y$$

$$M = y + \frac{y}{x} + \cos y$$

$$\frac{\partial M}{\partial y} = 1 + \frac{1}{x} - \sin y$$

$$N = x + \log x - x \sin y$$

$$\frac{\partial N}{\partial x} = 1 + \frac{1}{x} - \sin y$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Eqn (1) is an exact d.e

∴ The soln is,

$$\int \left(y\left(1 + \frac{1}{x}\right) + \cos y\right) dx + \int 0 dy = C$$

$$\Rightarrow y \int 1 \cdot dx + y \int \frac{1}{x} dx + \cos y \int 1 \cdot dx = C$$

$$\Rightarrow xy + y \log x + x \cos y = C$$

8. Solve $(x^2 + y^2 + x)dx + xy dy = 0$

$$\Rightarrow (x^2 + y^2 + x)dx + xy dy = 0$$

$$M dx + N dy$$

$$M = x^2 + y^2 + x$$

$$N = xy$$

$$\frac{\partial M}{\partial y} = 2y$$

$$\frac{\partial N}{\partial x} = y$$

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

let,

$$\frac{1}{N} \left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right] = \frac{1}{xy} (2y - y)$$

$$= \frac{y}{xy} = \frac{1}{x} = f(x)$$

$$\begin{aligned}
 \therefore IF &= e^{\int f(x) dx} \\
 &= e^{\int \frac{1}{x} dx} \\
 &= e^{\log_e x} \\
 &= x
 \end{aligned}$$

$$\begin{aligned}
 \text{It} \Rightarrow \text{Eqn (1)} \times IF &\Rightarrow x(x^2 + y^2 + x) dx + x \cdot xy dy = 0 \\
 &\Rightarrow (x^3 + xy^2 + x^2) dx + x^2 y dy = 0 \quad \text{--- (2)} \\
 &\Rightarrow M' dx + N' dy = 0
 \end{aligned}$$

$$\therefore \frac{\partial M'}{\partial y} = 2xy, \quad \frac{\partial N'}{\partial x} = 2xy$$

$$\therefore \frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x}$$

eqn (2) is an EDE

$\therefore$ The solution is,

$$\begin{aligned}
 &\int [x^3 + xy^2 + x^2] dx + \int 0 dy = C \\
 \Rightarrow &\int x^3 dx + y^2 \int x dx + \int x^2 dx = C \\
 \Rightarrow &\frac{x^4}{4} + y^2 \frac{x^2}{2} + \frac{x^3}{3} = C \\
 \Rightarrow &3x^4 + 6x^2 y^2 + 4x^3 = 12C
 \end{aligned}$$

$$\begin{aligned}
 \text{Q. } (4xy + 3y^2 - x) dx + x(x + 2y) dy &= 0 \\
 \Rightarrow (4xy + 3y^2 - x) dx + x(x + 2y) dy &= 0 \quad \text{--- (1)}
 \end{aligned}$$

$$M dx + N dy = 0$$

$$M = 4xy + 3y^2 - x$$

$$N = x(x + 2y) = x^2 + 2xy$$

$$\frac{\partial M}{\partial y} = 4x + 6y$$

$$\frac{\partial N}{\partial x} = 2x + 2y$$

Let,

$$\begin{aligned}\frac{1}{N} \left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right] &= \frac{1}{x(x+2y)} [2x+4y] \\ &= \frac{2(x+2y)}{x(x+2y)} \\ &= \frac{2}{x} = f(x)\end{aligned}$$

$$\begin{aligned}\text{I.F} &= e^{\int f(x) dx} \\ &= e^{\int \frac{2}{x} dx} \\ &= e^{2 \int \frac{1}{x} dx} \\ &= e^{2 \log_e x} \\ &= e^{\log_e x^2} \\ &= x^2\end{aligned}$$

$$\textcircled{1} \times \text{IF} \Rightarrow x^2(4xy + 3y^2 - x) dx + x^2 \cdot x(x+2y) dy = 0$$

$$\Rightarrow (4x^3y + 3x^2y^2 - x^3) dx + (x^4 + 2x^3y) dy = 0$$

$$\frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x}$$

eqn (2) is EDE

$\therefore$ The solution is,

$$\int (4x^3y + 3y^2x^2 - x^3) dx - \int 0 dy = C$$

$$\Rightarrow 4y \int x^3 dx + 3y^2 \int x^2 dx - \int x^3 dx = C$$

$$\Rightarrow \frac{4yx^4}{4} + \frac{3y^2x^3}{3} - \frac{x^4}{4} = C$$

$$\Rightarrow yx^4 + y^2x^3 - \frac{x^4}{4} = C$$

10. Solve $(x^3 + y^3 + 6x) dx + (xy)^2 dy = 0$

$\Rightarrow (x^3 + y^3 + 6x) dx + (xy)^2 dy = 0 \quad \text{---(1)}$

$M dx + N dy = 0$

$M = x^3 + y^3 + 6x$

$N = xy^2$

$\frac{\partial M}{\partial y} = 3y^2$

$\frac{\partial N}{\partial x} = y^2$

let, $\frac{1}{N} \left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right] = \frac{1}{xy^2} [3y^2 - y^2]$

$= \frac{1}{xy^2} [2y^2]$

$= \frac{2}{x} = f(x)$

I.F. = $e^{\int f(x) dx}$

$= e^{\int 2/x dx}$

$= e^{2 \int 1/x dx}$

$= e^{2 \log x}$

$= e^{\log_e x^2}$

$= x^2$

① $\times$ IF = $x^2 [x^3 + y^3 + 6x] dx + x^2 [xy^2] dy = 0$

$= x^5 + x^2 y^3 + 6x^3 dx + x^3 y^2 dy = 0 \quad \text{---(2)}$

$\therefore \frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x}$

eqn (2) is an EDE

The solution is,

$\int x^5 + x^2 y^3 + 6x^3 dx + \int 0 dy = C$

$\Rightarrow \int x^5 dx + y^3 \int x^2 dx + 6 \int x^3 dx = C$

$$\Rightarrow \frac{x^6}{6} + y^3 \frac{x^3}{3} + 6 \frac{x^4}{4} = C$$

$$\Rightarrow 2x^6 + 4x^3y^3 + 18x^4 = 12C$$

11. Solve $(y \log y) dx + (x - \log y) dy = 0$

$$\Rightarrow (y \log y) dx + (x - \log y) dy = 0 \quad \text{--- (1)}$$

$$M dx + N dy = 0$$

$$M = y \log y$$

$$N = x - \log y$$

$$\frac{\partial M}{\partial y} = 1 \cdot \log y + y \cdot \frac{1}{y}$$

$$\frac{\partial N}{\partial x} = 1$$

$$= \log y + 1$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$\text{let, } \frac{1}{M} \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right] = \frac{1}{y \log y} [1 - (\log y + 1)]$$

$$= \frac{1}{y \log y} [x - \log y - x]$$

$$= - \frac{\log y}{y \log y}$$

$$= - \frac{1}{y} = g(y)$$

$$\therefore IF = e^{\int g(y) dy}$$

$$= e^{\int -\frac{1}{y} dy}$$

$$= e^{-\log y}$$

$$= e^{\log(y^{-1})}$$

$$= e^{\log(1/y)}$$

$$= \frac{1}{y}$$

$$\textcircled{1} \times \text{I.F} \Rightarrow \frac{1}{y} [y \log y] dx + \frac{1}{y} [x - \log y] dy = 0$$

$$\Rightarrow \log y dx + \left[\frac{x}{y} - \frac{\log y}{y} \right] dy = 0$$

$$\Rightarrow \log y dx + \left[\frac{x}{y} - \frac{\log y}{y} \right] dy = 0$$

$$\frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x}$$

The solution is,

$$\int \log y dx + \int \left[-\frac{\log y}{y} \right] dy = C$$

$$\Rightarrow \log y \int 1 dx + \int \frac{1}{y} \log y dy = C \quad \text{--- (3)}$$

$$\text{Let, } I = \int \frac{1}{y} \log y$$

$$\text{formula :- } \int u v dx = u \int v dx - \int (u' \int v dx) dx$$

$$\Rightarrow I = \log y \cdot \int \frac{1}{y} dx - \int \left(\frac{1}{y} \int \frac{1}{y} dy \right) dx$$

$$\Rightarrow I = \log y \cdot \log y - \int \frac{1}{y} \log y dy$$

$$\Rightarrow I = (\log y)^2 - I$$

$$\Rightarrow I + I = (\log y)^2$$

$$\Rightarrow 2I = (\log y)^2$$

$$\Rightarrow I = \frac{1}{2} (\log y)^2$$

from eqn (3)

$$\Rightarrow x \log y - \frac{1}{2} (\log y)^2 = C$$

12. Solve $y(2xy+1)dx - x dy = 0$

$\Rightarrow y(2xy+1)dx - x dy = 0 \quad \text{---(1)}$

$Mdx + Ndy = 0$

$M = y(2xy+1)$

$N = -x$

$M = 2xy^2 + y$

$\frac{\partial N}{\partial x} = -1$

$\frac{\partial M}{\partial y} = 4xy + 1$

$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

let, $\frac{1}{M} \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right] = \frac{1}{y(2xy+1)} [-1 - 4xy - 1]$

$= \frac{-2 - 4xy}{y(2xy+1)}$

$= \frac{-2(1+2xy)}{y(1+2xy)}$

$= -\frac{2}{y} = g(y)$

$I.F = e^{\int g(y) dy}$

$= e^{-2 \int \frac{1}{y} dy}$

$= e^{-2 \log y}$

$= e^{\log_e (1/y^2)}$

$= \frac{1}{y^2}$

$\textcircled{1} \times I.F \Rightarrow \frac{1}{y^2} y(1+2xy)dx - \frac{x}{y^2} dy = 0$

$\Rightarrow \frac{1}{y} (1+2xy) dx - \frac{x}{y^2} dy = 0$

$$\Rightarrow \left[\frac{1}{y} + 2x \right] dx - \frac{x}{y^2} dy = 0$$

$$\frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x}$$

$\therefore$ The solution is,

$$\Rightarrow \int \left(\frac{1}{y} + 2x \right) dx + \int 0 dy = C$$

$$\Rightarrow \frac{1}{y} \int 1 dx + 2 \int x dx = C$$

$$\Rightarrow \frac{x}{y} + x^2 = C$$

Solution of Linear Differential Equation of First Order and First Degree.

Step 01 :- Write the given differential equation to the standard form.

$$\frac{dy}{dx} + P(x)y = Q(x)$$

Step 02 :- Identify P and Q and find the integrating factor
 $IF = e^{\int P(x) dx}$

Step 03 :- Write the soln as given below

$$y \times IF = \int Q(x) \times IF dx + C$$

Similarly, for the D.E

$$\frac{dx}{dy} + P(y)x = Q(y)$$

we get the soln

$$x \times IF = \int Q(y) \times IF dy + C$$

when, $IF = e^{\int P(y) dy}$

Bernoulli's Equation

Step 01 :- Define the Bernoulli's equation

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \text{ --- (1)}$$

Step 02 :- Divide y^n on b.s and reduce the equation as,

$$\text{eqn (1)} \Rightarrow \frac{1}{y^n} \frac{dy}{dx} + P(x) \cdot \frac{1}{y^{n-1}} = Q(x) \text{ --- (2)}$$

Step 03 :- Assume

$\frac{1}{y^{n-1}} = u$ and differentiate the eqn w.r to 'x'

$$\Rightarrow \frac{1}{y^n} \frac{dy}{dx} = \frac{1}{1-n} \frac{du}{dx}$$

Step 04 :- Reduce the eqn (2) to the linear form and follow the same procedure.

Step 05 :- The similar process can be applicable for the Bernoulli's equation.

$$\frac{dx}{dy} + P(y)x = Q(y)x^n$$

1. Solve $\frac{dy}{dx} - \frac{y}{x} = 2x^2$

$$\Rightarrow \frac{dy}{dx} - \frac{y}{x} = 2x^2$$

$$\Rightarrow \frac{dy}{dx} + \left[-\frac{1}{x}\right]y = 2x^2$$

where, $P = -\frac{1}{x}$ $Q = 2x^2$

$$\begin{aligned}
 \therefore \text{I.F} &= e^{\int P(x) dx} \\
 &= e^{\int -\frac{1}{x} dx} \\
 &= e^{-\log x} \\
 &= e^{\log_e(1/x)} \\
 &= \frac{1}{x}
 \end{aligned}$$

The solution is,

$$\begin{aligned}
 Y \times \text{IF} &= \int Q \times \text{IF} dx + C \\
 \Rightarrow Y \left[\frac{1}{x} \right] &= \int 2x^2 \cdot \frac{1}{x} dx + C \\
 \Rightarrow \frac{Y}{x} &= 2 \int x dx + C \\
 \Rightarrow \frac{Y}{x} &= 2 \frac{x^2}{2} + C \\
 \Rightarrow \frac{Y}{x} &= x^2 + C
 \end{aligned}$$

Q. Solve, $\frac{dy}{dx} + \frac{y}{x} = y^2 x$

$$\Rightarrow \frac{dy}{dx} + \frac{y}{x} = y^2 x \quad \text{--- (1)}$$

$$\Rightarrow \frac{dy}{dx} + \frac{1}{x} (y) = y^2 x$$

$$\Rightarrow \frac{1}{y^2} \frac{dy}{dx} + \frac{1}{x} \cdot \frac{1}{y} = \frac{xy^2}{y^2}$$

$$\Rightarrow \frac{1}{y^2} \frac{dy}{dx} + \frac{1}{x} \cdot \frac{1}{y} = x \quad \text{--- (2)}$$

$$\text{let, } \frac{1}{y^2} = u \quad \text{--- (3)}$$

diff (3) w.r to x

$$(3) \Rightarrow \frac{1}{y^2} \frac{dy}{dx} = \frac{du}{dx}$$

$$\Rightarrow \frac{1}{y^2} \frac{dy}{dx} = - \frac{du}{dx}$$

$$(2) \Rightarrow - \frac{du}{dx} + \frac{1}{x} u = x$$

$$\Rightarrow \frac{du}{dx} - \frac{1}{x} u = -x$$

$$\Rightarrow \frac{du}{dx} + \left[-\frac{1}{x} \right] u = -x$$

$$\therefore p = -\frac{1}{x}, \quad Q = -x$$

$$\begin{aligned} \text{I.F} &= e^{-\int \frac{1}{x} dx} \\ &= e^{-\log x} \\ &= e^{\log(1/x)} \\ &= \frac{1}{x} \end{aligned}$$

The solution is,

$$u \times \text{IF} = \int Q(x) \times \text{IF} dx + C$$

$$\Rightarrow u \left(\frac{1}{x} \right) = - \int x \cdot \frac{1}{x} dx + C$$

$$\Rightarrow \frac{u}{x} = - \int 1 dx + C$$

$$\Rightarrow \frac{u}{x} = -x + C$$

$$\Rightarrow \frac{1}{xy} = -x + C$$

$$\Rightarrow \frac{1}{xy} + x = C$$

3. Solve, $\frac{dy}{dx} + xy = xy^3$

$$\Rightarrow \frac{dy}{dx} + xy = xy^3 \text{ --- (1)}$$

$$\Rightarrow \frac{1}{y^3} \frac{dy}{dx} + \frac{xy}{y^3} = \frac{xy^3}{y^3}$$

$$\Rightarrow \frac{1}{y^3} \frac{dy}{dx} + \frac{x}{y^2} = x \text{ --- (2)}$$

Let, $\frac{1}{y^2} = u$ --- (3)

diff w.r to 'x'

$$(3) \Rightarrow -\frac{2}{y^3} \frac{dy}{dx} = \frac{du}{dx}$$

$$\Rightarrow \frac{1}{y^3} \frac{dy}{dx} = -\frac{1}{2} \frac{du}{dx}$$

$$(2) \Rightarrow -\frac{1}{2} \frac{du}{dx} + xu = x$$

$$\Rightarrow \frac{du}{dx} - 2xu = -2x \quad (\times 2)$$

$$\Rightarrow \frac{du}{dx} + (-2x)u = -2x$$

$$P = -2x, \quad Q = -2x$$

$$\therefore IF = e^{\int P(x) dx}$$

$$= e^{-2 \int x dx}$$

$$= e^{-2 \frac{x^2}{2}}$$

$$= e^{-x^2}$$

$\therefore$ The solution is,

$$u \times IF = \int Q(x) \times I.F dx + C$$

$$\Rightarrow ue^{-x^2} = \int (-2x) e^{-x^2} dx + C$$

$$\Rightarrow \frac{e^{-x^2}}{y^2} = \int e^{-x^2} (-2x) dx + c \quad \text{---(4)}$$

$$\text{let, } -x^2 = t$$

$$\Rightarrow -2x dx = dt$$

$$\therefore (4) \Rightarrow \frac{e^{-x^2}}{y^2} = \int e^t dt + c$$

$$\Rightarrow \frac{e^{-x^2}}{y^2} = e^t + c$$

$$\Rightarrow \frac{e^{-x^2}}{y^2} = e^{-x^2} + c$$

$$\Rightarrow e^{-x^2} = (e^{-x^2} + c) y^2$$

$$4. \text{ Solve } x \left[\frac{dy}{dx} \right] + y = x^3 y^6$$

$$\Rightarrow x \left[\frac{dy}{dx} \right] + y = x^3 y^6$$

$$\Rightarrow \frac{dy}{dx} + \frac{1}{x} y = x^2 y^6 \quad \text{---(1)}$$

$$\Rightarrow \frac{1}{y^6} \frac{dy}{dx} + \left[\frac{1}{x} \right] \left[\frac{1}{y^5} \right] = x^2 \quad \text{---(2)}$$

$$\text{let, } \frac{1}{y^5} = u$$

$$\Rightarrow -\frac{5}{y^6} \cdot \frac{dy}{dx} = \frac{du}{dx}$$

$$\Rightarrow \frac{1}{y^6} \frac{dy}{dx} = -\frac{1}{5} \frac{du}{dx}$$

$$(2) \Rightarrow -\frac{1}{5} \frac{du}{dx} + \frac{1}{x} u = x^2$$

$$\Rightarrow \frac{du}{dx} + \left[-\frac{5}{x} \right] u = -5x^2 \text{ --- (3)}$$

$$\therefore P = -\frac{5}{x}, \quad Q = -5x^2$$

$$\begin{aligned} \text{I.F} &= e^{\int P(x) dx} \\ &= e^{-5 \int \frac{1}{x} dx} \\ &= e^{-5 \log x} \\ &= e^{\log_e (1/x^5)} \\ &= \frac{1}{x^5} \end{aligned}$$

$\therefore$ The solution is,

$$u \times \text{I.F} = \int Q \times \text{I.F} dx + C$$

$$\Rightarrow u \left(\frac{1}{x^5} \right) = - \int 5x^2 \cdot \frac{1}{x^5} dx + C$$

$$\Rightarrow \frac{u}{x^5} = -5 \int \frac{1}{x^3} dx + C$$

$$\Rightarrow \frac{u}{x^5} = -5 \frac{x^{-3+1}}{-3+1} + C$$

$$\Rightarrow \frac{u}{x^5} = \frac{-5}{-2} x^{-2} + C$$

$$\Rightarrow \frac{1}{x^5 y^5} = \frac{5}{2} x^2 + C$$

5. Solve, $\frac{dy}{dx} - y \tan x = y^2 \sec x$

$$\Rightarrow \frac{dy}{dx} - y \tan x = y^2 \sec x$$

$$\Rightarrow \frac{1}{y^2} \frac{dy}{dx} - (\tan x) \left(\frac{1}{y}\right) = \sec x \quad \text{--- (1)}$$

$$\text{let, } \frac{1}{y} = u$$

$$\Rightarrow -\frac{1}{y^2} \frac{dy}{dx} = \frac{du}{dx}$$

$$\Rightarrow \frac{1}{y^2} \frac{dy}{dx} = -\frac{du}{dx}$$

$$(1) \Rightarrow -\frac{du}{dx} - (\tan x)u = \sec x$$

$$\Rightarrow \frac{du}{dx} + \tan x \cdot u = -\sec x \quad \text{--- (2)}$$

$$p = \tan x, \quad Q = -\sec x$$

$$\begin{aligned} \therefore \text{I.F} &= e^{\int \tan x dx} \\ &= e^{\log_e |\sec x|} \\ &= \sec x \end{aligned}$$

$\therefore$ The solution is,

$$u \sec x = -\int \sec x \cdot \sec x dx + C$$

$$\Rightarrow \frac{1}{y} \sec x = -\int \sec^2 x dx + C$$

$$\Rightarrow \frac{1}{y} \sec x = -\tan x + C$$

$$\Rightarrow \frac{1}{y} \sec x + \tan x = C$$

$$6. \text{ solve, } \frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$$

$$\Rightarrow \frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$$

$$\Rightarrow \frac{1}{\cos^2 y} \frac{dy}{dx} + 2x \frac{\sin y \cos y}{\cos^2 y} = x^3$$

$$\Rightarrow \sec^2 y \frac{dy}{dx} + 2x \frac{\sin y}{\cos y} = x^3$$

$$\Rightarrow \sec^2 y \frac{dy}{dx} + 2x \tan y = x^3 \quad \text{--- (1)}$$

$$\text{let, } \tan y = u$$

$$\sec^2 y \cdot \frac{dy}{dx} = \frac{du}{dx}$$

$$\Rightarrow \frac{du}{dx} + 2x u = x^3$$

$$\therefore P = 2x, \quad Q = x^3$$

$$\therefore IF = e^{\int 2x dx}$$

$$= e^{x^2}$$

The solution is ;

$$u e^{x^2} = \int x^3 e^{x^2} dx + C$$

$$\Rightarrow e^{x^2} \tan y = \int x^2 \cdot e^{x^2} \cdot x dx + C \quad \text{--- (2)}$$

$$\text{let, } x^2 = t$$

$$2x dx = dt$$

$$x dx = \frac{1}{2} dt$$

$$(2) \Rightarrow e^{x^2} \tan y = \frac{1}{2} \int t \cdot e^t dt + C$$

$$\Rightarrow e^{x^2} \tan y = \frac{1}{2} e^t (t-1) + c$$

$$\Rightarrow e^{x^2} \tan y = \frac{1}{2} e^{x^2} (x^2-1) + c$$

$$\text{Solve } xy(1+xy^2) \frac{dy}{dx} = 1$$

$\Rightarrow$ Given,

$$\Rightarrow xy(1+xy^2) \frac{dy}{dx} = 1$$

$$\Rightarrow (xy + x^2y^3) \frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{xy + x^2y^3}$$

$$\Rightarrow \frac{dx}{dy} = xy + x^2y^3$$

$$\Rightarrow \frac{dx}{dy} - xy = x^2y^3$$

$$\text{let, } \frac{1}{x} = u$$

$$\Rightarrow -\frac{1}{x^2} \frac{dx}{dy} = \frac{du}{dy}$$

$$\Rightarrow \frac{1}{x^2} \frac{dx}{dy} = -\frac{du}{dy}$$

$$\therefore (i) \Rightarrow -\frac{du}{dy} - yu = y^3$$

$$\Rightarrow \frac{du}{dy} + yu = -y^3$$

$$\therefore p(y) = y, \quad Q(y) = -y^3$$

$$\begin{aligned} \text{I. } I &= e^{\int P(y) dy} \\ &= e^{\int y dy} \\ &= e^{y^2/2} \end{aligned}$$

$\therefore$ The solution is,

$$\begin{aligned} u e^{y^2/2} &= -\int y^3 \cdot e^{y^2/2} dy + c \\ \Rightarrow \frac{1}{x} e^{y^2/2} &= -\int y^2 \cdot e^{y^2/2} \cdot y dy + c \quad \text{---(2)} \end{aligned}$$

$$\begin{aligned} \text{Let, } \frac{y^2}{2} &= t \\ \Rightarrow y^2 &= 2t \\ \Rightarrow 2y dy &= 2 dt \\ \Rightarrow y dy &= dt \end{aligned}$$

$$\begin{aligned} \therefore (2) \Rightarrow \frac{1}{x} e^{y^2/2} &= -\int y^2 \cdot e^t dt + c \\ \Rightarrow \frac{1}{x} e^{y^2/2} &= -\frac{y^3}{3} e^t + c \end{aligned}$$

Orthogonal Trajectories :-

If the given two curves can intersect perpendicularly then those curves are said to be orthogonally to each other.

Orthogonal Trajectory In Cartesian Form :-

Step 1 :- Let the given curve be $F(x, y, c) = 0$ ---(1)

Step 2 :- Diff eqn (1) w.r to 'x' and get the differential equation as

$$f\left[x, y, \frac{dy}{dx}\right] = 0 \quad \text{--- (2)}$$

Step 3 :- To get orthogonally trajectory, replace

$$\frac{dy}{dx} = -\frac{dx}{dy} \text{ in eqn (2), we get}$$

$$(2) \Rightarrow f\left[x, y, -\frac{dx}{dy}\right] = 0 \quad \text{--- (3)}$$

Step 4 :- Solve eqn(3) and get the solution, say $G(x, y, c') = 0$, which is called the orthogonal trajectory of the given curve.

Orthogonal Trajectory In Polar Form :-

Step 1 :- Let the given polar curve be

$$F(r, \theta, c) = 0 \quad \text{--- (1)}$$

Step 2 :- Diff eqn(1) w.r to ' θ ' and get the

$$F\left(r, \theta, \frac{dr}{d\theta}\right) = 0 \quad \text{--- (2)}$$

Step 3 :- To get orthogonally trajectory, replace

$$\frac{dr}{d\theta} = -r^2 \frac{d\theta}{dr}, \text{ we get}$$

$$(2) \Rightarrow f\left(r, \theta, -r^2 \frac{d\theta}{dr}\right) = 0 \quad \text{--- (3)}$$

Step 4 :- Solve eqn(3) and get the solution $G(r, \theta, c') = 0$ which is called the orthogonal trajectory of the polar curves.

Problems :-

1. Find the orthogonally trajectory of the parabola $y^2 = 4ax$, where 'a' is the parameter.

$\Rightarrow$ Given,

$$y^2 = 4ax \text{ --- (1)}$$

diff (1) w.r to 'x'

$$\therefore (1) \Rightarrow 2y \frac{dy}{dx} = 4a$$

$$\Rightarrow y^2 = 2y \frac{dy}{dx} \cdot x$$

$$\Rightarrow y = 2x \frac{dy}{dx} \text{ --- (2)}$$

replace , $\frac{dy}{dx} = -\frac{dx}{dy}$

$$\therefore (2) \Rightarrow y = 2x \left[-\frac{dx}{dy} \right]$$

$$\Rightarrow y = -2x \frac{dx}{dy}$$

$$\Rightarrow y dy = -2x \cdot dx$$

$$\Rightarrow \int y dy = -2 \int x dx$$

$$\Rightarrow \frac{y^2}{2} = -2 \frac{x^2}{2} + C$$

$$\Rightarrow x^2 + \frac{y^2}{2} = C$$

28. Show that the parabola $y^2 = 4a(x+a)$ is self-orthogonal.

$$\Rightarrow y^2 = 4a(x+a) \text{ --- (1)}$$

diff (1) w.r to 'x'

$$(1) \Rightarrow 2yy_1 = 4a(1+0)$$

$$\Rightarrow yy_1 = 2a$$

$$\Rightarrow a = \frac{yy_1}{2}$$

$$\therefore (1) \Rightarrow y^2 = 4 \left[\frac{yy_1}{2} \right] \left[x + \frac{yy_1}{2} \right]$$

$$\Rightarrow y^2 = 4 \frac{yy_1}{2} \left[\frac{2x + yy_1}{2} \right]$$

$$\Rightarrow y^2 = yy_1 (2x + yy_1)$$

$$\Rightarrow y = y_1 (2x + yy_1) \text{ --- (2) Replacing, } y_1 = -\frac{1}{y}$$

$$(2) \Rightarrow y = 2x \left[-\frac{1}{y} \right] + y \left[-\frac{1}{y} \right]^2$$

$$\Rightarrow y = -\frac{2x}{y} + \frac{y}{y^2}$$

$$\Rightarrow y = \frac{-2xy + y}{y^2}$$

$$\Rightarrow yy^2 = -2xy + y$$

$$\Rightarrow y = 2xy + yy^2 \text{ --- (3)}$$

Eqn (2) & (3) are equal

$\therefore$ Eqn is self orthogonally.

3. Find the orthogonal trajectory of $\frac{x^2}{a^2} + \frac{y^2}{a^2 + \lambda} = 1$ where ' λ ' is the parameter.

$$\Rightarrow \text{Given: } \frac{x^2}{a^2} + \frac{y^2}{a^2 + \lambda} = 1 \quad \text{--- (1)}$$

diff (1) w.r to ' x '

$$\therefore (1) \Rightarrow \frac{2x}{a^2} + \frac{2yy_1}{a^2 + \lambda} = 0$$

$$\Rightarrow \frac{x}{a^2} + \frac{yy_1}{a^2 + \lambda} = 0$$

$$\Rightarrow \frac{yy_1}{a^2 + \lambda} = -\frac{x}{a^2}$$

$$\Rightarrow \frac{1}{a^2 + \lambda} = \frac{-x}{a^2 yy_1}$$

$$\Rightarrow a^2 + \lambda = \frac{a^2 yy_1}{-x}$$

$$\therefore (1) \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{\frac{-a^2 yy_1}{x}} = 1$$

$$\Rightarrow x^2 - \frac{y}{y_1/x} = a^2$$

$$\Rightarrow x^2 - \frac{xy}{y_1} = a^2 \quad \text{--- (2)}$$

$$\text{let, } y_1 = -\frac{1}{y_1}$$

$$\therefore (2) \Rightarrow x^2 - \frac{xy}{(-1/y_1)} = a^2$$

$$\Rightarrow x^2 - xy y_1 = a^2$$

$$\Rightarrow yy_1 = \frac{a^2 - x^2}{x}$$

$$\Rightarrow y \frac{dy}{dx} = \frac{a^2}{x} - \frac{x^2}{x}$$

$$\Rightarrow y dy = \left[\frac{a^2}{x} - x \right] dx$$

$$\Rightarrow \int y \cdot dy = a^2 \int \frac{1}{x} dx - \int x \cdot dx$$

$$\Rightarrow \frac{y^2}{2} = a^2 \log x - \frac{x^2}{2} + C$$

$$\Rightarrow \frac{x^2}{2} + \frac{y^2}{2} = a^2 \log x + C$$

4. Show that $\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} = 1$ is self orthogonal where ' λ ' is the parameter.

$$\Rightarrow \text{Given :- } \frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} = 1 \quad \text{--- (1)}$$

diff (1) w. r to ' x '

$$\therefore (1) \Rightarrow \frac{2x}{a^2 + \lambda} + \frac{2yy_1}{b^2 + \lambda} = 0$$

$$\Rightarrow \frac{x}{a^2 + \lambda} + \frac{yy_1}{b^2 + \lambda} = 0$$

$$\Rightarrow (b^2 + \lambda)x + (a^2 + \lambda)yy_1 = 0$$

$$\Rightarrow b^2x + \lambda x + a^2yy_1 + \lambda yy_1 = 0$$

$$\Rightarrow (x + yy_1)\lambda = -(b^2x + a^2yy_1)$$

$$\Rightarrow \lambda = \frac{-(b^2x + a^2yy_1)}{x + yy_1}$$

$$\therefore a^2 + \lambda = \frac{a^2 - (b^2x + a^2yy_1)}{x + yy_1}$$

$$= \frac{a^2(x + yy_1) - (b^2x + a^2yy_1)}{x + yy_1}$$

$$= \frac{a^2x + a^2yy_1 - b^2x - a^2yy_1}{x + yy_1}$$

$$= \frac{a^2x + \cancel{a^2yy_1} - b^2x - \cancel{a^2yy_1}}{x + yy_1}$$

$$= \frac{(a^2 - b^2)x}{x + yy_1}$$

$$\therefore b^2 + \lambda = \frac{b^2 - (b^2x + a^2yy_1)}{x + yy_1}$$

$$= \frac{b^2x + b^2yy_1 - b^2x - a^2yy_1}{x + yy_1}$$

$$= \frac{-(a^2 - b^2)yy_1}{x + yy_1}$$

$$\therefore (1) \Rightarrow \frac{x^2}{\frac{(a^2 - b^2)x}{x + yy_1}} + \frac{y^2}{\frac{-(a^2 - b^2)yy_1}{x + yy_1}} = 1$$

$$\Rightarrow \frac{x(x + yy_1)}{1} - \frac{y(x + yy_1)}{y_1} = a^2 - b^2$$

$$\Rightarrow (x + yy_1) \left(x - \frac{y}{y_1} \right) = a^2 - b^2 \quad \text{--- (2)}$$

$$\text{Let, } y_1 = -\frac{1}{y_1}$$

$$\therefore (2) \Rightarrow \left[x + y \left[-\frac{1}{y_1} \right] \right] \left[x - \frac{y}{\left(-\frac{1}{y_1} \right)} \right] = a^2 - b^2$$

$$\Rightarrow \left[x - \frac{y}{y_1} \right] [x + yy_1] = a^2 - b^2 \text{ --- (3)}$$

$\therefore$ Eqn (2) & (3) are equal

$\therefore$ The eqn is self orthogonal.

5. Find the orthogonal trajectory of the given polar curve.

if $r = a(1 - \cos \theta)$

$\Rightarrow$ Given: $r = a(1 - \cos \theta)$ --- (1)
diff. w. r to ' θ '

$$\therefore (1) \Rightarrow \frac{dr}{d\theta} = a(0 + \sin \theta)$$

$$\Rightarrow \frac{dr}{d\theta} = a \sin \theta$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{a \sin \theta}{r}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{a \sin \theta}{a(1 - \cos \theta)}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{2 \sin(\theta/2) \cos(\theta/2)}{2 \sin^2(\theta/2)}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \cot(\theta/2) \text{ --- (2)}$$

$$\text{but, } \frac{dr}{d\theta} = -r^2 \frac{d\theta}{dr}$$

$$\therefore (2) \Rightarrow \frac{1}{r} \left[-r^2 \frac{d\theta}{dr} \right] = \cot(\theta/2)$$

$$\Rightarrow -r \frac{d\theta}{dr} = \cot\left(\frac{\theta}{2}\right)$$

$$\Rightarrow -\frac{1}{r} dr = -\frac{1}{\cot(\frac{\theta}{2})} d\theta$$

$$\Rightarrow \int \frac{1}{r} dr = - \int \tan(\theta/2) d\theta$$

$$\Rightarrow \log r = -\frac{\log |\sec(\theta/2)|}{1/2} + \log k$$

$$\Rightarrow \log r = -2 \log |\sec(\theta/2)| + \log k$$

$$\Rightarrow \log r = \log |\cos^2(\theta/2)| + \log k$$

$$\Rightarrow \log r = \log |k \cos^2(\theta/2)|$$

$$\Rightarrow r = k \cos^2\left[\frac{\theta}{2}\right]$$

$$\Rightarrow r = k \left[\frac{1 + \cos\theta}{2} \right]$$

$$\Rightarrow r = \frac{k}{2} (1 + \cos\theta)$$

$$\Rightarrow r = b (1 + \cos\theta)$$

$$\text{iii) } r = a (1 + \cos\theta)$$

$$\Rightarrow r = a (1 + \cos\theta) \text{ ————— (1)}$$

diff (1) w.r to ' θ '

$$\Rightarrow \frac{dr}{d\theta} = a(0 - \sin\theta)$$

$$\Rightarrow \frac{dr}{d\theta} = -a \sin \theta$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = -\frac{a \sin \theta}{r}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = -\frac{a \sin \theta}{a(1 + \cos \theta)}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = -\frac{2 \sin(\theta/2) \cos(\theta/2)}{2 \cos^2(\theta/2)}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = -\tan(\theta/2) \quad \text{--- (2)}$$

$$\text{but, } \frac{dr}{d\theta} = -r^2 \frac{d\theta}{dr}$$

$$\therefore (2) \Rightarrow \frac{1}{r} \left[-r^2 \frac{d\theta}{dr} \right] = -\tan(\theta/2)$$

$$\Rightarrow -r \frac{d\theta}{dr} = -\tan(\theta/2)$$

$$\Rightarrow -\frac{1}{r} dr = -\tan(\theta/2) d\theta$$

$$\Rightarrow \int -\frac{1}{r} dr = \int -\tan(\theta/2) d\theta$$

$$\Rightarrow \log r = \frac{\log |\sec(\theta/2)|}{1/2} + \log K$$

$$\Rightarrow \log r = 2 \log |\sec(\theta/2)| + \log K$$

$$\Rightarrow \log r = \log \sec^2(\theta/2) + \log K$$

$$\Rightarrow \log r = \log K \sec^2(\theta/2)$$

$$\Rightarrow r = K \sec^2(\theta/2)$$

$$\text{iii) } r^n = a^n \sin n\theta$$

$$\Rightarrow r^n = a^n \sin n\theta \text{ --- (1)}$$

diff (1) w.r to ' θ '

$$\therefore (1) \Rightarrow n r^{n-1} \cdot \frac{dr}{d\theta} = a^n \cos n\theta \cdot n$$

$$\Rightarrow \frac{r^n}{r} \cdot \frac{dr}{d\theta} = a^n \cos n\theta$$

$$\Rightarrow \frac{1}{r} \cdot \frac{dr}{d\theta} = \frac{a^n \cos n\theta}{r^n}$$

$$\Rightarrow \frac{1}{r} \cdot \frac{dr}{d\theta} = \frac{a^n \cos n\theta}{a^n \sin n\theta}$$

$$\Rightarrow \frac{1}{r} \cdot \frac{dr}{d\theta} = \cot n\theta \text{ --- (2)}$$

$$\therefore \text{ but } \frac{dr}{d\theta} = -r^2 \frac{d\theta}{dr}$$

$$\therefore (2) \Rightarrow \frac{1}{r} \left[-r^2 \frac{d\theta}{dr} \right] = \cot n\theta$$

$$\Rightarrow -r \frac{d\theta}{dr} = \cot n\theta$$

$$\Rightarrow \int \frac{1}{r} dr = - \int \frac{1}{\cot n\theta} d\theta$$

$$\Rightarrow \log r = - \int \tan n\theta d\theta$$

$$\Rightarrow \log r = - \frac{\log |\sec n\theta|}{n} + \log b$$

$$\Rightarrow n \log r = - \log |\sec n\theta| + n \log b$$

$$\Rightarrow \log r^n = \log |\cos n\theta| + \log b^n$$

$$\Rightarrow \log(r^n) = \log(b^n \cos n\theta)$$

$$\Rightarrow r^n = b^n \cos n\theta$$

$$\text{iv) } \frac{2a}{r} = 1 - \cos \theta$$

$$\Rightarrow 2a = r(1 - \cos \theta) \text{ --- (1)}$$

diff w.r to ' θ '

$$\Rightarrow 0 = r(a + \sin \theta) + (1 - \cos \theta) \frac{dr}{d\theta}$$

$$\Rightarrow 0 = r \sin \theta + (1 - \cos \theta) \frac{dr}{d\theta}$$

$$\Rightarrow \frac{-r \sin \theta}{1 - \cos \theta} = \frac{dr}{d\theta}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = - \frac{\sin \theta}{1 - \cos \theta}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = - \frac{2 \sin(\theta/2) \cos(\theta/2)}{2 \sin^2(\theta/2)}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = - \cot(\theta/2) \text{ --- (2)}$$

$$\text{But, } \frac{dr}{d\theta} = -r^2 \frac{d\theta}{dr}$$

$$\therefore (2) \Rightarrow \frac{1}{r} \left[-r^2 \frac{d\theta}{dr} \right] = - \cot(\theta/2)$$

$$\Rightarrow -r \frac{d\theta}{dr} = - \cot(\theta/2)$$

$$\Rightarrow \int \frac{1}{r} \frac{d\theta}{dr} = \int \frac{1}{\cot(\theta/2)} d\theta$$

$$\Rightarrow \log r = \int \tan(\theta/2) d\theta$$

$$\Rightarrow \log r = 2 \log |\sec(\theta/2)| + \log K$$

$$\Rightarrow \log r = \log \sec^2(\theta/2) + \log K$$

$$\Rightarrow \log r = \log |\sec^2(\theta/2) \cdot K|$$

$$\Rightarrow r = K \sec^2(\theta/2)$$

6.

$$\Rightarrow \text{Given : } r = 2a (\cos \theta + \sin \theta) \text{ ———(1)}$$

$$\Rightarrow \frac{dr}{d\theta} = 2a (-\sin \theta + \cos \theta) \text{ ———(2)}$$

$$(2) \div (1) \Rightarrow \frac{dr/d\theta}{r} = \frac{2a (\cos \theta - \sin \theta)}{2a (\cos \theta + \sin \theta)}$$

$$\Rightarrow \frac{1}{r} \cdot \frac{dr}{d\theta} = \frac{1 - \tan \theta}{1 + \tan \theta}$$

$$\Rightarrow \frac{1}{r} \cdot \frac{dr}{d\theta} = \tan \left[\frac{\pi}{4} - \theta \right]$$

$$\text{but } \frac{dr}{d\theta} = -r^2 \frac{d\theta}{dr}$$

$$\therefore (2) \Rightarrow \frac{1}{r} \left[-r^2 \cdot \frac{d\theta}{dr} \right] = \tan \left[\frac{\pi}{4} - \theta \right]$$

$$\Rightarrow -r \cdot \frac{d\theta}{dr} = \tan \left[\frac{\pi}{4} - \theta \right]$$

$$\Rightarrow \frac{1}{r} \cdot \frac{d\theta}{dr} = \frac{1}{\tan \left[\frac{\pi}{4} - \theta \right]} \cdot d\theta$$

$$\Rightarrow \int \frac{1}{r} \cdot \frac{d\theta}{dr} = \int \cot \left(\frac{\pi}{4} - \theta \right) \cdot d\theta$$

$$\Rightarrow \log r = \frac{-\log |\sec (\pi/4 - \theta)|}{-1} + \log b$$

$$\Rightarrow \log r = \log |\sin (\pi/4 - \theta)| + \log b$$

$$\Rightarrow \log r = \log |b \sin (\pi/4 - \theta)|$$

$$\Rightarrow r = b \sin (\pi/4 - \theta)$$

$$\Rightarrow r = b \left[\sin \pi/4 \cos \theta - \cos \frac{\pi}{4} \sin \theta \right]$$

$$\Rightarrow r = \frac{b}{\sqrt{2}} (\cos \theta - \sin \theta)$$

Applications of Differential Equations

Newton's law of cooling :-

Let 'T' be the temperature of the body and 'T₀' be the surrounding medium / room temperature at any time 't', then the Newton's law can state that "The rate of change of body temperature is directly proportional to the difference of the temperature of the body and its surrounding medium."

$$\therefore \frac{dT}{dt} \propto (T - T_0)$$

$$\Rightarrow \frac{dT}{dt} = -k(T - T_0)$$

$$\Rightarrow \frac{1}{T - T_0} dT = -k dt$$

$$\Rightarrow \int \frac{1}{T - T_0} dT = -kt + C$$

$$\Rightarrow \log_e [T - T_0] = -kt + C$$

$$\Rightarrow T - T_0 = e^{-kt + C}$$

$$\Rightarrow T - T_0 = e^{-kt} \cdot e^C$$

$$\Rightarrow T = T_0 + e^C \cdot e^{-kt}$$

$$\Rightarrow \boxed{T = T_0 + \lambda e^{-kt}}$$

1. A body originally 80°C cools down to 60°C in 20 min the temp of the air being 40°C , what will be the temperature of the body after 40 min from the original.

$\Rightarrow$ Given,

Air temperature $[T_0] = 40^{\circ}\text{C}$

WKT, $T = T_0 + \lambda e^{-kt}$ — (1)

$\Rightarrow T = 40 + \lambda e^{-kt}$ — (2)

Also given,

The body given temp $[T] = 80^{\circ}\text{C}$ at $t=0$

$\therefore (2) \Rightarrow 80 = 40 + \lambda e^{-kt}$

$\Rightarrow 80 - 40 = \lambda$

$\Rightarrow \lambda = 40$

$\therefore (2) \Rightarrow T = 40 + 40 e^{-kt}$ — (3)

Also given, The body temp is reduced to 60°C after 20 min.

$\therefore T = 60^{\circ}\text{C}$

(3) $\Rightarrow 60 = 40 + 40 e^{-20k}$

$\Rightarrow 40 e^{-20k} = 20$

$\Rightarrow e^{-20k} = \frac{20}{40}$

$\Rightarrow e^{-20k} = 0.5$

$\Rightarrow -20k = \log_e(0.5)$

$\Rightarrow -20k = -0.69314$

$\Rightarrow k = \frac{0.69314}{20} = 0.03465$

$$\therefore (2) \Rightarrow T = 40 + 40 e^{-(0.03465)t}$$

$$\therefore \text{At, } t=40$$

$$\Rightarrow T = 40 + 40 e^{-(0.03465)(40)}$$

$$\Rightarrow T = 40 + 40 e^{-1.386}$$

$$\Rightarrow T = 40 + 40(0.25)$$

$$\Rightarrow T = 40 + 10$$

$$\Rightarrow \boxed{T = 50^\circ\text{C}}$$

2. If the temperature of air is 30°C and a metal ball cools from 100°C to 70°C in 15 min. Find how long will it take for the metal ball to reach the temperature of 40°C ?

$\Rightarrow$ Given,

$$\text{Air temperature } [T_0] = 30^\circ\text{C}$$

$$\text{WKT, } T = T_0 + \lambda e^{-kt} \text{ ———(1)}$$

$$\Rightarrow T = 30 + \lambda e^{-kt} \text{ ———(2)}$$

Given,

$$\text{Temperature of metal ball } [T] = 100^\circ\text{C at } t=0$$

$$\therefore \text{eqn (2) :- } \Rightarrow 100 = 30 + \lambda e^0$$

$$\Rightarrow 100 - 30 = \lambda$$

$$\Rightarrow \lambda = 70$$

$$(2) \Rightarrow T = 30 + \lambda e^{-kt}$$

$$T = 30 + 70 e^{-kt} \text{ ———(3)}$$

Also given, the metal ball temp reduced to 70°C after 15 min.

$$(3) \Rightarrow 70 = 30 + 70 e^{-15k}$$

$$\Rightarrow 70 e^{-15K} = 40$$

$$\Rightarrow e^{-15K} = \frac{40}{70}$$

$$\Rightarrow e^{-15K} = 0.5714$$

$$\Rightarrow -15K = \log_e(0.5714)$$

$$\Rightarrow -15K = -0.5596$$

$$\Rightarrow K = \frac{0.5596}{15}$$

$$\Rightarrow K = 0.0373$$

$$\therefore (3) \Rightarrow T = 30 + 70 e^{-(0.0373)t} \quad \text{--- (4)}$$

$$\text{At } T = 40^\circ\text{C}$$

$$(4) \Rightarrow 40 = 30 + 70 e^{-(0.0373)t}$$

$$\Rightarrow 70 e^{-(0.0373)t} = 10$$

$$\Rightarrow e^{-(0.0373)t} = \frac{10}{70}$$

$$\Rightarrow e^{-(0.0373)t} = 0.1428$$

$$\Rightarrow -(0.0373)t = \log_e(0.1428)$$

$$\Rightarrow -(0.0373)t = -1.9463$$

$$\Rightarrow t = \frac{1.9463}{0.0373}$$

$$\Rightarrow t = 52.18 \text{ min}$$

3. Water at temperature 10°C takes 5 min to warm upto 20°C at room temperature of 40°C . Find the temperature of water after 20 min.

$\Rightarrow$ Given,

room temperature $[T_0] = 40^{\circ}\text{C}$

WKT, The Newton's law of warming.

$$T = T_0 + \lambda e^{kt}$$

$$\Rightarrow T = 40 + \lambda e^{kt} \quad \text{---(1)}$$

Given, that the water is at 10°C at $t=0$

$$(1) \Rightarrow T = 40 + \lambda e^0$$

$$\Rightarrow 10 = 40 + \lambda$$

$$\Rightarrow \lambda = -30$$

$$\therefore (1) \Rightarrow T = 40 - 30e^{-kt} \quad \text{---(2)}$$

Also given, the water was heated to 20°C after 5 min

$$\therefore (2) \Rightarrow 20 = 40 - 30e^{5k}$$

$$\Rightarrow -30e^{5k} = -20$$

$$\Rightarrow e^{5k} = \frac{20}{30}$$

$$\Rightarrow e^{5k} = 0.6666$$

$$\Rightarrow 5k = \log_e(0.6666)$$

$$\Rightarrow 5k = -0.4055$$

$$\Rightarrow k = -0.0811$$

$$(2) \Rightarrow T = 40 - 30e^{-(0.0811)t} \quad \text{---(3)}$$

$\therefore$ At $t=20$ min

$$(3) \Rightarrow T = 40 - 30 e^{-(0.0811)(20)}$$

$$\Rightarrow T = 40 - 30 e^{-1.622}$$

$$\Rightarrow T = 40 - 30(0.1975)$$

$$\Rightarrow T = 40 - 5.9250$$

$$\Rightarrow \boxed{T = 34.07^\circ\text{C}}$$

4. A body is heated at 110°C and placed in the air at 10°C after an hour its temperature becomes 60°C . How much additional time is required to cool 30°C at $t=0$.

$\Rightarrow$ Given,

$$T_0 = 10^\circ\text{C}$$

WKT,

$$T = T_0 + \lambda e^{-kt}$$

$$\Rightarrow T = 10 + \lambda e^{-kt} \text{ --- (1)}$$

given the water is cooled to 60°C at $t = 1 \text{ hour} = 60 \text{ min}$

$$\Rightarrow 60 = 10 + \lambda e^{-60k}$$

also given :-

$$\therefore T = 110^\circ\text{C} \text{ at } t = 0 \text{ min}$$

$$\therefore (2) \Rightarrow 110 = 10 + \lambda e$$

$$\lambda = 110 - 10$$

$$\lambda = 100$$

$$\therefore (2) \Rightarrow T = 10 + 100 e^{-kt} \text{ --- (3)}$$

also given, $T = 60^\circ\text{C}$ at $t = 60 \text{ min}$

$$(3) \Rightarrow 60 - 10 = 100 e^{-60k}$$

$$\Rightarrow \frac{50}{100} = e^{-60k}$$

$$\Rightarrow e^{-60K} = 0.5$$

$$\Rightarrow -60K = \log_e(0.5)$$

$$\Rightarrow K = \frac{0.69}{60}$$

$$\Rightarrow K = 0.0115$$

$$\therefore (3) \Rightarrow T = 10 + 100e^{-0.0115t}$$

$$\text{At } T = 30^\circ$$

$$\Rightarrow 30 = 10 + 100e^{-(0.0115)t}$$

$$\Rightarrow 30 - 10 = 100e^{-(0.0115)t}$$

$$\Rightarrow e^{-(0.0115)t} = \frac{20}{100}$$

$$\Rightarrow -(0.0115)t = -0.2$$

$$\Rightarrow t = \frac{0.2}{0.0115}$$

$$\Rightarrow t = 17.39 \text{ min}$$

5. A copper ball at 80°C at cools down to 60°C in 20 min. If the temperature of the room is being 40°C , What will be the temperature of the ball after 40 min from the original.

$\Rightarrow$ Given,

The room temperature, $T_0 = 40^\circ\text{C}$

WKT,

$$T = T_0 + \lambda e^{-kt} \text{ ——— (1)}$$

$$\Rightarrow T = 40 + \lambda e^{-kt} \text{ ——— (2)}$$

also given, $T = 80^\circ\text{C}$ at $t = 0$

$$(2) \Rightarrow 80 = 40 \lambda e^0$$

$$\Rightarrow \lambda = 40$$

$$\therefore (2) \Rightarrow T = 40 + 40 e^{-kt} \text{ ————— (3)}$$

and given, $T = 60^\circ\text{C}$ at $t = 20 \text{ min}$

$$\therefore (3) \Rightarrow 60 = 40 + 40 e^{-20k}$$

$$\Rightarrow 40 e^{-20k} = 20$$

$$\Rightarrow e^{-20k} = \frac{20}{40}$$

$$\Rightarrow e^{-20k} = 0.5$$

$$\Rightarrow -20k = \log_e(0.5)$$

$$\Rightarrow -20k = -0.6931$$

$$\Rightarrow k = \frac{0.6931}{20}$$

$$\Rightarrow k = 0.03465$$

$$\therefore (3) \Rightarrow T = 40 + 40 e^{-0.03465t} \text{ ————— (4)}$$

$\therefore$ At $t = 40 \text{ min}$

$$\therefore T = 40 + 40 e^{(-0.03465)(40)}$$

$$\Rightarrow T = 40 + 40 e^{(-1.386)}$$

$$\Rightarrow T = 40 + 40 (0.250)$$

$$\Rightarrow T = 40 + 10$$

$$\Rightarrow \boxed{T = 50^\circ\text{C}}$$

6. A body is at 25°C , whose from 100°C to 75°C in one min. Find the temperature of the body at 3min.

⇒ Given,

$$T_0 = 25^\circ\text{C}$$

WKT,

$$T = T_0 + \lambda e^{-kt} \quad \text{---(1)}$$

$$\Rightarrow T = 25 + \lambda e^{-kt} \quad \text{---(2)}$$

$$\text{at } T = 100, \quad t = 0$$

$$\therefore (2) \Rightarrow 100 = 25 + \lambda e^0$$

$$\Rightarrow 75 = \lambda$$

$$(2) \Rightarrow T = 25 + 75 e^{-kt} \quad \text{---(3)}$$

also given, at $T = 75^\circ$, $t = 1 \text{ min}$

$$(3) \Rightarrow 75 = 25 + 75 e^{-k(1)}$$

$$\Rightarrow 75 - 25 = 75 e^{-k}$$

$$\Rightarrow 50 = 75 e^{-k}$$

$$\Rightarrow e^{-k} = \frac{50}{75}$$

$$\Rightarrow e^{-k} = 0.6666$$

$$\Rightarrow -k = \log_e(0.6666)$$

$$\Rightarrow -k = -0.4055$$

$$\Rightarrow k = 0.4055$$

$$\therefore (3) \Rightarrow T = 25 + 75 e^{(-0.4055)t} \quad \text{---(4)}$$

$$\text{at } t = 3 \text{ min}$$

$$\therefore (4) \Rightarrow T = 25 + 75 e^{(-0.4055)(3)}$$

$$= 25 + 75 e^{-1.2165}$$

$$= 25 + 75 (0.2962)$$

$$= 25 + 22.215$$

$$= 47.215$$

Flow of Electricity

Non-linear differential Equations :-

Solvable for p method

1. Let $y=f(x)$ be a function and let $p = dy/dx$, then the general non-linear differential equation can be defined as

$$A_0 p^n + A_1 p^{n-1} + A_2 p^{n-2} + \dots + A_n = 0 \quad \text{---(1)}$$

where $A_0, A_1, A_2, \dots, A_n$ are the functions of the variables x & y and also eqn (1) is a polynomial of degree n , it follows as

2. (1) $\Rightarrow [p - f_1(x, y)] [p - f_2(x, y)] [p - f_3(x, y)] \dots [p - f_n(x, y)] = 0$

$$\Rightarrow p - f_1(x, y) = 0, \quad p - f_2(x, y) = 0, \quad p - f_3(x, y) = 0 \dots p - f_n(x, y) = 0$$

$$\Rightarrow p = f_1(x, y), \quad p = f_2(x, y), \quad p = f_3(x, y) \dots p = f_n(x, y)$$

3. Solve the above equations and get the solutions

$$F_1(x, y, C_1) = 0, \quad F_2(x, y, C_2) = 0, \quad F_3(x, y, C_3) = 0 \dots F_n(x, y, C_n) = 0$$

$\therefore$ The complete solution of given non-linear differential equation can be written as,

$$f_1(x, y, C_1) \cdot f_2(x, y, C_2) \cdot f_3(x, y, C_3) \dots f_n(x, y, C_n) = 0$$

1. Solve $xy p^2 - (x^2 + y^2) p + xy = 0$

$\Rightarrow$ Given :- $xy p^2 - (x^2 + y^2) p + xy = 0$ ———(1)

$\Rightarrow xy p^2 - x^2 p = y^2 p + xy = 0$

$\Rightarrow xp (yp - x) - y (yp - x) = 0$

$\Rightarrow (px - y) (py - x) = 0$

$\Rightarrow px - y = 0$, $py - x = 0$

$\Rightarrow px = y$, $py = x$

$\Rightarrow p = \frac{y}{x}$, $p = \frac{x}{y}$

case (1) : $p = \frac{y}{x}$

$\Rightarrow \frac{dy}{dx} = \frac{y}{x}$

$\Rightarrow \int \frac{1}{y} \cdot dy = \int \frac{1}{x} \cdot dx$

$\Rightarrow \log y = \log x + \log C_1$

$\Rightarrow \log y = \log (C_1 x)$

$\Rightarrow y = C_1 x$

$\Rightarrow y - C_1 x = 0$

case (2) : $p = \frac{x}{y}$

$\Rightarrow \frac{dy}{dx} = \frac{x}{y}$

$\Rightarrow \int y \cdot dy = \int x \cdot dx$

$\Rightarrow \frac{y^2}{2} = \frac{x^2}{2} + C_2$

$$\Rightarrow y^2 = x^2 + 2C_2$$

$$\Rightarrow y^2 - x^2 - 2C_2 = 0$$

$\therefore$ The complete solution of equation (1) is

$$(y - C_1 x)(y^2 - x^2 - 2C_2) = 0$$

2. Solve $\frac{dy}{dx} - \frac{dx}{dy} = \frac{x}{y} - \frac{y}{x}$

$\Rightarrow$ Given, $\frac{dy}{dx} - \frac{dx}{dy} = \frac{x}{y} - \frac{y}{x} \quad \text{---(1)}$

but $\frac{dy}{dx} = p$

$\therefore (1) \Rightarrow p - \frac{1}{p} = \frac{x^2 - y^2}{xy}$

$\Rightarrow \frac{p^2 - 1}{p} = \frac{x^2 - y^2}{xy}$

$\Rightarrow (p^2 - 1)xy = (x^2 - y^2)p$

$\Rightarrow xyp^2 - (x^2 - y^2)p - xy = 0$

$\Rightarrow xyp^2 - x^2p + y^2p - xy = 0$

$\Rightarrow xp(y p - x) + y(y p - x) = 0$

$\Rightarrow (yp - x)(xp + y) = 0$

$\Rightarrow yp - x = 0, \quad xp + y = 0$

$\Rightarrow yp = x, \quad xp = -y$

$\Rightarrow p = \frac{x}{y}, \quad p = -\frac{y}{x}$

case (1) :- $p = \frac{x}{y}$

$$\Rightarrow \frac{dy}{dx} = \frac{x}{y}$$

$$\Rightarrow \int y \cdot dy = \int x \cdot dx$$

$$\Rightarrow \frac{y^2}{2} = \frac{x^2}{2} + C_1$$

$$\Rightarrow y^2 = x^2 + 2C_1$$

$$\Rightarrow y^2 - x^2 - 2C_1 = 0$$

case (2) :

$$p = -y/x$$

$$\Rightarrow \frac{dy}{dx} = -\frac{y}{x}$$

$$\Rightarrow \int \frac{1}{y} dy = - \int \frac{1}{x} dx$$

$$\Rightarrow \log y = -\log x + \log C_2$$

$$\Rightarrow \log y = \log \left[\frac{C_2}{x} \right]$$

$$\Rightarrow y = \frac{C_2}{x}$$

$$\Rightarrow xy = C_2$$

$$\Rightarrow xy - C_2 = 0$$

$\therefore$ The solution is $(y^2 - x^2 - C_1)(xy - C_2) = 0$

3. solve, $p^2 + 2py \cot x - y^2 = 0$

$$\Rightarrow \text{Given, } p^2 + 2py \cot x - y^2 = 0$$

$$\therefore a = 1, \quad b = 2y \cot x, \quad c = -y^2$$

$$\therefore p = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$p = \frac{-2y \cot x \pm \sqrt{(2y \cot x)^2 - 4(1)(-y^2)}}{2(1)}$$

$$p = \frac{-2y \cot x \pm \sqrt{(4y^2 \cot^2 x) + 4y^2}}{2}$$

$$p = \frac{-2y \cot x \pm 2y \sqrt{\cot^2 x + 1}}{2}$$

$$p = -y \cot x \pm y \operatorname{cosec} x$$

case (i) :-

$$p = -y \cot x + y \operatorname{cosec} x$$

$$\Rightarrow \frac{dy}{dx} = y [\operatorname{cosec} x - \cot x]$$

$$\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = [-\cot x + \operatorname{cosec} x] dx$$

$$\Rightarrow \frac{1}{y} \cdot dy = \left[\frac{1}{\sin x} - \frac{\cos x}{\sin x} \right] dx$$

$$\Rightarrow \frac{1}{y} \cdot dy = \left[\frac{1 - \cos x}{\sin x} \right] dx$$

$$\Rightarrow \frac{1}{y} \cdot dy = \frac{2 \sin^2 \frac{x}{2}}{2 \sin(\frac{x}{2}) \cos(\frac{x}{2})} dx$$

$$\Rightarrow \int \frac{1}{y} \cdot dy = \int \tan(\frac{x}{2}) dx$$

$$\Rightarrow \log y = \frac{\log |\sec(\frac{x}{2})|}{1/2} + \log C_1$$

$$\Rightarrow \log y = 2 \log |\sec(x/2)| + \log C_1$$

$$\Rightarrow \log y = \log |C_1 \sec^2(x/2)|$$

$$\Rightarrow y = C_1 \sec^2(x/2)$$

$$\Rightarrow y = \frac{C_1}{\cos^2(x/2)}$$

$$\Rightarrow y = \frac{C_1}{\left(\frac{1+\cos x}{2}\right)}$$

$$\Rightarrow y = \frac{2C_1}{1+\cos x}$$

$$\Rightarrow y(1+\cos x) = 2C_1$$

$$\Rightarrow y(1+\cos x) - 2C_1 = 0$$

Case (2):-

$$\frac{1}{y} dy = -(\cot x + \operatorname{cosec} x)$$

$$\Rightarrow \frac{1}{y} dy = -\left[\frac{\cos x}{\sin x} + \frac{1}{\sin x}\right] dx$$

$$\Rightarrow \frac{1}{y} dy = -\frac{(1+\cos x)}{\sin x} dx$$

$$\Rightarrow \frac{1}{y} dy = \frac{-2\cos^2 x}{2\sin(x/2)\cos(x/2)} dx$$

$$\Rightarrow \int \frac{1}{y} \cdot dy = -\int \cot(x/2) dx$$

$$\Rightarrow \log y = \frac{-\log |\sin(x/2)|}{1/2} + \log C_2$$

$$\Rightarrow \log y = -2 \log |\sin(x/2)| + \log C_2$$

$$\Rightarrow \log y = -\log |\sin^2(x/2)| + \log C_2$$

$$\Rightarrow \log y = \log \left[\frac{C_2}{\sin^2(x/2)} \right]$$

$$\Rightarrow y = \frac{C_2}{\sin^2(x/2)}$$

$$\Rightarrow y = \frac{C_2}{\left(\frac{1 - \cos x}{2}\right)}$$

$$\Rightarrow y = \frac{2C_2}{1 - \cos x}$$

$$\Rightarrow y(1 - \cos x) = 2C_2$$

$$\Rightarrow y(1 - \cos x) - 2C_2 = 0$$

4. Solve $p^3 + 2xp^2 - y^2p^2 - 2xy^2p = 0$

$$\Rightarrow p^3 + 2xp^2 - y^2p^2 - 2xy^2p = 0$$

$$\Rightarrow p[p^2 + 2xp - y^2p - 2xy^2] = 0$$

$$\Rightarrow p=0, p^2 + 2xp - y^2p - 2xy^2 = 0$$

$$\Rightarrow p=0, p[p + 2x] - y^2[p + 2x] = 0$$

$$\Rightarrow p=0, (p - y^2)(p + 2x) = 0$$

$$\Rightarrow p=0, p - y^2 = 0, p + 2x = 0$$

$$\Rightarrow p=0, p = y^2, p = -2x$$

case (1) :- $p=0$

$$\Rightarrow \frac{dy}{dx} = 0$$

$$\Rightarrow \int dy = \int 0 dx$$

$$\Rightarrow y = C_1$$

$$\Rightarrow y - C_1 = 0$$

case (2) :-

$$p = y^2$$

$$\Rightarrow \frac{dy}{dx} = y^2$$

$$\Rightarrow \int \frac{1}{y^2} dy = \int dx$$

$$\Rightarrow -\frac{1}{y} = x + C_2$$

$$\Rightarrow x + \frac{1}{y} + C_2 = 0$$

case (3) :-

$$p = -2x$$

$$\Rightarrow \frac{dy}{dx} = -2x$$

$$\Rightarrow \int dy = -2 \int x dx$$

$$\Rightarrow y = -2 \frac{x^2}{2} + C_3$$

$$\Rightarrow y = -x^2 + C_3$$

$$\Rightarrow y + x^2 - C_3 = 0$$

$\therefore$ The solution is, $(y - C_1)(x + \frac{1}{y} + C_2)(y + x^2 - C_3) = 0$

$$5. \text{ Solve, } y \left[\frac{dy}{dx} \right]^2 + (x-y) \frac{dy}{dx} - x = 0$$

$$\Rightarrow y \left[\frac{dy}{dx} \right]^2 + (x-y) \frac{dy}{dx} - x = 0$$

$$\Rightarrow yp^2 + (x-y)p - x = 0$$

$$\Rightarrow yp^2 - xp - yp - x = 0$$

$$\Rightarrow p(yp+x) - 1(yp+x) = 0$$

$$\Rightarrow (p-1)(yp+x) = 0$$

$$\Rightarrow p-1=0, \quad yp+x=0$$

$$\Rightarrow p=1, \quad yp=-x$$

$$\Rightarrow p=1, \quad p = -\frac{x}{y}$$

$$\text{case (1) :- } p=1$$

$$\Rightarrow \frac{dy}{dx} = 1$$

$$\Rightarrow \int dy = \int dx$$

$$\Rightarrow y = x + C_1$$

$$\Rightarrow y - x - C_1 = 0$$

$$\text{case (2) :- } p = -\frac{x}{y}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x}{y}$$

$$\Rightarrow y dy = -x dx$$

$$\Rightarrow \int y dy = -\int x dx$$

$$\Rightarrow \frac{y^2}{2} = -\frac{x^2}{2} + C_2$$

$$\Rightarrow y^2 + x^2 - 2C_2 = 0$$

Clairaut's Equation:-

1. Write the given non-linear differential eqn to be in the form of $y = px + f(p)$, it is called the Clairaut's Equation.

2. To get the general solution for Clairaut's equation, substitute $p=c$ in the Clairaut's equation, we get

$$y = cx + f(c) \text{ --- (1)}$$

3. To get similar solution, diff (1) partially w.r to 'c' and hence substitute the 'c' value in the general solution.

1. Solve the Clairaut's equation $y = px + \frac{a}{p}$

$$\Rightarrow \text{Given:- } y = px + \frac{a}{p} \text{ --- (1)}$$

$\therefore$ eqn (1) is in Clairaut's form

The solution is

$$y = cx + \frac{a}{c} \text{ --- (2)}$$

diff (2) partially w.r to 'c'

$$\therefore (2) \Rightarrow 0 = x - \frac{a}{c^2}$$

$$\Rightarrow \frac{a}{c^2} = x$$

$$\Rightarrow c^2 = \frac{a}{x}$$

$$\Rightarrow c = \sqrt{\frac{a}{x}}$$

$$\therefore (2) \Rightarrow y = \sqrt{\frac{a}{x}} \cdot x + \frac{a}{\sqrt{\frac{a}{x}}}$$

$$\Rightarrow y = \frac{\sqrt{a}}{\sqrt{x}} \sqrt{x} \sqrt{x} + \frac{\sqrt{a}\sqrt{a}}{\sqrt{a}/\sqrt{x}}$$

$$\Rightarrow y = \sqrt{a} \sqrt{x} + \sqrt{a}\sqrt{x}$$

$$\Rightarrow y = 2\sqrt{ax}$$

2. Show that equation $xp^2 + px - py - y + 1 = 0$ is a Clairaut's equation and hence find its general and singular solution.

$$\Rightarrow \text{Given, } xp^2 + px - py + 1 = 0$$

$$\Rightarrow xp^2 + px - py - y = -1$$

$$\Rightarrow xp(p+1) - y(p+1) = -1$$

$$\Rightarrow (xp - y)(p+1) = -1$$

$$\Rightarrow (xp - y) = \frac{-1}{p+1}$$

$$\Rightarrow -y = -px - \frac{1}{p+1}$$

$$\Rightarrow y = px + \left[\frac{1}{p+1} \right] \text{ ——— (1)}$$

$\therefore$ eqn (1) is in Clairaut's form

$\therefore$ The general solution is

$$y = cx + \frac{1}{c+1} \text{ ——— (2)}$$

diff (2) w.r to 'c' partially

$$\therefore (2) \Rightarrow 0 = x - \frac{1}{(c+1)^2}$$

$$\Rightarrow \frac{1}{(c+1)^2} = x$$

$$\Rightarrow (c+1)^2 = \frac{1}{x}$$

$$\Rightarrow c+1 = \frac{1}{\sqrt{x}}$$

$$\Rightarrow c = \frac{1}{\sqrt{x}} - 1$$

$\therefore$ The singular solution is

$$y = \left(\frac{1}{\sqrt{x}} - 1 \right) x + \frac{1}{1/\sqrt{x}}$$

$$\Rightarrow y = \frac{x}{\sqrt{x}} - x + \sqrt{x}$$

$$\Rightarrow y = \sqrt{x} - x + \sqrt{x}$$

$$\Rightarrow y = 2\sqrt{x} - x$$

3. Show that the equation $xp^3 - yp^2 + 1 = 0$ is the clauwit's eqn and hence find its solution.

$$\Rightarrow \text{Given, } xp^3 - yp^2 + 1 = 0$$

$$\Rightarrow xp^3 - yp^2 = -1$$

$$\Rightarrow p^2 (px - y) = -1$$

$$\Rightarrow px - y = -\frac{1}{p^2}$$

$$\Rightarrow y = px + \frac{1}{p^2} \text{ --- (1)}$$

$\therefore$ Eqn (1) is in claurit's form

$\therefore$ The general solution

$$(1) \Rightarrow y = cx + \frac{1}{c^2} \text{ --- (2)}$$

diff (2) w.r to 'c' partially

$$(2) \Rightarrow 0 = x - \frac{1}{c}$$

$$\Rightarrow \frac{1}{c} = x$$

$$\Rightarrow c = 1/x$$

$\therefore$ The singular soln is

$$y = \left(\frac{1}{x}\right)x + x^2$$

$$\Rightarrow y = 1 + x^2$$

$$\Rightarrow x^2 - y + 1 = 0$$

4. Reduce the equation $(px-y)(py-x) = 2p$ to the claurit's form, taking the substitution $X=x^2$ and $Y=y^2$ and hence find its solution.

$\Rightarrow$ Given:-

$$(px-y)(py-x) = 2p \text{ --- (1)}$$

$$X = x^2 \text{ --- (2)}$$

$$Y = y^2 \text{ --- (3)}$$

$$\therefore \frac{dX}{dx} = 2x, \quad \frac{dY}{dy} = 2y$$

$$\Rightarrow dX = 2x dx, \quad dY = 2y dy$$

$$\Rightarrow \frac{dY}{dX} = \frac{2y dy}{2x dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x} \left[\frac{dy}{dx} \right]$$

$$\Rightarrow p = \frac{\sqrt{y}}{\sqrt{x}} \quad b$$

$$\Rightarrow b = \frac{\sqrt{x}}{\sqrt{y}} p$$

$$\therefore (1) \Rightarrow \left[\frac{\sqrt{x}}{\sqrt{y}} p \sqrt{x} - \sqrt{y} \right] \left[\frac{\sqrt{x}}{\sqrt{y}} p \cdot \sqrt{y} + \sqrt{x} \right] = 2 \frac{\sqrt{x}}{\sqrt{y}} p$$

$$\Rightarrow \left[\frac{px}{\sqrt{y}} - \sqrt{y} \right] \sqrt{x} (p+1) = 2 \frac{\sqrt{x}}{\sqrt{y}} p$$

$$\Rightarrow \left[\frac{px - y}{\sqrt{y}} \right] \sqrt{x} (p+1) = 2 \frac{\sqrt{x}}{\sqrt{y}} p$$

$$\Rightarrow (px - y) (p+1) = 2p$$

$$\Rightarrow px - y = \frac{2p}{p+1}$$

$$\Rightarrow -y = -px + \frac{2p}{p+1}$$

$$\Rightarrow y = px + \frac{2p}{p+1}$$

$\therefore$ The solution is,

$$y = cx - \frac{2c}{c+1}$$

$$\Rightarrow y^2 = cx^2 - \frac{2c}{c+1}$$

5. Reduce the equation to the claurit's form $x^2(y - px) = p^2y$ taking the substitutions $X = x^2$, $Y = y^2$ and hence find the general and singular solution?

$$\Rightarrow x^2(y - px) = p^2y \quad \text{---(1)}$$

$$X = x^2 \quad \text{---(2)}$$

$$Y = y^2 \quad \text{---(3)}$$

$$\therefore \frac{dX}{dx} = 2x \quad \frac{dY}{dy} = 2y$$

$$\Rightarrow dx = 2x dx, \quad dY = 2y dy$$

$$\therefore \frac{dY}{dX} = \frac{y}{x} \left[\frac{dy}{dx} \right]$$

$$\Rightarrow p = \frac{\sqrt{Y}}{\sqrt{X}} p$$

$$\Rightarrow \beta = \frac{\sqrt{X}}{\sqrt{Y}} p$$

$$(1) \Rightarrow x \left[\sqrt{Y} - \frac{\sqrt{X}}{\sqrt{Y}} p \cdot \sqrt{X} \right] = \left[\frac{\sqrt{X}}{\sqrt{Y}} p \right]^2 \sqrt{Y}$$

$$\Rightarrow x \left[\frac{Y - pX}{\sqrt{Y}} \right] = \frac{x}{Y} p^2 \sqrt{Y}$$

$$\Rightarrow Y - pX = \frac{p^2 x}{Y}$$

$$\Rightarrow Y - pX = p^2$$

$$\Rightarrow Y = pX + p^2$$

$\therefore$ The soln is

$$Y = CX + C^2 \quad \text{---(4)}$$

$$\Rightarrow y^2 = cx^2 + c^2 \text{ --- (5)}$$

diff w.r to 'c' partially

$$(5) \Rightarrow 0 = x^2 + 2c$$

$$\Rightarrow 2c = -x^2$$

$$\Rightarrow c = -\frac{x^2}{2}$$

$$(5) \Rightarrow y^2 = \left[-\frac{x^2}{2}\right][x^2] + \left[-\frac{x^2}{2}\right]^2$$

$$\Rightarrow y^2 = -\frac{x^4}{2} + \frac{x^4}{4}$$

$$\Rightarrow y^2 = -\frac{x^4}{4}$$

$$\Rightarrow x^4 + 4y^2 = 0$$

6. Find the general soln of the equation $(px-y)(py+x) = a^2p$ by reducing into claurit's form by taking the substitution $x = x^2, Y = y^2$

$$\Rightarrow (px-y)(py+x) = a^2p \text{ --- (1)}$$

$$X = x^2 \text{ --- (2)}$$

$$Y = y^2 \text{ --- (3)}$$

$$\therefore \frac{dX}{dx} = 2x, \quad \frac{dY}{dy} = 2y$$

$$\Rightarrow dX = 2x dx, \quad dY = 2y dy$$

$$\therefore \frac{dY}{dX} = \frac{y}{x} \left[\frac{dy}{dx} \right]$$

$$\Rightarrow p = \frac{\sqrt{Y}}{\sqrt{X}} \beta \quad \Rightarrow \beta = \frac{\sqrt{X}}{\sqrt{Y}} p$$

$$(1) \Rightarrow \left[\frac{\sqrt{x}}{\sqrt{y}} p \sqrt{x} - \sqrt{y} \right] \left[\frac{\sqrt{x}}{\sqrt{y}} p \sqrt{y} + \sqrt{x} \right] = a^2 \frac{\sqrt{x}}{\sqrt{y}} p$$

$$\Rightarrow \left[\frac{px}{\sqrt{y}} - \sqrt{y} \right] \left[\sqrt{x} [p+1] \right] = a^2 \frac{\sqrt{x}}{\sqrt{y}} p$$

$$\Rightarrow \left[\frac{px - y}{\sqrt{y}} \right] \sqrt{x} (p+1) = a^2 \frac{\sqrt{x}}{\sqrt{y}} p$$

$$\Rightarrow (px - y)(p+1) = a^2 p$$

$$\Rightarrow px - y = \frac{a^2 p}{p+1}$$

$$\Rightarrow -y = -px + \frac{a^2 p}{p+1}$$

$$\Rightarrow y = px + \frac{a^2 p}{p+1}$$

$\therefore$ The solution is ;

$$y = \frac{cx - a^2 c}{c+1}$$

$$\Rightarrow y^2 = \frac{cx^2 - a^2 c}{c+1}$$