

MODULE 5

Matrix :-

The set of mn number of elements or objects arranged as a 'm' number of rows and 'n' number of columns is called the matrix.

The matrix can be denoted by the capitals of alphabets and the order of the matrix can be denoted as $m \times n$.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}_{m \times n}$$

(or)

$$A = [a_{ij}]_{m \times n} \quad 1 \leq i \leq m, \quad 1 \leq j \leq n$$

Rank of a Matrix

The number of linearly independent rows or linearly independent columns of a matrix is called the rank of a matrix.

Also, the rank of a matrix in simple words may be explained as the number of non-zero rows (or) columns of a non-zero matrix is called the rank of a matrix.

The highest order of non-zero minor of a matrix is also said to be a rank of the matrix.

The rank of the matrix 'A' can be denoted by $\rho(A)$ and which is $\rho(A) = r$ and it should be always $1 \leq \rho(A) \leq m$ (or) $1 \leq r \leq m$

Generally, the rank of a matrix can be evaluated in various ways, they are echelon form.

Rank of a Matrix Using Echelon form.

1. Let 'A' be a matrix of a order $m \times n$.

2. There are any zero rows then they should be placed below non-zero rows.

3. The number of zero in front of any row increases according to the row number.
4. The non-zero rows of a echelon matrix are that matrix's linearly independent row vectors.
5. Hence, the number of non-zero rows of a matrix reduced in echelon form is called the rank of the matrix and which is $\rho(A)$ or $r(A)$

$$A = \begin{bmatrix} \overset{\text{Key}}{\circledast} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & \overset{\text{Key}}{\circledast} a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & \overset{\text{Key}}{\circledast} a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

1. Find the rank of a matrix.

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$$

$$\Rightarrow \text{Let, } A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$$

$$R_2 = R_2 - 2R_1$$

$$R_3 : R_3 - 3R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & -1 & -2 \end{bmatrix}$$

$$R_3 : R_3 - R_2$$

2. Find the rank of the matrix.

$$\begin{bmatrix} 2 & 3 & 0 & 1 \\ 1 & 0 & 1 & 2 \\ -1 & 1 & 1 & -2 \\ 1 & 5 & 3 & -1 \end{bmatrix}$$

$\Rightarrow$ Let,

$$A = \begin{bmatrix} 2 & 3 & 0 & 1 \\ 1 & 0 & 1 & 2 \\ -1 & 1 & 1 & -2 \\ 1 & 5 & 3 & -1 \end{bmatrix}$$

$$R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 2 \\ 2 & 3 & 0 & 1 \\ -1 & 1 & 1 & -2 \\ 1 & 5 & 3 & -1 \end{bmatrix}$$

$$R_2: R_2 - 2R_1$$

$$R_3: R_3 + R_1$$

$$R_4: R_4 - R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 3 & -2 & -3 \\ 0 & 1 & 2 & 0 \\ 0 & 5 & 2 & -3 \end{bmatrix}$$

$$R_2 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 3 & -2 & -3 \\ 0 & 5 & 2 & -3 \end{bmatrix}$$

$$R_3: R_3 - 3R_2$$

$$R_4: R_4 - 5R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -8 & -3 \\ 0 & 0 & -8 & -3 \end{bmatrix} \quad R_4 : R_4 - R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -8 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\rho(A) = 3$$

3. Find the rank of the matrix

$$\begin{bmatrix} 4 & 0 & 2 & 1 \\ 2 & 1 & 3 & 4 \\ 2 & 3 & 4 & 7 \\ 2 & 3 & 1 & 4 \end{bmatrix}$$

$\Rightarrow$ Let,

$$A = \begin{bmatrix} 4 & 0 & 2 & 1 \\ 2 & 1 & 3 & 4 \\ 2 & 3 & 4 & 7 \\ 2 & 3 & 1 & 4 \end{bmatrix} \quad R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 2 & 1 & 3 & 4 \\ 4 & 0 & 2 & 1 \\ 2 & 3 & 4 & 7 \\ 2 & 3 & 1 & 4 \end{bmatrix} \quad \begin{array}{l} R_2 : R_2 - R_1 \\ R_3 : R_3 - R_1 \\ R_4 : R_4 - R_1 \end{array}$$

$$\sim \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & -2 & -4 & -7 \\ 0 & 2 & 1 & 3 \\ 0 & 2 & -2 & 0 \end{bmatrix}$$

$R_3: R_3 + R_2$
 $R_4: R_4 + R_2$

$$\sim \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & -2 & -4 & -7 \\ 0 & 0 & -3 & -4 \\ 0 & 0 & -6 & -7 \end{bmatrix}$$

$R_4: R_4 - 2R_3$

$$\sim \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & -2 & -4 & -7 \\ 0 & 0 & -3 & -4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\rho(A) = 4$$

4. Find the rank of the matrix

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 1 & -1 & 0 \\ 3 & 3 & 2 & 1 \\ 2 & 4 & 6 & 2 \end{bmatrix}$$

$\Rightarrow$ Let,

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 1 & -1 & 0 \\ 3 & 3 & 2 & 1 \\ 2 & 4 & 6 & 2 \end{bmatrix}$$

$$R_2: R_2 - 2R_1$$

$$R_3: R_3 - 3R_1$$

$$R_4: R_4 - 2R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -3 & -7 & -2 \\ 0 & -3 & -7 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 : R_3 - R_2$$

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -3 & -7 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \rho(A) = 2$$

5. Find the rank of the matrix

$$\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

$$\Rightarrow \text{Let, } A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix} \quad R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

$$R_3 : R_3 - 3R_1$$

$$R_4 : R_4 - R_1$$

$$2 \quad \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix}$$

$R_3 : R_3 - R_2$
 $R_4 : R_4 - R_2$

$$2 \quad \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \rho(A) = 2$$

6. Find the rank of the matrix

$$\begin{bmatrix} 1 & 2 & 4 & 3 \\ 2 & 4 & 6 & 8 \\ 4 & 8 & 12 & 16 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

$$\Rightarrow \text{Let, } A = \begin{bmatrix} 1 & 2 & 4 & 3 \\ 2 & 4 & 6 & 8 \\ 4 & 8 & 12 & 16 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

$R_2 : R_2 - 2R_1$
 $R_3 : R_3 - 4R_1$
 $R_4 : R_4 - R_1$

$$2 \quad \begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & -4 & 4 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$R_3 \longleftrightarrow R_4$

$$\sim \begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -4 & 4 \end{bmatrix} \quad R_4: R_4 - 4R_3$$

$$\sim \begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\rho(A) = 3$$

7. Find the rank of the matrix

$$\begin{bmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 8 & 4 & 7 & 13 \\ 8 & 4 & -3 & -1 \end{bmatrix}$$

$\Rightarrow$ Let,

$$A = \begin{bmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 8 & 4 & 7 & 13 \\ 8 & 4 & -3 & -1 \end{bmatrix}$$

$$R_2: R_2 - 2R_1$$

$$R_3: R_3 - 4R_1$$

$$R_4: R_4 - 4R_1$$

$$\sim \begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & -15 & -19 \end{bmatrix}$$

$$R_4: R_4 - 3R_3$$

$$\sim \begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$\rho(A) = 4$$

8. Find the rank of the matrix

$$\begin{bmatrix} 11 & 12 & 13 & 14 \\ 12 & 13 & 14 & 15 \\ 13 & 14 & 15 & 16 \\ 14 & 15 & 16 & 17 \end{bmatrix}$$

$$\Rightarrow \text{Let } A = \begin{bmatrix} 11 & 12 & 13 & 14 \\ 12 & 13 & 14 & 15 \\ 13 & 14 & 15 & 16 \\ 14 & 15 & 16 & 17 \end{bmatrix}$$

$$\sim \begin{bmatrix} 11 & 12 & 13 & 14 \\ 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \end{bmatrix}$$

$$R_2: R_2 - R_1$$

$$R_3: R_3 - R_1$$

$$R_4: R_4 - R_1$$

$$\sim \begin{bmatrix} 11 & 12 & 13 & 14 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_3: R_3 - 2R_2$$

$$R_4: R_4 - 3R_2$$

$$\rho(A) = 2$$

Testing of Consistency for System of Linear Equations :-

1. Let 3 system of linear equations with 3-unknown are :

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

$$\Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\Rightarrow AX = B$$

where, $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is called co-efficient matrix.

$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ the variable (or) unknown matrix

$B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ is called the constant matrix.

2. Write an augmented matrix

$$[A:B] = [A/B] = [AB] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ and reduce it to echelon form.}$$

3. If $\rho(A) = \rho(AB) = r = n$ [no. of unknowns], then the system of equations are consistency and may have unique solutions.

If $\rho(A) = \rho(AB) < n$, then the system of equations are consistency and may have infinite number of solutions.

If $\rho(A) \neq \rho(AB)$ [or] $\rho(A) < \rho(AB)$, then the system of equations are said to be inconsistency and no solutions.

1. Investigate the values of λ and μ , so that the equation $2x + 3y + 5z = 9$, $7x + 3y - 2z = 8$ and $2x + 3y + \lambda z = \mu$ may have

- i. Unique Solution
- ii. Many Solution
- iii. No Solution

$$\Rightarrow \begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ \mu \end{bmatrix}$$

$$\Rightarrow AX = B$$

$$\therefore [A:B] = \begin{bmatrix} 2 & 3 & 5 & : & 9 \\ 7 & 3 & -2 & : & 8 \\ 2 & 3 & \lambda & : & \mu \end{bmatrix}$$

$$R_2 : 2R_2 - 7R_1$$

$$R_3 : R_3 - R_1$$

$$\sim \begin{bmatrix} 2 & 3 & 5 & : & 9 \\ 0 & -15 & -39 & : & -47 \\ 0 & 0 & \lambda-5 & : & \mu-9 \end{bmatrix}$$

- i) The given system of equation may have unique solution only at,
 $\lambda \neq 5, \mu \neq 9$
- ii) The system of equation may have infinite number of solution only at,
 $\lambda = 5, \mu = 9$
- iii) The system of equation are inconsistency and with no solution only at,
 $\lambda = 5, \mu \neq 9$

2. For what values of λ and μ , the system of equations, $x+2y+3z=6$, $x+3y+5z=9$, $2x+5y+\lambda z=\mu$ have

- i) Unique Solutions
 ii) Infinite
 iii) No Solution

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 2 & 5 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \\ \mu \end{bmatrix}$$

$$\Rightarrow AX = B$$

$$\therefore [A:B] = \begin{bmatrix} 1 & 2 & 3 & : & 6 \\ 1 & 3 & 5 & : & 9 \\ 2 & 5 & \lambda & : & \mu \end{bmatrix}$$

$$R_2 : R_2 - R_1$$

$$R_3 : R_3 - 2R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & : & 6 \\ 0 & 1 & 2 & : & 3 \\ 0 & 1 & \lambda-6 & : & \mu-12 \end{bmatrix} \quad R_3: R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & : & 6 \\ 0 & 1 & 2 & : & 3 \\ 0 & 0 & \lambda-8 & : & \mu-15 \end{bmatrix}$$

- i. The given solution of equations may have unique solution only at
 $\lambda \neq 8, \mu \neq 15$
- ii. The given system of equations may have infinite solutions only at,
 $\lambda = 8, \mu = 15$
- iii. The given system of equations are inconsistency only at,
 $\lambda = 8, \mu \neq 15$

Gauss - Elimination Method

1. Test for consistency and solve $x+y+z=6$,
 $x-y+2z=5, 3x+y+z=8$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 8 \end{bmatrix}$$

$$\Rightarrow AX = B$$

$$\therefore [A:B] = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 1 & -1 & 2 & : & 5 \\ 3 & 1 & 1 & : & 8 \end{bmatrix}$$

$$R_2 : R_2 - R_1$$

$$R_3 : R_3 - 3R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & -2 & 1 & : & -1 \\ 0 & -2 & -2 & : & -10 \end{bmatrix}$$

$$R_3 : R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & -2 & 1 & : & -1 \\ 0 & 0 & -3 & : & -9 \end{bmatrix}$$

$\therefore$ The given equations are consistency and follows unique solution.

$$\therefore AX = B$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -1 \\ -9 \end{bmatrix}$$

$$\Rightarrow x + y + z = 6 \text{ --- (1)}$$

$$\Rightarrow -2y + z = -1 \text{ --- (2)}$$

$$\Rightarrow -3z = -9 \text{ --- (3)}$$

$$\therefore z = 3$$

$$(2) \Rightarrow -2y + 3 = -1$$

$$\Rightarrow -2y = -4$$

$$\Rightarrow y = 2$$

$$(1) \Rightarrow x + y + z = 6$$

$$\Rightarrow x + 2 + 3 = 6$$

$$\Rightarrow x = 6 - 5$$

$$\Rightarrow x = 1$$

$$\therefore x = 1, y = 2, z = 3$$

2. Solve the system of equation by Gauss-elimination method $x+y+z=9$, $2x+y-z=0$, $2x+5y+7z=52$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 2 & 5 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \\ 52 \end{bmatrix}$$

$$AX = B$$

$$\therefore [A:B] = \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 2 & 1 & -1 & : & 0 \\ 2 & 5 & 7 & : & 52 \end{bmatrix}$$

$$R_2: R_2 - 2R_1$$

$$R_3: R_3 - 2R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 0 & -1 & -3 & : & -18 \\ 0 & 3 & 5 & : & 34 \end{bmatrix}$$

$$R_3: R_3 + 3R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 0 & -1 & -3 & : & -18 \\ 0 & 0 & -4 & : & -20 \end{bmatrix}$$

$$\rho(A) = \rho(AB) = 3 = n$$

$\therefore$ The system of equation follows unique solution.

$$\therefore AX = B$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -3 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ -18 \\ -20 \end{bmatrix}$$

$$\Rightarrow x + y + z = 9 \text{ --- (1)}$$

$$\Rightarrow -y - 3z = -18 \text{ --- (2)}$$

$$\Rightarrow -4z = -20 \text{ --- (3)}$$

$$\therefore z = 5$$

$$(2) \Rightarrow -y - 3(5) = -18$$

$$-y = -3$$

$$y = 3$$

$$(1) \Rightarrow x + y + z = 9$$

$$x + 3 + 5 = 9$$

$$x = 9 - 8$$

$$x = 1$$

$$\therefore x = 1, y = 3, z = 5$$

3. Test for consistency and solve $5x + 3y + 7z = 4$,
 $3x + 26y + 2z = 9$, $7x + 2y + 10z = 5$

$$\Rightarrow \begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix}$$

$$AX = B$$

$$\therefore [A:B] = \begin{bmatrix} 5 & 3 & 7 & : & 4 \\ 3 & 26 & 2 & : & 9 \\ 7 & 2 & 10 & : & 5 \end{bmatrix}$$

$$R_2 : 5R_2 - 3R_1$$

$$R_3 : 5R_3 - 7R_1$$

$$\sim \begin{bmatrix} 5 & 3 & 7 & : & 4 \\ 0 & 11 & -1 & : & 33 \\ 0 & -11 & 1 & : & 3 \end{bmatrix}$$

$$R_2 = \frac{1}{11} R_2$$

$$\sim \begin{bmatrix} 5 & 3 & 7 & : & 4 \\ 0 & 11 & -1 & : & 3 \\ 0 & -11 & 1 & : & -3 \end{bmatrix}$$

$$R_3: R_3 + R_2$$

$$\sim \begin{bmatrix} 5 & 3 & 7 & : & 4 \\ 0 & 11 & -1 & : & 3 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

$$\rho(A) = \rho(AB) = 2 = n < 3$$

$\therefore$ The given equation follows infinite solutions

$$AX = B$$

$$\Rightarrow \begin{bmatrix} 5 & 3 & 7 \\ 0 & 11 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$$

$$\therefore 5x + 3y + 7z = 4 \quad \text{--- (1)}$$

$$11y - z = 3 \quad \text{--- (2)}$$

$$\text{Let, } z = k$$

$$(2) \Rightarrow 11y - k = 3$$

$$11y = k + 3$$

$$y = \frac{1}{11}(k+3)$$

$$(1) \Rightarrow 5x + 3 \left[\frac{1}{11}(k+3) \right] + 7k = 4$$

$$\Rightarrow 5x + \frac{3}{11}(k+3) + 7k = 4$$

$$\Rightarrow 55x + 3k + 9 + 7k = 44$$

$$\Rightarrow 55x + 10k = 44 - 9$$

$$\Rightarrow 55x = 35 - 10k$$

$$\Rightarrow x = \frac{1}{55}(35 - 10k)$$

4. Solve the system of equation by Gauss - elimination method, $x+2y+z=3$, $3x+2y+z=3$, $x-2y-5z=1$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 1 \\ 1 & -1 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}$$

$$AX = B$$

$$[A:B] = \begin{bmatrix} 1 & 2 & 1 & : & 3 \\ 3 & 2 & 1 & : & 3 \\ 1 & -1 & -5 & : & 1 \end{bmatrix}$$

$R_2: R_2 - 3R_1$
 $R_3: R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 2 & 1 & : & 3 \\ 0 & -4 & -2 & : & -6 \\ 0 & -4 & -6 & : & -2 \end{bmatrix}$$

$R_3: 4R_3 + 3R_2$

$$\sim \begin{bmatrix} 1 & 2 & 1 & : & 3 \\ 0 & -4 & -2 & : & -6 \\ 0 & 0 & 2 & : & 2 \end{bmatrix}$$

$$\rho(A) = \rho(AB) = 3$$

$\therefore$ The given equations follows unique solutions.

$$AX = B$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & -2 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \\ 2 \end{bmatrix}$$

$$x + 2y + z = 3 \quad \text{---(1)}$$

$$-4y - 2z = -6 \quad \text{---(2)}$$

$$2z = -2 \quad \text{---(3)}$$

$$\therefore z = -1$$

$$(2) \Rightarrow -4y - 2(-1) = -6$$

$$-4y + 2 = -6$$

$$-4y = -8$$

$$y = 2$$

$$\therefore x = 0, y = 2, z = -1$$

$$(3) \Rightarrow x + 2(2) + (-1) = 3$$

$$x + 4 - 1 = 3$$

$$x + 3 = 3$$

$$x = 0$$

Guass - Jordan Method

1. Solve the system of equation by Guass-Jordan method $x + y + z = 9$, $x - 2y + 3z = 8$, $2x + y - z = 3$.

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & 3 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ 3 \end{bmatrix}$$

$$AX = B$$

$$\therefore [A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 1 & -2 & 3 & 8 \\ 2 & 1 & -1 & 3 \end{array} \right]$$

$$R_2: R_2 - R_1$$

$$R_3: R_3 - 2R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -3 & 2 & -1 \\ 0 & -1 & -3 & -15 \end{array} \right]$$

$$R_2: R_2 - R_1$$

$$R_3: R_3 - 2R_1$$

$$\sim \begin{bmatrix} 3 & 3 & 3 & : & 27 \\ 0 & -3 & 2 & : & -1 \\ 0 & 3 & 9 & : & 45 \end{bmatrix}$$

$$R_1: 3R_1$$

$$R_3: -3R_3$$

$$\sim \begin{bmatrix} 3 & 0 & 5 & : & 26 \\ 0 & -3 & 2 & : & -1 \\ 0 & 0 & 11 & : & 44 \end{bmatrix}$$

$$R_3: \frac{1}{11}R_3$$

$$\sim \begin{bmatrix} 3 & 0 & 5 & : & 26 \\ 0 & -3 & 2 & : & -1 \\ 0 & 0 & 1 & : & 4 \end{bmatrix}$$

$$R_1: R_1 - 5R_3$$

$$R_2: R_2 - 2R_3$$

$$\sim \begin{bmatrix} 3 & 0 & 0 & : & 6 \\ 0 & -3 & 0 & : & -9 \\ 0 & 0 & 1 & : & 4 \end{bmatrix}$$

$$R_1: \frac{1}{3}R_1$$

$$R_2: -\frac{1}{3}R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & : & 2 \\ 0 & 1 & 0 & : & 3 \\ 0 & 0 & 1 & : & 4 \end{bmatrix}$$

$$\therefore P(A) : P(AB) = 3 = n$$

$$AX = B$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

$$\therefore x = 2$$

$$y = 3$$

$$z = 4$$

2. Solve the system of equation by Gauss-Jordan method $x+y+z=11$, $3x-y+2z=12$, $2x+y-z=3$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 2 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ 12 \\ 3 \end{bmatrix}$$

$$AX=B$$

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & : & 11 \\ 3 & -1 & 2 & : & 12 \\ 2 & 1 & -1 & : & 3 \end{bmatrix}$$

$$R_2 : R_2 - 3R_1$$

$$R_3 : R_3 - 2R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 11 \\ 0 & -4 & -1 & : & -21 \\ 0 & -1 & -3 & : & -19 \end{bmatrix}$$

$$R_1 : 4R_1$$

$$R_3 : -4R_3$$

$$\sim \begin{bmatrix} 4 & 4 & 4 & : & 44 \\ 0 & -4 & -1 & : & -21 \\ 0 & 4 & 12 & : & 76 \end{bmatrix}$$

$$R_1 : R_1 + R_2$$

$$R_3 : R_3 + R_2$$

$$\sim \begin{bmatrix} 4 & 0 & 3 & : & 23 \\ 0 & -4 & -1 & : & -21 \\ 0 & 0 & 11 & : & 55 \end{bmatrix}$$

$$R_2 : (-1)R_2$$

$$R_3 : \frac{1}{11} R_3$$

$$\sim \begin{bmatrix} 4 & 0 & 3 & : & 23 \\ 0 & 4 & 1 & : & 21 \\ 0 & 0 & 1 & : & 5 \end{bmatrix}$$

$$R_1: R_1 - 3R_3$$

$$R_2: R_2 - R_3$$

$$\sim \begin{bmatrix} 4 & 0 & 0 & : & 8 \\ 0 & 4 & 0 & : & 16 \\ 0 & 0 & 1 & : & 5 \end{bmatrix}$$

$$R_1: \frac{1}{4} R_1$$

$$R_2: \frac{1}{4} R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & : & 2 \\ 0 & 1 & 0 & : & 4 \\ 0 & 0 & 1 & : & 5 \end{bmatrix}$$

$$\rho(A) = \rho(AB) = 3 = n$$

$$AX = B$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}$$

$$\therefore x = 2, y = 4, z = 5$$

3. Solve the system of equation by Gauss-Jordan method $x + y + z = 10$, $2x - y + 3z = 19$, $x + 2y + 3z = 22$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 19 \\ 22 \end{bmatrix}$$

$$AX = B$$

$$[A:B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 10 \\ 2 & -1 & 3 & 19 \\ 1 & 2 & 3 & 22 \end{array} \right]$$

$$R_2: R_2 - 2R_1$$

$$R_3: R_3 - R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 10 \\ 0 & -3 & 1 & -1 \\ 0 & 1 & 2 & 12 \end{array} \right]$$

$$R_1: 3R_1$$

$$R_3: 3R_3$$

$$\sim \left[\begin{array}{ccc|c} 3 & 3 & 3 & 30 \\ 0 & -3 & 1 & -1 \\ 0 & 3 & 6 & 36 \end{array} \right]$$

$$R_1: R_1 + R_2$$

$$R_3: R_3 + R_2$$

$$\sim \left[\begin{array}{ccc|c} 3 & 0 & 4 & 29 \\ 0 & -3 & 1 & -1 \\ 0 & 0 & 7 & 35 \end{array} \right]$$

$$R_3: \frac{1}{7} R_3$$

$$\sim \left[\begin{array}{ccc|c} 3 & 0 & 4 & 29 \\ 0 & -3 & 1 & -1 \\ 0 & 0 & 1 & 5 \end{array} \right]$$

$$R_1: R_1 - 4R_3$$

$$R_2: R_2 - R_3$$

$$\sim \left[\begin{array}{ccc|c} 3 & 0 & 0 & 9 \\ 0 & -3 & 0 & -6 \\ 0 & 0 & 1 & 5 \end{array} \right]$$

$$R_1: \frac{1}{3} R_1$$

$$R_2: -\frac{1}{3} R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & : & 3 \\ 0 & 1 & 0 & : & 2 \\ 0 & 0 & 1 & : & 5 \end{bmatrix}$$

$$\rho(A) = \rho(AB) = 3 = n$$

$$AX = B$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$$

$$\therefore x=3, y=2, z=5$$

4. Solve the system of equation by Gauss-Jordan method $x+y+z=8$, $-x-y+2z=-4$, $3x+5y-7z=14$

$$\Rightarrow \Rightarrow x+y+z=8 \text{ --- (1)}$$

$$\Rightarrow -x-y+2z=-4$$

$$\Rightarrow x+y-2z=4 \text{ --- (2)}$$

$$\Rightarrow 3x+5y-7z=14 \text{ --- (3)}$$

$$\Rightarrow [A:B] = \begin{bmatrix} 1 & 1 & 1 & : & 8 \\ 1 & 1 & -2 & : & 4 \\ 3 & 5 & -7 & : & 14 \end{bmatrix}$$

$$R_2: R_2 - R_1$$

$$R_3: R_3 - 3(R_1)$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 8 \\ 0 & 0 & -3 & : & -4 \\ 0 & 2 & -10 & : & -10 \end{bmatrix}$$

$$R_2 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 8 \\ 0 & 2 & -10 & : & -10 \\ 0 & 0 & -3 & : & -4 \end{bmatrix} \quad R_2 : \frac{1}{2} R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 8 \\ 0 & 1 & -5 & : & -5 \\ 0 & 0 & -3 & : & -4 \end{bmatrix} \quad R_1 : R_1 - R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 6 & : & 13 \\ 0 & 1 & -5 & : & -5 \\ 0 & 0 & -3 & : & -4 \end{bmatrix} \quad \begin{array}{l} R_1 : R_1 + 2R_3 \\ R_2 : 3R_2 - 5R_3 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & : & 5 \\ 0 & 3 & 0 & : & 5 \\ 0 & 0 & -3 & : & -4 \end{bmatrix} \quad \begin{array}{l} R_2 : \frac{1}{3} R_2 \\ R_3 : -\frac{1}{3} R_3 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & : & 5 \\ 0 & 1 & 0 & : & 5/3 \\ 0 & 0 & 1 & : & 4/3 \end{bmatrix}$$

$$\rho(A) = \rho(AB) = 3 = n$$

$$AX = B$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 5/3 \\ 4/3 \end{bmatrix}$$

$$x = 5, \quad y = 5/3, \quad z = 4/3$$

5. Solve the equation by using Gauss-Jordan method $x+y+z=9$, $2x+y-z=0$, $2x+5y+7z=52$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 2 & 5 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \\ 52 \end{bmatrix}$$

$$AX=B$$

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 2 & 1 & -1 & : & 0 \\ 2 & 5 & 7 & : & 52 \end{bmatrix}$$

$$R_2: R_2 - 2R_1$$

$$R_3: R_3 - 2R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 0 & -1 & -3 & : & -18 \\ 0 & 3 & 5 & : & 34 \end{bmatrix}$$

$$R_3: R_3 + 3R_2$$

$$R_1: R_1 + R_2$$

$$\sim \begin{bmatrix} 1 & 0 & -2 & : & -9 \\ 0 & -1 & -3 & : & -18 \\ 0 & 0 & -4 & : & -20 \end{bmatrix}$$

$$R_3: \left[-\frac{1}{4}\right] R_3$$

$$\sim \begin{bmatrix} 1 & 0 & -2 & : & -9 \\ 0 & -1 & -3 & : & -18 \\ 0 & 0 & 1 & : & 5 \end{bmatrix}$$

$$R_1: R_1 + 2R_3$$

$$R_2: R_2 + 3R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & : & 1 \\ 0 & -1 & 0 & : & -3 \\ 0 & 0 & 1 & : & 5 \end{bmatrix}$$

$$R_2: (-1)R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & : & 1 \\ 0 & 1 & 0 & : & 3 \\ 0 & 0 & 1 & : & 5 \end{bmatrix}$$

$$AX = B$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

$$x=1, y=3, z=5$$

6. Solve the equation by using Gauss-Jordan method $x+2y+z=3$, $2x+3y+2z=5$, $3x-5y+5z=2$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 3 & -5 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}$$

$$AX = B$$

$$[A:B] = \begin{bmatrix} 1 & 2 & 1 & : & 3 \\ 2 & 3 & 2 & : & 5 \\ 3 & -5 & 5 & : & 2 \end{bmatrix}$$

$$R_2 : R_2 - 2R_1$$

$$R_3 : R_3 - 3R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & : & 3 \\ 0 & -1 & 0 & : & -1 \\ 0 & -11 & 2 & : & -7 \end{bmatrix}$$

$$R_1 : R_1 + 2R_2$$

$$R_3 : R_3 - 11R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & : & 1 \\ 0 & -1 & 0 & : & -1 \\ 0 & 0 & 2 & : & 4 \end{bmatrix}$$

$$R_1 : 2R_1 - R_3$$

$$\sim \begin{bmatrix} 2 & 0 & 0 & : & -2 \\ 0 & -1 & 0 & : & -1 \\ 0 & 0 & 2 & : & 4 \end{bmatrix} \quad \begin{array}{l} R_1 : \frac{1}{2} R_1 \\ R_2 : (-1) R_2 \\ R_3 : (\frac{1}{2}) R_3 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & : & -1 \\ 0 & 1 & 0 & : & 1 \\ 0 & 0 & 1 & : & 2 \end{bmatrix}$$

$$AX = B$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

$$\therefore x = -1, y = 1, z = 2$$

Gauss - Seidel Method (or)

Gauss - Seidel Iterative Method

1. Let the 3 system of linear equations are

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \text{ ——— (1)}$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \text{ ——— (2)}$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \text{ ——— (3)}$$

2. check the property of diagonal dominant as given below.

$$|a_{11}| > |a_{12}| + |a_{13}|$$

$$|a_{22}| > |a_{21}| + |a_{23}|$$

$$|a_{33}| > |a_{31}| + |a_{32}|$$

3. If the diagonal elements are not in diagonal dominant, then rearrange the equation.

4. Write the unknown x_1, x_2, x_3 from the 3 eqn (1) (2) & (3)

$$(1) \Rightarrow x_1 = \frac{1}{a_{11}} |b_1 - a_{12}x_2 - a_{13}x_3|$$

$$(2) \Rightarrow x_2 = \frac{1}{a_{22}} |b_2 - a_{21}x_1 - a_{23}x_3|$$

$$(3) \Rightarrow x_3 = \frac{1}{a_{33}} |b_3 - a_{31}x_1 - a_{32}x_2|$$

5. Take the initial condition as $x_1=0, x_2=0$ and $x_3=0$ and start iterations by updating the values of x_1, x_2, x_3 and continue the same until to reach the solution (approximately)

1. Solve the system of equation by using Gauss-Seidel method.

$$10x + y + z = 12, \quad x + 10y + z = 12, \quad x + y + 10z = 12$$

$$\Rightarrow 10x + y + z = 12 \text{ --- (1)}$$

$$x + 10y + z = 12 \text{ --- (2)}$$

$$x + y + 10z = 12 \text{ --- (3)}$$

$\therefore$ The given equations are diagonally dominant

$$(1) \Rightarrow x = \frac{1}{10} [12 - y - z]$$

$$(2) \Rightarrow y = \frac{1}{10} [12 - x - z]$$

$$(3) \Rightarrow z = \frac{1}{10} [12 - x - y]$$

$$I-(1) \Rightarrow x^{(1)} = \frac{1}{10} [12 - 0 - 0] = 1.2$$

$$y^{(1)} = \frac{1}{10} [12 - 1.2 - 0] = 1.08$$

$$z^{(1)} = \frac{1}{10} [12 - 1.2 - 1.08] = 0.972$$

$$I-(2) \Rightarrow x^{(2)} = \frac{1}{10} [12 - 1.08 - 0.972] = 0.9948$$

$$y^{(2)} = \frac{1}{10} [12 - 0.9948 - 0.972] = 1.0033$$

$$z^{(2)} = \frac{1}{10} [12 - 0.9948 - 1.0033] = 1.0002$$

$$I-(3) \Rightarrow x^{(3)} = \frac{1}{10} [12 - 1.0033 - 1.0002] = 1$$

$$y^{(3)} = \frac{1}{10} [12 - 1 - 1.0002] = 1$$

$$z^{(3)} = \frac{1}{10} [12 - 1 - 1] = 1$$

$$I-(4) \Rightarrow x^{(4)} = \frac{1}{10} [12 - 1 - 1] = 1$$

$$y^{(4)} = \frac{1}{10} [12 - 1 - 1] = 1$$

$$z^{(4)} = \frac{1}{10} [12 - 1 - 1] = 1$$

$\therefore$ The solution is

$$x=1, y=1, z=1$$

2. Solve the following equation by using Gauss-Seidel method.

$5x + 2y + z = 12$, $x + 4y + 2z = 15$, $x + 2y + 5z = 20$
taking $(0, 0, 0)$ as an initial approximation
[carry out 4 iteration].

$$\Rightarrow 5x + 2y + z = 12 \text{ --- (1)}$$

$$x + 4y + 2z = 15 \text{ --- (2)}$$

$$x + 2y + 5z = 20 \text{ --- (3)}$$

$\therefore$ The given equations are diagonally dominant

$$(1) \Rightarrow x = \frac{1}{5} [12 - 2y - z]$$

$$(2) \Rightarrow y = \frac{1}{4} [15 - x - 2z]$$

$$(3) \Rightarrow z = \frac{1}{5} [20 - x - 2y]$$

$$I - (1) \Rightarrow x^{(1)} = \frac{12}{5} = 2.4$$

$$y^{(1)} = \frac{1}{4} [15 - 2.4 - 0] = 3.15$$

$$z^{(1)} = \frac{1}{5} [20 - 2.4 - 2(3.15)] = 2.26$$

$$I - (2) \Rightarrow x^{(2)} = \frac{1}{5} [12 - 2(3.15) - 2.26] = 0.688$$

$$y^{(2)} = \frac{1}{4} [15 - 0.688 - 2(2.26)] = 2.448$$

$$z^{(2)} = \frac{1}{5} [20 - 0.688 - 2(2.448)] = 2.8832$$

$$I-(3) \Rightarrow x^{(3)} = \frac{1}{5} [12 - 2(2.448) - 2.8832] = 0.84416$$

$$y^{(3)} = \frac{1}{4} [15 - 0.84416 - 2(2.8832)] = 2.09736$$

$$z^{(3)} = \frac{1}{5} [20 - 0.84416 - 2(2.09736)] = 2.9922$$

$$I-(4) \Rightarrow x^{(4)} = \frac{1}{5} [12 - 2(2.09736) - 2.9922] = 0.9626$$

$$y^{(4)} = \frac{1}{4} [15 - 0.9626 - 2(2.9922)] = 2.0132$$

$$z^{(4)} = \frac{1}{5} [20 - 0.9626 - 2(2.0132)] = 3.0022$$

$\therefore$ The solution is

$$x = 0.9626 \sim 1$$

$$y = 2.0132 \sim 2$$

$$z = 3.0022 \sim 3$$

3. $2x - 3y + 20z = 25$, $20x + y - 2z = 17$, $3x + 20y - z = -18$
using Gauss-Seidel taking $(0,0,0)$ as an initial approximation.

$\Rightarrow$ The given equations are not in diagonally dominant form. By reordering the given equations, we have

$$20x + y - 2z = 17 \quad \text{---(1)}$$

$$3x + 20y - z = -18 \quad \text{---(2)}$$

$$2x - 3y + 20z = 25 \quad \text{---(3)}$$

$$(1) \Rightarrow x = \frac{1}{20} [17 - y + 2z]$$

$$(2) \Rightarrow y = \frac{1}{20} [-18 - 3x + z]$$

$$(3) \Rightarrow z = \frac{1}{20} [25 - 2x + 3y]$$

$$I - (1) \Rightarrow x^{(1)} = \frac{17}{20} = 0.85$$

$$y^{(1)} = \frac{1}{20} [-18 - 3(0.85) + 0] = -1.0275$$

$$z^{(1)} = \frac{1}{20} [25 - 2(0.85) + 3(-1.0275)] = 1.0108$$

$$I - (2) \Rightarrow x^{(2)} = \frac{1}{20} [17 + 1.0275 + 2(1.0108)] = 1.0025$$

$$y^{(2)} = \frac{1}{20} [-18 - 3(1.0025) + (1.0108)] = -0.998$$

$$z^{(2)} = \frac{1}{20} [25 - 2(1.0025) + 3(-0.998)] = 0.9998$$

$$I - (3) \Rightarrow x^{(3)} = \frac{1}{20} [17 + 0.998 + 2(0.9998)] = 1$$

$$y^{(3)} = \frac{1}{20} [-18 - 3(1) + 0.9998] = -1$$

$$z^{(3)} = \frac{1}{20} [25 - 2(1) + 3(-1)] = 1$$

$$I - (4) \Rightarrow x^{(4)} = \frac{1}{20} [17 + 1 + 12] = 1$$

$$y^{(4)} = \frac{1}{20} [-18 - 3 + 1] = -1$$

$$z^{(4)} = \frac{1}{20} [25 - 2 - 3] = 1$$

$\therefore$ The solution is $x=1, y=-1, z=1$

11. $12x + y + z = 31$, $2x + 8y - z = 24$, $3x + 4y + 10z = 58$
using Gauss-Seidel taking $(0,0,0)$ as an
initial approximation.

$$\Rightarrow \begin{aligned} 12x + y + z &= 31 & \text{---(1)} \\ 2x + 8y - z &= 24 & \text{---(2)} \\ 3x + 4y + 10z &= 58 & \text{---(3)} \end{aligned}$$

$$(1) \Rightarrow x = \frac{1}{12} [31 - y - z]$$

$$(2) \Rightarrow y = \frac{1}{8} [24 - 2x + z]$$

$$(3) \Rightarrow z = \frac{1}{10} [58 - 3x - 4y]$$

$$I - (1) \Rightarrow x^{(1)} = \frac{31}{12} = 2.5833$$

$$y^{(1)} = \frac{1}{8} [24 - 2(2.5833) + 0] = 2.3542$$

$$z^{(1)} = \frac{1}{10} [58 - 3(2.5833) - 4(2.3542)] = 4.0833$$

$$I - (2) \Rightarrow x^{(2)} = \frac{1}{12} [31 - 2.3542 - 4.0833] = 2.0468$$

$$y^{(2)} = \frac{1}{8} [24 - 2(2.7274) + 4.0833] = 1.1923$$

$$z^{(2)} = \frac{1}{10} [58 - 3(2.7274) - 4(1.1923)] = 4.8686$$

$$I - (3) \Rightarrow x^{(3)} = \frac{1}{12} [31 - 1.1923 - 4.8686] = 2.0787$$

$$y^{(3)} = \frac{1}{8} [24 - 2(2.8892) + 4.8686] = 2.8855$$

$$z^{(3)} = \frac{1}{10} [58 - 3(2.8892) - 4(2.8855)] = 3.7190$$

$\therefore$ The solution is $x=2$, $y=3$, $z=4$

Eigen Values And Eigen Vectors

Eigen values are a special set of scalar associated with the linear system of equations they are also known as characteristics roots and characteristics values and they can be determined by taking

$$(A - \lambda I) X = 0$$

where, A = Square matrix

I = Unit matrix

X = Variable matrix having single column

And the determinant of $A - \lambda I$ can be used to find the eigen values and which follows as

$|A - \lambda I| = 0$ called characteristic equation and it can provide the roots of ' λ '.

Rayleigh's Power Method (or) Power Method

To determine the largest eigen value and the respective eigen vector we used to follow the given - working rule.

1. Let the given square matrix as ' A ' with the initial eigen vector as ' X ' follows:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \text{ or } \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

2. Take the product of 'A' and the initial eigen vector and bring out the numerically largest value, say it as ' λ '

$$\therefore AX^{(0)} = X^{(1)}\lambda^{(1)}$$

3. Continue the same process with the resultant eigen vectors until to reach an equal eigen value

$$\therefore AX^{(1)} = X^{(2)}\lambda^{(2)}$$

$$AX^{(2)} = X^{(3)}\lambda^{(3)}$$

$$AX^{(3)} = X^{(4)}\lambda^{(4)}$$

⋮

⋮

4. Finally, the obtained ' λ ' is called the largest eigen value and the vector ' x ' is called respective eigen vector.

1. Find the largest Eigen Value of the matrix

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

with the initial vector $[1 \ 0 \ 0]^T$

$\Rightarrow$ Let,

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \quad x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore AX^{(0)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$= 2 \begin{bmatrix} 1 \\ 0 \\ 0.5 \end{bmatrix} = \lambda^{(1)} x^{(1)}$$

$$\therefore AX^{(1)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 0 \\ 2 \end{bmatrix}$$

$$= 2.5 \begin{bmatrix} 1 \\ 0 \\ 0.8 \end{bmatrix} = \lambda^{(2)} x^{(2)}$$

$$\therefore AX^{(2)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.8 \end{bmatrix} = \begin{bmatrix} 2.8 \\ 0 \\ 2.6 \end{bmatrix}$$

$$= 2.8 \begin{bmatrix} 1 \\ 0 \\ 0.9286 \end{bmatrix} = \lambda^{(3)} x^{(3)}$$

$$\therefore AX^{(3)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.9286 \end{bmatrix} = \begin{bmatrix} 2.9286 \\ 0 \\ 2.8571 \end{bmatrix}$$

$$= 2.9286 \begin{bmatrix} 1 \\ 0 \\ 0.9756 \end{bmatrix} = \lambda^{(4)} x^{(4)}$$

$$\therefore AX^{(4)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.9756 \end{bmatrix} = \begin{bmatrix} 2.9756 \\ 0 \\ 2.9512 \end{bmatrix}$$

$$= 2.9756 \begin{bmatrix} 1 \\ 0 \\ 0.9918 \end{bmatrix} = \lambda^{(5)} x^{(5)}$$

$\therefore$ The largest eigen value is $2.9756 \sim 3$
and eigen vector is $\begin{bmatrix} 1 \\ 0 \\ 0.9918 \end{bmatrix} \sim \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

2. Using Rayleigh's Power method find the dominant eigen value and the corresponding eigen vector of $\begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}$ by taking $[1 \ 0 \ 0]^T$ as an initial eigen vector.

$\Rightarrow$ Let, $A = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}$ $x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$\therefore AX^{(0)} = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ -2 \end{bmatrix}$$

$$= 4 \begin{bmatrix} 1 \\ 0.5 \\ -0.5 \end{bmatrix} = \lambda^{(1)} x^{(1)}$$

$$\therefore AX^{(1)} = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5 \\ -0.5 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ -4 \end{bmatrix}$$

$$= 5 \begin{bmatrix} 1 \\ 0.8 \\ -0.8 \end{bmatrix} = \lambda^{(2)} X^{(2)}$$

$$\therefore AX^{(2)} = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0.8 \\ -0.8 \end{bmatrix} = \begin{bmatrix} 5.6 \\ 5.2 \\ -5.2 \end{bmatrix}$$

$$= 5.6 \begin{bmatrix} 1 \\ 0.9286 \\ -0.9286 \end{bmatrix} = \lambda^{(3)} X^{(3)}$$

$$\therefore AX^{(3)} = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0.9286 \\ -0.9286 \end{bmatrix} = \begin{bmatrix} 5.8571 \\ 5.7144 \\ -5.7144 \end{bmatrix}$$

$$= 5.8571 \begin{bmatrix} 1 \\ 0.9756 \\ -0.9756 \end{bmatrix} = \lambda^{(4)} X^{(4)}$$

$$\therefore AX^{(4)} = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0.9756 \\ -0.9756 \end{bmatrix} = \begin{bmatrix} 5.9513 \\ 5.9024 \\ -5.9024 \end{bmatrix}$$

$$= 5.9513 \begin{bmatrix} 1 \\ 0.9918 \\ -0.9918 \end{bmatrix} = \lambda^{(5)} X^{(5)}$$

$$\therefore AX^{(5)} = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0.9918 \\ -0.9918 \end{bmatrix} = \begin{bmatrix} 5.9836 \\ 5.9672 \\ -5.9672 \end{bmatrix}$$

$$= 5.9836 \begin{bmatrix} 1 \\ 0.9973 \\ -0.9973 \end{bmatrix}$$

$\therefore$ The largest eigen value is 5.9836 ~ 6
and eigen vector is $\begin{bmatrix} 1 \\ 0.9973 \\ -0.9973 \end{bmatrix} \sim \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

3. $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ with the initial approximate eigen vector $[1 \ 0 \ 0]^T$ and carry out 4 iterations.

$$\Rightarrow \text{Let, } A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \quad x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore AX^{(0)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

$$= 2 \begin{bmatrix} 1 \\ -0.5 \\ 0 \end{bmatrix} = \lambda^{(1)} x^{(1)}$$

$$\therefore AX^{(1)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.5 \\ -2 \\ 0.5 \end{bmatrix}$$

$$= 2.5 \begin{bmatrix} 1 \\ 0.8 \\ 0.2 \end{bmatrix} = \lambda^{(2)} X^{(2)}$$

$$\therefore AX^{(2)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.8 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 2.8 \\ -2.8 \\ 1.8 \end{bmatrix}$$

$$= 2.8 \begin{bmatrix} 1 \\ -1 \\ 0.4285 \end{bmatrix} = \lambda^{(3)} X^{(3)}$$

$$\therefore AX^{(3)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0.4285 \end{bmatrix} = \begin{bmatrix} 3 \\ -3.4286 \\ 1.8572 \end{bmatrix}$$

$$= 3.4286 \begin{bmatrix} 0.8750 \\ -1 \\ 0.5417 \end{bmatrix} = \lambda^{(4)} X^{(4)}$$

$\therefore$ The largest eigen value is 3.4286 and eigen vector is $\begin{bmatrix} 0.8750 \\ -1 \\ 0.5417 \end{bmatrix}$

4. Using Rayleigh's Power method find the dominant eigen value and the corresponding eigen vector of $\begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}$ by taking $[1 \ 0 \ 0]^T$ as the initial eigen vector

$$\Rightarrow \text{let, } A = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \quad x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore AX^{(0)} = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 25 \\ 1 \\ 2 \end{bmatrix}$$

$$= 25 \begin{bmatrix} 0.04 \\ 1 \\ 0.08 \end{bmatrix} = \lambda^{(1)} x^{(1)}$$

$$\therefore AX^{(1)} = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} 0.04 \\ 1 \\ 0.08 \end{bmatrix} = \begin{bmatrix} 25.2 \\ 1.120 \\ 1.68 \end{bmatrix}$$

$$= 25.2 \begin{bmatrix} 1 \\ 0.0444 \\ 0.0667 \end{bmatrix} = \lambda^{(2)} x^{(2)}$$

$$\therefore AX^{(2)} = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0.0444 \\ 0.0667 \end{bmatrix} = \begin{bmatrix} 25.1777 \\ 1.1332 \\ 1.7332 \end{bmatrix}$$

$$= 25.1777 \begin{bmatrix} 1 \\ 0.045 \\ 0.0688 \end{bmatrix} = \lambda^{(3)} x^{(3)}$$

$$\therefore AX^{(3)} = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0.045 \\ 0.0688 \end{bmatrix} = \begin{bmatrix} 25.1826 \\ 1.135 \\ 1.7248 \end{bmatrix}$$

$$= 25.1826 \begin{bmatrix} 1 \\ 0.0451 \\ 0.0684 \end{bmatrix} = \lambda^{(4)} x^{(4)}$$

$$\therefore AX^{(4)} = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0.0451 \\ 0.0684 \end{bmatrix} = \begin{bmatrix} 25.1821 \\ 1.1353 \\ 1.7264 \end{bmatrix}$$

$$= 25.1821 \begin{bmatrix} 1 \\ 0.0451 \\ 0.0685 \end{bmatrix} = \lambda^{(5)} x^{(5)}$$

$$\therefore AX^{(5)} = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0.0451 \\ 0.0685 \end{bmatrix} = \begin{bmatrix} 25.1821 \\ 1.1353 \\ 1.7260 \end{bmatrix}$$

$$= 25.1821 \begin{bmatrix} 1 \\ 0.0451 \\ 0.0685 \end{bmatrix}$$

$\therefore$ The largest eigen value is 25.1821
and eigen vector is $\begin{bmatrix} 1 \\ 0.0451 \\ 0.0685 \end{bmatrix}$

5. Using Power method find the largest eigen value and corresponding eigen vector of the matrix.

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \text{ by taking } [1 \ 1 \ 1]^T$$

$$\Rightarrow A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \quad X = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\therefore AX^{(0)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 4 \end{bmatrix}$$

$$= 6 \begin{bmatrix} 1 \\ 0 \\ 0.6667 \end{bmatrix} = \lambda^{(1)} X^{(1)}$$

$$\therefore AX^{(1)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.6667 \end{bmatrix} = \begin{bmatrix} 7.3334 \\ -2.6667 \\ 4.0001 \end{bmatrix}$$

$$= 7.3334 \begin{bmatrix} 1 \\ -0.3636 \\ 0.5455 \end{bmatrix} = \lambda^{(2)} X^{(2)}$$

$$\therefore AX^{(2)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -0.3636 \\ 0.5455 \end{bmatrix} = \begin{bmatrix} 7.8182 \\ -3.6363 \\ 4.0001 \end{bmatrix}$$

$$= 7.8182 \begin{bmatrix} 1 \\ -0.4651 \\ 0.5116 \end{bmatrix} = \lambda^{(3)} X^{(3)}$$

$$\therefore AX^{(3)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -0.4651 \\ 0.5116 \end{bmatrix} = \begin{bmatrix} 7.9534 \\ -0.4651 \\ 0.5116 \end{bmatrix}$$

$$= 7.9534 \begin{bmatrix} 1 \\ -0.4912 \\ 0.5029 \end{bmatrix} = \lambda^{(4)} X^{(4)}$$

$$\therefore AX^{(4)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -0.4912 \\ 0.5024 \end{bmatrix} = \begin{bmatrix} 7.9882 \\ -3.9765 \\ 3.4999 \end{bmatrix}$$

$$= 7.9882 \begin{bmatrix} 1 \\ -0.4978 \\ 0.5007 \end{bmatrix}$$

$\therefore$ The largest eigen value is $7.9882 \sim 8$
 and eigen vector is $\begin{bmatrix} 1 \\ -0.4978 \\ 0.5007 \end{bmatrix} \sim \begin{bmatrix} 1 \\ -0.5 \\ 0.5 \end{bmatrix}$

Additional Problems:-

1. For What values of λ and μ , the system of equations $x+y+z=6$, $x+2y+3z=10$, $x+2y+\lambda z=\mu$ has

- i. No Solution
- ii. Unique Solution
- iii. Infinite Solution

$$\Rightarrow \begin{aligned} x+y+z &= 6 & \text{---(1)} \\ x+2y+3z &= 10 & \text{---(2)} \\ x+2y+\lambda z &= \mu & \text{---(3)} \end{aligned}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

$$AX=B$$

where,

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

$$\begin{aligned} [A:B] &= \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 1 & 2 & 3 & : & 10 \\ 1 & 2 & \lambda & : & \mu \end{bmatrix} & R_2 : R_2 - R_1 \\ &\sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 1 & \lambda-1 & : & \mu-6 \end{bmatrix} & R_3 : R_3 - R_2 \\ & & R_3 : R_3 - R_2 \end{aligned}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 0 & (\lambda-3) & : & (\mu-10) \end{bmatrix}$$

- i. The system of equation may have unique solution only at $\lambda \neq 3$ and any value of μ .
- ii. The system of equation may have infinite number of solution at $\lambda = 3$ and $\mu = 10$
- iii. The system of equation may have no solution only at $\lambda = 3$ and $\mu \neq 10$.

2. Solve the following system of equations by using Gauss-elimination methods $3x + y + 2z = 3$, $2x - 3y - z = -3$ and $x + 2y + z = 4$

$$\Rightarrow \begin{aligned} 3x + y + 2z &= 3 \quad \text{---(1)} \\ 2x - 3y - z &= -3 \quad \text{---(2)} \\ x + 2y + z &= 4 \quad \text{---(3)} \end{aligned}$$

$$\Rightarrow \begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix}$$

$$AX = B$$

$$[A:B] \Rightarrow \begin{bmatrix} 3 & 1 & 2 & : & 3 \\ 2 & -3 & -1 & : & -3 \\ 1 & 2 & 1 & : & 4 \end{bmatrix}$$

$$R_1 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & : & 4 \\ 2 & -3 & -1 & : & -3 \\ 3 & 1 & 2 & : & 3 \end{bmatrix}$$

$$R_2: R_2 - 2R_1$$

$$R_3: R_3 - 3R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & : & 4 \\ 0 & -7 & -3 & : & -11 \\ 0 & -5 & -1 & : & -9 \end{bmatrix}$$

$$R_2: (-5)R_2$$

$$R_3: (-7)R_3$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & : & 4 \\ 0 & 35 & 15 & : & 55 \\ 0 & 35 & 7 & : & 63 \end{bmatrix}$$

$$R_3: R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & : & 4 \\ 0 & 35 & 15 & : & 55 \\ 0 & 0 & -8 & : & 8 \end{bmatrix}$$

$$\therefore \rho(A) = \rho(AB) = 3 = n$$

$\therefore$ The equations may have unique solutions

$$AX = B$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 35 & 15 \\ 0 & 0 & -8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 55 \\ 8 \end{bmatrix}$$

$$x + 2y + z = 4 \quad \text{--- (4)}$$

$$35y + 15z = 55 \quad \text{--- (5)}$$

$$7y + 3z = 11 \quad \text{--- (5)}$$

$$-8z = 8 \quad \text{--- (6)}$$

$$\boxed{z = -1}$$

$$\begin{aligned}
 (5) \Rightarrow 7y + 3(1) &= 11 \\
 7y &= 11 - 3 \\
 7y &= 8 \\
 \boxed{y &= 2}
 \end{aligned}$$

$$\begin{aligned}
 (6) \Rightarrow x + 2(2) + (-1) &= 4 \\
 x + 4 - 1 &= 4 \\
 \boxed{x &= 1}
 \end{aligned}$$

$\therefore$ The solution is $x=1, y=2, z=-1$

3. Solve the system of equations by Gauss-Jordan method.

$$x + y + z = 9, \quad x - 2y + 3z = 8, \quad 2x + y - z = 3$$

$$\begin{aligned}
 \Rightarrow x + y + z &= 9 \quad \text{--- (1)} \\
 x - 2y + 3z &= 8 \quad \text{--- (2)} \\
 2x + y - z &= 3 \quad \text{--- (3)}
 \end{aligned}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & 3 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ 3 \end{bmatrix}$$

$$AX = B$$

$$[A:B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 1 & -2 & 3 & 8 \\ 2 & 1 & -1 & 3 \end{array} \right]$$

$$R_2: -R_2 - R_1$$

$$R_3: R_3 - 2R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -3 & 2 & -1 \\ 0 & -1 & -3 & -15 \end{array} \right]$$

$$R_1: 3R_1 + R_2$$

$$R_3: 3R_2 - R_2 \Rightarrow R_3: -\frac{1}{11} R_3$$

$$\sim \left[\begin{array}{ccc|c} 3 & 0 & 5 & 26 \\ 0 & -3 & 2 & -1 \\ 0 & 0 & -11 & -44 \end{array} \right]$$

$$R_1: -R_1 - 5R_3$$

$$R_2: R_2 - 2R_3$$

$$\sim \begin{bmatrix} 3 & 0 & 0 & : & 6 \\ 0 & -3 & 0 & : & -9 \\ 0 & 0 & 1 & : & 4 \end{bmatrix}$$

$$R_1: \frac{1}{3} R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & : & 2 \\ 0 & 1 & 0 & : & 3 \\ 0 & 0 & 1 & : & 4 \end{bmatrix} \quad R_2: -\frac{1}{3} R_2$$

$$P(A) = P(AB) = 3 = n$$

$$AX = B$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

$$\boxed{x=2}, \boxed{y=3}, \boxed{z=4}$$

4. Solve the system of equations by Gauss - Elimination method.

$$x + 2y + z = 3, \quad 3x + 2y + z = 3, \quad x - 2y - 5z = 1$$

$$\Rightarrow x + 2y + z = 3 \quad \text{---(1)}$$

$$3x + 2y + z = 3 \quad \text{---(2)}$$

$$x - 2y - 5z = 1 \quad \text{---(3)}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 1 \\ 1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix} \Rightarrow AX = B$$

$$[A:B] = \begin{bmatrix} 1 & 2 & 1 & : & 3 \\ 3 & 2 & 1 & : & 3 \\ 1 & -2 & -5 & : & 1 \end{bmatrix}$$

$$R_2 : R_2 - 3R_1$$

$$R_3 : R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & : & 3 \\ 0 & -4 & -2 & : & -6 \\ 0 & -4 & -6 & : & -2 \end{bmatrix}$$

$$R_2 : (-\frac{1}{2})R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & : & 3 \\ 0 & 2 & 1 & : & 3 \\ 0 & 2 & 3 & : & 1 \end{bmatrix}$$

$$R_3 : (-\frac{1}{2})R_3$$

$$R_3 : R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & : & 3 \\ 0 & 2 & 1 & : & 3 \\ 0 & 0 & 2 & : & -2 \end{bmatrix}$$

$$\rho(A) = \rho(AB) = 3 = n$$

$\therefore$ The given equations are unique solutions;

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -2 \end{bmatrix}$$

$$AX = B$$

$$x + 2y + z = 3 \quad \text{---(1)}$$

$$2y + z = 3 \quad \text{---(2)}$$

$$2z = -2 \quad \text{---(3)}$$

$$\boxed{z = -1}$$

$$(2) \Rightarrow 2y + (-1) = 3$$

$$2y - 1 = 3$$

$$2y = 4$$

$$\boxed{y = 2}$$

$$(3) \Rightarrow x + 2y + z = 3$$

$$x + 2(2) - 1 = 3$$

$$x + 4 - 1 = 3$$

$$x = 3 - 3$$

$$\boxed{x = 0}$$

5. $83x + 11y - 4z = 95$, $3x + 8y + 29z = 71$, $7x + 52y + 13z = 104$ using Gauss-Seidel taking $(0,0,0)$ as initial approximation.

$$\Rightarrow 83x + 11y - 4z = 95$$

$$3x + 8y + 29z = 71$$

$$7x + 52y + 13z = 104$$

The given equations are not in diagonally dominant form. By reordering the given equations we have,

$$83x + 11y - 4z = 95 \text{ --- (1)}$$

$$7x + 52y + 13z = 104 \text{ --- (2)}$$

$$3x + 8y + 29z = 71 \text{ --- (3)}$$

$$(1) \Rightarrow x = \frac{1}{83} [95 - 11y + 4z]$$

$$\Rightarrow y = \frac{1}{52} [104 - 7x - 13z]$$

$$\Rightarrow z = \frac{1}{29} [71 - 3x - 8y]$$

$$I-(1) \quad x^{(1)} = \frac{95}{83} = 1.1446$$

$$y^{(1)} = \frac{1}{52} [104 - 7(1.1446) - 13(0)] = 1.8459$$

$$z^{(1)} = \frac{1}{29} [71 - 3(1.1446) - 8(1.8459)] = 1.8206$$

$T = 0$

$$x^{(0)} = \frac{1}{83} [96 - 11(1.8459) + 4(1.8006)] = 0.9846$$

$$y^{(0)} = \frac{1}{58} [104 - 7(0.9846) - 12(1.8006)] = 1.4119$$

$$z^{(0)} = \frac{1}{89} [71 - 3(0.9846) - 8(1.4119)] = 1.9566$$

$T = 1$

$$x^{(1)} = \frac{1}{83} [96 - 11(1.4119) + 4(1.9566)] = 1.0515$$

$$y^{(1)} = \frac{1}{58} [104 - 7(1.0515) - 12(1.9566)] = 1.2692$$

$$z^{(1)} = \frac{1}{89} [71 - 3(1.0515) - 8(1.2692)] = 1.9614$$

$T = 2$

$$x^{(2)} = \frac{1}{83} [96 - 11(1.2692) + 4(1.9614)] = 1.0546$$

$$y^{(2)} = \frac{1}{58} [104 - 7(1.0546) - 12(1.9614)] = 1.2642$$

$$z^{(2)} = \frac{1}{89} [71 - 3(1.0546) - 8(1.2642)] = 1.9614$$

The solution is $\therefore x = 1.0546$

$$y = 1.2642$$

$$z = 1.9614$$

6. $27x + 6y - z = 85$, $6x + 15y + 2z = 72$, $x + y + 54z = 110$, Using Gauss - Seidel taking $(0,0,0)$ as initial approximation.

$$\Rightarrow 27x + 6y - z = 85 \text{ --- (1)}$$

$$6x + 15y + 2z = 72 \text{ --- (2)}$$

$$x + y + 54z = 110 \text{ --- (3)}$$

$$(1) \Rightarrow x = \frac{1}{27} [85 - 6y + z]$$

$$(2) \Rightarrow y = \frac{1}{15} [72 - 6x - 2z]$$

$$(3) \Rightarrow z = \frac{1}{54} [110 - x - y]$$

I - (1)

$$x^{(1)} = \frac{1}{27} [85 - 6(0) + 0] = 3.1481$$

$$y^{(1)} = \frac{1}{15} [72 - 6(3.1481) - 2(0)] = 3.5407$$

$$z^{(1)} = \frac{1}{54} [110 - 3.1481 - 3.5407] = 1.9131$$

II - (2)

$$x^{(2)} = \frac{1}{27} [85 - 6(3.5407) + 1.9131] = 2.4321$$

$$y^{(2)} = \frac{1}{15} [72 - 6(2.4321) - 2(1.9131)] = 3.5720$$

$$z^{(2)} = \frac{1}{54} [110 - 2.4321 - 3.5720] = 1.92585$$

I = (3)

$$x^{(3)} = \frac{1}{27} [85 - 6(3.5720) + 1.9258] = 2.4256$$

$$y^{(3)} = \frac{1}{15} [72 - 6(2.4256) - 2(1.9258)] = 3.5729$$

$$z^{(3)} = \frac{1}{54} [110 - 2.4256 - 3.5729] = 1.9259$$

I = (4)

$$x^{(4)} = \frac{1}{27} [85 - 6(3.5729) + 1.9259] = 2.4255$$

$$y^{(4)} = \frac{1}{15} [72 - 6(2.4255) - 2(1.9258)] = 3.5730$$

$$z^{(4)} = \frac{1}{54} [110 - 2.4255 - 3.5730] = 1.9259$$

∴ The solution are $x = 2.4255$

$$y = 3.5730$$

$$z = 1.9259$$