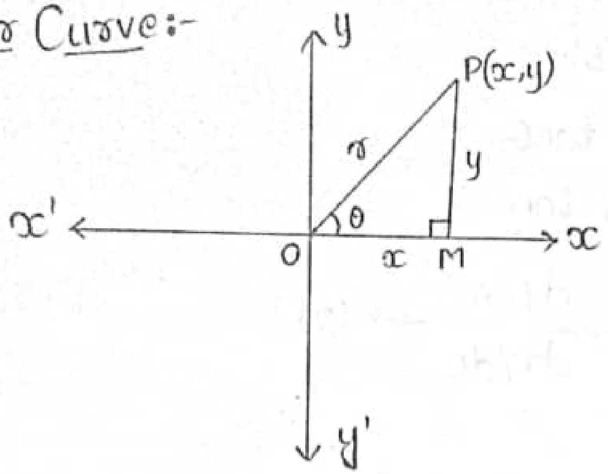
MODULE-1

Module - 1

Polar Curve:-



$$-\sin\theta = \frac{PM}{OP} = \frac{y}{r}$$

$$\Rightarrow y = r\sin\theta$$

$$-\cos\theta = \frac{OM}{OP} = \frac{x}{r}$$

$$\Rightarrow x = r\cos\theta$$

$$\Rightarrow r = \sqrt{x^2+y^2}$$

$$\Rightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

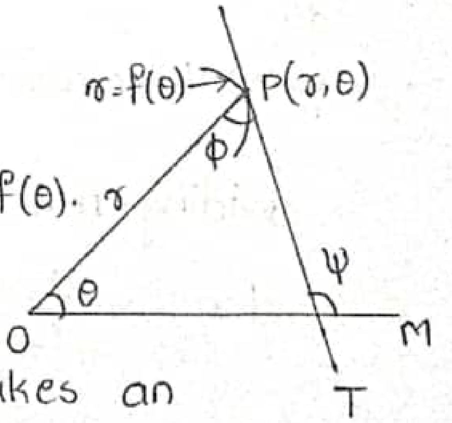
Let $P(x,y)$ be any point in xy plane such that $OP = r$ and θ be the angle between X-axis and OP. Draw PM perpendicular to X-axis, the P has a new coordinates (r,θ) is called polar coordinates.

Here, O is called pole.

OM is initial line.

OP is radius vector.

$\Rightarrow$ P.T with usual notations $\tan\phi = r.\frac{d\theta}{dr}$, (or) Find angle between radius vector and tangent.



Let O be the pole and OM be initial line.

Let $p(r,\theta)$ be any point on polar curve $r = f(\theta)$.

Let $OP = r$ be the radius vector, draw a tangent PT at the point P and which makes an angle ψ with the initial line.

Let θ be the angle between radius vector and initial line.

ϕ be the angle between radius vector and tangent.

Here we have, $\psi = \phi + \theta$

Applying tan on both sides.

$$\tan\psi = \tan(\phi+\theta)$$

$$\tan\psi = \frac{\tan\phi + \tan\theta}{1-\tan\phi.\tan\theta} \rightarrow (1)$$

Slope is given by $\tan\psi = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} \rightarrow (2)$

WKT, $x = r\cos\theta$, $y = r\sin\theta$

differentiate with respect to θ

$$\frac{dx}{d\theta} = r(-\sin\theta) + \cos\theta\frac{dr}{d\theta} \qquad \left(\text{Let } \frac{dr}{d\theta} = r'\right)$$

$$\Rightarrow \frac{dx}{d\theta} = r'\cos\theta - r\sin\theta$$

$y = r\sin\theta$

d.w.r.t θ

$$\Rightarrow \frac{dy}{d\theta} = r\cos\theta + \sin\theta\frac{dr}{d\theta}$$

$$\frac{dy}{d\theta} = r\cos\theta + r'\sin\theta$$

$\therefore$ eq-(2) becomes $\tan\psi = \frac{r\cos\theta + r'\sin\theta}{r'\cos\theta - r\sin\theta}$

dividing num. & deno with $r'\cos\theta$

$$\tan\psi = \frac{\frac{r}{r'} + \tan\theta}{1 - \frac{r}{r'}\tan\theta} \rightarrow (3)$$

From eqn. (1) & (3) LHS are same

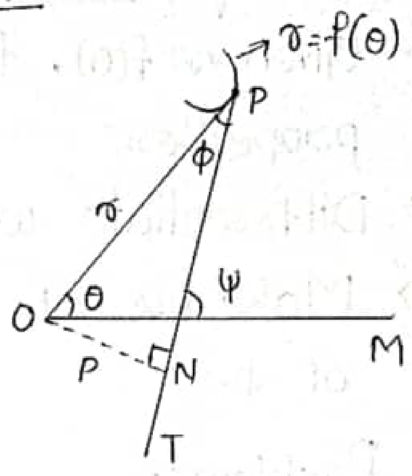
$$\therefore \tan\phi = \frac{r}{r'} = r.\frac{d\theta}{dr} \qquad \Rightarrow \cot\phi = \frac{1}{r}.\frac{dr}{d\theta}$$

Length of perpendicular from pole to tangent :-

Let O be the pole and OM be initial line,

$P(r,\theta)$ be any point on polar curve $r = f(\theta)$.

Draw a tangent PT at point P, draw ON perpendicular to the tangent from pole.



Let $OP = r$, $ON = p$

$$\sin\phi = \frac{ON}{OP} = \frac{p}{r}$$

$$p = r\sin\phi$$

$$\frac{1}{p} = \frac{1}{r\sin\phi}$$

$$\frac{1}{p} = \frac{1}{r}\operatorname{cosec}\phi$$

squaring on both sides

$$\frac{1}{p^2} = \frac{1}{r^2}\operatorname{cosec}^2\phi$$

$$\frac{1}{p^2} = \frac{1}{r^2}\left(1+\cot^2\phi\right)$$

$$= \frac{1}{r^2}\left(1+\left(\frac{1}{r}\cdot\frac{dr}{d\theta}\right)^2\right)$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

Note:-

Given $r = f(\theta) \rightarrow (1)$

W.K.T length of perpendicular from pole to tangent, $p = r\sin\phi \rightarrow (2)$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

If it possible to eliminate θ from eq-(1) and (2) or from eq-(1) & (3) then resulting equation contains p and r. such equation is called "Pedal Equation".

Working procedure to find 'ϕ' :-

1) Given $r = f(\theta)$, take log on both sides and apply logarithm properties.

2) Differentiate with respect to θ.

3) Make use of allied angles if require to get the value of ϕ.

Problems:-

1. Find the angle between radius vector and tangent, given that $r = a(1-\cos\theta)$.

Sol:- $r = a(1-\cos\theta)$

Take log o.b.s

$$\log r = \log(a(1-\cos\theta))$$

$$\log r = \log a + \log(1-\cos\theta)$$

d.w.r.t θ.

$$\frac{1}{r}\cdot\frac{dr}{d\theta} = 0 + \frac{0-(-\sin\theta)}{1-\cos\theta}$$

$$\cot\phi = \frac{\sin\theta}{1-\cos\theta}$$

$$\cot\phi = \frac{\cancel{2}\sin\theta/2\cdot\cos\theta/2}{\cancel{2}\sin^{\cancel{2}}\theta/2}$$

$$\cot\phi = \cot\theta/2$$

$$\boxed{\phi = \theta/2}$$

2. $r = a(1+\cos\theta)$

Take log o.b.s

$$\log r = \log a + \log(1+\cos\theta)$$

d.w.r.t θ

$$\frac{1}{r}\cdot\frac{dr}{d\theta} = 0 + \frac{0+(-\sin\theta)}{1+\cos\theta}$$

$$\cot\phi = \frac{-\sin\theta}{1+\cos\theta}$$

$$= \frac{-\cancel{2}\sin\theta/2\cdot\cancel{\cos\theta/2}}{\cancel{2}\cos^{\cancel{2}}\theta/2}$$

$$\cot\phi = -\tan\theta/2$$

$$\cot\phi = \cot\left(\pi/2+\theta/2\right)$$

$$\boxed{\phi = \frac{\pi}{2}+\theta/2}$$

3. $r^2\cos 2\theta = a^2$

Take log o.b.s

$$\log(r^2\cos 2\theta) = \log a^2$$

$$2\log r + \log\cos 2\theta = 2\log a$$

$$\frac{2}{r}\cdot\frac{dr}{d\theta} + \frac{0+(-\sin 2\theta)2}{\cos 2\theta} = 0$$

$$\cancel{2}\cot\phi = \cancel{2}\tan 2\theta$$

$$\cot\phi = \tan 2\theta$$

$$\cot\phi = \cot\left(\frac{\pi}{2}-2\theta\right)$$

$$\boxed{\phi = \frac{\pi}{2}-2\theta}$$

4. $r = a\sin^3(\theta/3)$

Take log o.b.s

$$\log r = \log a + 3\log\sin(\theta/3)$$

d.w.r.t θ

$$\frac{1}{r}\cdot\frac{dr}{d\theta} = 0 + \frac{\cancel{3}\cdot\cos(\theta/3)\cdot\frac{1}{\cancel{3}}}{\sin(\theta/3)}$$

$$\cot\phi = \cot(\theta/3)$$

$$\boxed{\phi = \theta/3}$$

5. $\frac{2a}{r} = 1-\cos\theta$

Take log o.b.s

$$\log\left(\frac{2a}{r}\right) = \log(1-\cos\theta)$$

$$\log 2a - \log r = \log(1-\cos\theta)$$

d.w.r.t θ

$$0 - \frac{1}{r}\cdot\frac{dr}{d\theta} = \frac{0-(-\sin\theta)}{1-\cos\theta}$$

$$-\cot\phi = \frac{\cancel{2}\cancel{\sin}\theta/2\cdot\cos\theta/2}{\cancel{2}\sin^{\cancel{2}}\theta/2}$$

$$-\cot\phi = \cot\theta/2$$

$$\boxed{-\phi = \theta/2}$$

6. $r^m = a^m(\cos m\theta + \sin m\theta) \rightarrow (1)$ Find the pedal equation of the curve.

Take log o.b.s

$$\log r^m = \log a^m + \log(\cos m\theta + \sin m\theta)$$

$$m\log r = m\log a + \log(\cos m\theta + \sin m\theta)$$

d.w.r.t. θ

$$\cancel{m}.\frac{1}{r}.\frac{dr}{d\theta} = 0 + \frac{(-\sin m\theta + \cos m\theta).\cancel{m}}{(\cos m\theta + \sin m\theta)}$$

$$\cot\phi = \frac{\cos m\theta - \sin m\theta}{\cos m\theta + \sin m\theta}$$

$\div \cos m\theta$ in num. & deno.

$$\cot\phi = \frac{1 - \tan m\theta}{1 + \tan m\theta}$$

$$\cot\phi = \tan\left(\frac{\pi}{4} - m\theta\right)$$

$$\cot\phi = \cot\left(\frac{\pi}{2} - \left(\frac{\pi}{4} - m\theta\right)\right)$$

$$\cot\phi = \cot\left(\frac{\pi}{4} + m\theta\right)$$

$$\boxed{\phi = \frac{\pi}{4} + m\theta}$$

$$P = r\sin\phi$$

$$= r\sin\left(\frac{\pi}{4} + m\theta\right)$$

$$= r\left(\sin\frac{\pi}{4}.\cos m\theta + \cos\frac{\pi}{4}.\sin m\theta\right)$$

$$= r\left(\frac{1}{\sqrt{2}}\cos m\theta + \frac{1}{\sqrt{2}}\sin m\theta\right)$$

$$P = \frac{r}{\sqrt{2}}(\cos m\theta + \sin m\theta)$$

$$\cos m\theta + \sin m\theta = \frac{P\sqrt{2}}{r} \rightarrow (2)$$

from eq-(1), $r^m = a^m(\cos m\theta + \sin m\theta)$

$$r^m = a^m \cdot \frac{p\sqrt{2}}{r}$$

$$p = \frac{r^{m+1}}{a^m\sqrt{2}}$$

$$\boxed{p \propto r^{m+1}}$$

7. Find the pedal equation of the curve $\frac{2a}{r} = 1+\cos\theta$

Sol:- $\frac{2a}{r} = 1+\cos\theta \longrightarrow (1)$

Take log o.b.s

$$\log 2a - \log r = \log(1+\cos\theta)$$

d.w.r.t θ.

$$0 - \frac{1}{r}\cdot\frac{dr}{d\theta} = \frac{0+(-\sin\theta)}{1+\cos\theta}$$

$$\cancel{-}\cot\phi = \cancel{-}\frac{\sin\theta}{1+\cos\theta}$$

$$= \frac{\cancel{2}\sin\theta/2\cos\theta/2}{\cancel{2}\cos^{\cancel{2}}\theta/2}$$

$$\cot\phi = \tan\theta/2$$

$$\cot\phi = \cot\left(\frac{\pi}{2} - \theta/2\right)$$

$$\boxed{\phi = \frac{\pi}{2} - \theta/2}$$

WKT, $p = r\sin\phi$

$$= r\sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right)$$

$$p = r\cos(\theta/2) \longrightarrow (2)$$

From eq-(1) $\frac{2a}{r} = 1+\cos\theta.$

$$\frac{\cancel{2}a}{r} = \cancel{2}\cos^2\theta/2$$

$$\cos(\theta/2) = \sqrt{\frac{a}{r}}$$

eq-(2) becomes, $P = r \cdot \sqrt{\frac{a}{r}}$

$$P = \sqrt{r} \cdot \cancel{\sqrt{r}} \cdot \frac{\sqrt{a}}{\cancel{\sqrt{r}}}$$

$$P = \sqrt{a} \cdot \sqrt{r}$$

$$P \propto \sqrt{r} \quad (\because \sqrt{a} \text{ is constant})$$

8. Find the pedal equation of the curve $r^2 = a^2 \sec 2\theta$.

$r^2 = a^2 \sec 2\theta \longrightarrow (1)$

Take log o.b.s

$2\log r = 2\log a + \log \sec 2\theta$

d.w.r.t. θ

$$\cancel{2} \cdot \frac{1}{r} \cdot \frac{dr}{d\theta} = 0 + \frac{\cancel{2}(\cancel{\sec 2\theta} \cdot \tan 2\theta)}{\cancel{\sec 2\theta}}$$

$\cot\phi = \tan 2\theta$

$\cot\phi = \cot\left(\frac{\pi}{2} - 2\theta\right)$

$$\boxed{\phi = \frac{\pi}{2} - 2\theta}$$

WKT, $P = r\sin\phi$

$= r\sin\left(\frac{\pi}{2} - 2\theta\right)$

$P = r\cos 2\theta \longrightarrow (2)$

From eq-(1), $r^2 = a^2 \sec 2\theta$

$$r^2 = a^2 \frac{1}{\cos 2\theta}$$

$$\cos 2\theta = \frac{a^2}{r^2}$$

eq-(2) becomes, $P = \cancel{r} \cdot \frac{a^2}{r^{\cancel{2}}}$

$$P \propto \frac{1}{r}$$

9. Find the pedal equation $r^n = a^n \cos n\theta$

$$r^n = a^n \cos n\theta \longrightarrow (1)$$

Take log o.b.s

$$n \log r = n \log a + \log \cos n\theta$$

d.w.r.t. θ

$$\cancel{n}.\frac{1}{r}.\frac{dr}{d\theta} = 0 + \frac{-\sin n\theta . \cancel{n}}{\cos n\theta}$$

$$\cot\phi = -\tan n\theta$$

$$\cot\phi = \cot\left(\frac{\pi}{2} + n\theta\right)$$

$$\phi = \frac{\pi}{2} + n\theta$$

WKT,
$$P = r\sin\phi$$
$$= r\sin\left(\frac{\pi}{2} + n\theta\right)$$

$$P = r\cos n\theta \rightarrow (2)$$

From eq-(1) $r^n = a^n \cos n\theta$

$$\cos n\theta = \frac{r^n}{a^n}$$

eq-(2) becomes,

$$P = r.\frac{r^n}{a^n}$$

$$P = \frac{r^{n+1}}{a^n} \Rightarrow \boxed{P \propto r^{n+1}}$$

10. P.T pedal equation of the curve $r^n = a^n \cos n\theta + b^n \sin n\theta$ in the form of $p^2(a^{2n} + b^{2n}) = r^{2n+2}$.

Sol:- $r^n = a^n \cos n\theta + b^n \sin n\theta$

Take log o.b.s

$$n \log r = \log(a^n \cos n\theta + b^n \sin n\theta)$$

d.w.r.t. θ

$$n.\frac{1}{r}.\frac{dr}{d\theta} = \frac{a^n(-\sin n\theta).n + b^n(\cos n\theta)n}{a^n \cos n\theta + b^n \sin n\theta}$$

$$\cancel{r}\cot\phi = \frac{\cancel{r}\left(b^n\cos n\theta - a^n\sin n\theta\right)}{a^n\cos n\theta + b^n\sin n\theta}$$

$$\cot\phi = \frac{b^n\cos n\theta - a^n\sin n\theta}{a^n\cos n\theta + b^n\sin n\theta}$$

WKT, $$\frac{1}{p^2} = \frac{1}{r^2}\left(1 + \left(\frac{1}{r}\cdot\frac{dr}{d\theta}\right)^2\right)$$

$$= \frac{1}{r^2}\left(1+\cot^2\phi\right)$$

$$\frac{1}{p^2} = \frac{1}{r^2}\left[1 + \left(\frac{b^n\cos n\theta - a^n\sin n\theta}{a^n\cos n\theta + b^n\sin n\theta}\right)^2\right]$$

$$= \frac{1}{r^2}\left[1 + \frac{\left(b^n\cos n\theta - a^n\sin n\theta\right)^2}{\left(a^n\cos n\theta + b^n\sin n\theta\right)^2}\right]$$

$$= \frac{1}{r^2}\left[\frac{\left(a^n\cos n\theta + b^n\sin n\theta\right)^2 + \left(b^n\cos n\theta - a^n\sin n\theta\right)^2}{\left(a^n\cos n\theta + b^n\sin n\theta\right)^2}\right]$$

$$= \frac{1}{r^2}\cdot\frac{1}{r^n}\left[\begin{aligned}&a^{2n}\cos^2 n\theta + b^{2n}\sin^2 n\theta + 2a^n\cos n\theta . b^n\sin n\theta + \\ &b^{2n}\cos^2 n\theta + a^{2n}\sin^2 n\theta - 2a^n\sin n\theta\, b^n\cos n\theta\end{aligned}\right]$$

→ Show that the curve $r\sin^2(\theta/2)=a$, the length of the perpendicular from pole to tangent at the point $(2a, \pi/2)$ is equal to $a\sqrt{2}$.

Sol:- $r\sin^2(\theta/2)=a$

$\log r + 2\log\sin(\theta/2) = \log a$

d.w.r.t. θ

$$\frac{1}{r}\cdot\frac{dr}{d\theta} + \frac{\cancel{2}(\cos\theta/2)(1/\cancel{2})}{\sin\theta/2} = 0$$

$\cot\phi + \cot\theta/2 = 0$

$\cot\phi = -\cot\theta/2$

$\phi = -\theta/2$

$P = r\sin\phi$

$P = r\sin(-\theta/2)$

$P = 2a\sin\left(\frac{-\pi/2}{2}\right)$

$P = -2a\sin\left(\frac{\pi}{4}\right)$

$= -2a\frac{1}{\sqrt{2}} \Rightarrow P = -a\sqrt{2}.\cancel{\sqrt{2}}\cdot\frac{1}{\cancel{\sqrt{2}}}$

$|P| = |-a\sqrt{2}|$

$P = a\sqrt{2}$

→ Show that the length of cardiodes $r=a(1+\cos\theta)$ at the points $\theta=\frac{\pi}{3}$, $\theta=\frac{2\pi}{3}$ are respectively parallel & perpendicular to the initial line.

Sol:- $r=a(1+\cos\theta)$

$\log r = \log a + \log(1+\cos\theta)$

d.w.r.t θ.

$$\frac{1}{r}\cdot\frac{dr}{d\theta} = \frac{(-\sin\theta)}{1+\cos\theta} = \frac{-\cancel{2}\sin\theta/2\cdot\cancel{\cos\theta/2}}{\cancel{2}\cos^{\cancel{2}}\theta/2} = \cot\phi = -\tan(\theta/2)$$

$\cot\phi = \cot\left(\frac{\pi}{2}+\frac{\theta}{2}\right)$

$\phi = \frac{\pi}{2}+\frac{\theta}{2}$

→ When $\theta = \frac{\pi}{3}$,

$$\phi = \frac{\pi}{2} + \frac{\pi}{6} \Rightarrow \phi = \frac{2\pi}{3}$$

→ at $\theta = \frac{2\pi}{3}$

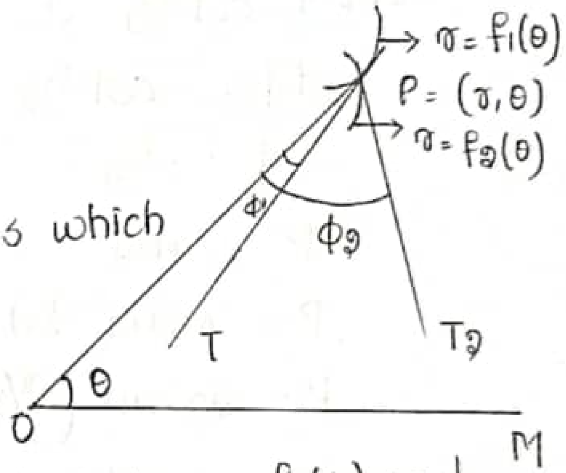
$$\phi = \frac{2\pi}{2} + \frac{1}{2} \cdot \frac{2\pi}{3} = \frac{5\pi}{3}$$

$$\psi = \theta + \phi$$

at $\theta = \frac{\pi}{3} \Rightarrow \psi = \frac{\pi}{3} + \frac{2\pi}{3} = \frac{\cancel{3}\pi}{\cancel{3}} = \pi$

$$\tan\psi = \tan\pi$$

$$\tan\psi = 0$$



<u>Angle between two polar curves</u>:-

Let $r = f_1(\theta)$, $r = f_2(\theta)$ be two polar curves which intersects at point P.

Let OM be initial line and OP be radius vector and 'O' be the pole.

Draw a tangent PT_1 & PT_2 at the point P to $r = f_1(\theta)$ and $r = f_2(\theta)$ respectively.

∴ Angle between radius vector and PT_1 is ϕ_1.

Angle between radius vector and PT_2 is ϕ_2.

∴ Acute angle between the tangents PT_1 and PT_2 is given by $|\phi_2 - \phi_1|$.

<u>Note</u>:- $r = f_1(\theta)$ & $r = f_2(\theta)$ are said to be orthogonal if

$$\tan\phi_1 \cdot \tan\phi_2 = 1.$$

1. Find the angle between the curves $r = 2\sin\theta$, $r = \sin\theta + \cos\theta$?

Sol:- $r = 2\sin\theta$

Take log

$\log r = \log(2\sin\theta)$

$\log r = \log 2 + \log\sin\theta$

d.w.r.t. θ

$\frac{1}{r}\cdot\frac{dr}{d\theta} = 0 + \frac{\cos\theta}{\sin\theta}$

$\cot\phi = \cot\theta$

$\boxed{\phi_1 = \theta}$

$r = \sin\theta + \cos\theta$

$\log r = \log(\sin\theta + \cos\theta)$

d.w.r.t. θ

$\frac{1}{r}\cdot\frac{dr}{d\theta} = \frac{\cos\theta - \sin\theta}{\sin\theta + \cos\theta}$

$\cot\phi = \frac{1-\tan\theta}{1+\tan\theta}$

$\cot\phi = \tan\left(\frac{\pi}{4} - \theta\right)$

$\cot\phi = \cot\left(\frac{\pi}{2} - \left(\frac{\pi}{4} - \theta\right)\right)$

$\cot\phi = \cot\left(\frac{\pi}{4} + \theta\right)$

$\boxed{\phi_2 = \frac{\pi}{4} + \theta}$

$\therefore$ Angle between two polar curves

$= |\phi_2 - \phi_1| = \frac{\pi}{4} + \cancel{\theta} - \cancel{\theta} = \frac{\pi}{4}$

2. PT angle between polar curves is $\frac{\pi}{2} + \frac{1}{2}\cos^{-1}\left(\frac{1}{3}\right)$ given $r = a(1-\cos\theta)$ & $r = 2a\cos\theta$.

Sol:- $r = a(1-\cos\theta)$

$\log r = \log a + \log(1-\cos\theta)$

$\frac{1}{r}\cdot\frac{dr}{d\theta} = \frac{\sin\theta}{1-\cos\theta}$

$\cot\phi = \frac{\cancel{2}\sin\theta/2\cdot\cos\theta/2}{\cancel{2}\sin^{\cancel{2}}\theta/2}$

$\cot\phi = \cot\theta/2$

$\boxed{\phi_1 = \theta/2}$

$r = 2a\cos\theta$

$\log r = \log 2a + \log\cos\theta$

$\frac{1}{r}\cdot\frac{dr}{d\theta} = \frac{-\sin\theta}{\cos\theta}$

$\cot\phi = -\tan\theta$

$\cot\phi = \cot\left(\frac{\pi}{2} + \theta\right)$

$\boxed{\phi_2 = \frac{\pi}{2} + \theta}$

$|\phi_2 - \phi_1| = \left|\frac{\pi}{2} + \theta - \frac{\theta}{2}\right|$

$= \frac{\pi}{2} + \frac{\theta}{2} \longrightarrow (1)$

Given $r = a(1-\cos\theta)$, $r = 2a\cos\theta$

since, LHS are equal

$\therefore\ 2\cancel{a}\cos\theta = \cancel{a}(1-\cos\theta)$

$2\cos\theta = 1-\cos\theta$

$2\cos\theta + \cos\theta = 1$

$3\cos\theta = 1$

$\theta = \cos^{-1}\left(\frac{1}{3}\right)$

$\therefore$ (1) becomes $|\phi_2 - \phi_1| = \frac{\pi}{2} + \frac{1}{2}\cos^{-1}\left(\frac{1}{3}\right)$

3. $r^n = a^n\cos n\theta$, $r^n = b^n\sin n\theta$

Sol:- $r^n = a^n\cos n\theta$

$n\log r = n\log a + \log\cos n\theta$

$\cancel{n}\cdot\frac{1}{r}\cdot\frac{dr}{d\theta} = \frac{\cancel{n}(-\sin n\theta)}{\cos n\theta}$

$\cot\phi = -\tan n\theta$

$\cot\phi = \cot\left(\frac{\pi}{2}+n\theta\right)$

$\boxed{\phi_1 = \frac{\pi}{2}+n\theta}$

$r^n = b^n\sin n\theta$

$n\log r = n\log b + \log\sin n\theta$

$\cancel{n}\cdot\frac{1}{r}\cdot\frac{dr}{d\theta} = \frac{\cancel{n}(\cos n\theta)}{\sin n\theta}$

$\cot\phi = \cot n\theta$

$\boxed{\phi_2 = n\theta}$

$|\phi_2 - \phi_1| = \left|n\theta - \left(\frac{\pi}{2}+n\theta\right)\right|$

$= \left|\cancel{n\theta} - \frac{\pi}{2} - \cancel{n\theta}\right| = \frac{\pi}{2}$

4. $r = 4\sec^2(\theta/2)$, $r = 9\operatorname{cosec}(\theta/2)$

Sol:- $r = 4\sec^2(\theta/2)$

$\log r = \log 4 + 2\log\sec(\theta/2)$

d.w.r.t. θ

$\frac{1}{r}\cdot\frac{dr}{d\theta} = \frac{\cancel{2}\left(\cancel{\sec}\left(\frac{\theta}{2}\right)\cdot\tan(\theta/2)\right)\frac{1}{\cancel{2}}}{\cancel{\sec}(\theta/2)}$

$\cot\phi = \tan\theta/2$

$\cot\phi = \cot\left(\frac{\pi}{2}-\theta/2\right) \Rightarrow \phi_1 = \frac{\pi}{2}-\theta/2$

$$r = 9\,\mathrm{cosec}^2(\theta/2)$$

$$\log r = \log 9 + 2\log \mathrm{cosec}(\theta/2)$$

d.w.r.t. θ

$$\frac{1}{r}\cdot\frac{dr}{d\theta} = \frac{\cancel{2}\left(-\cancel{\mathrm{cosec}}\,\theta/2\cdot\cot\theta/2\right)\frac{1}{\cancel{2}}}{\cancel{\mathrm{cosec}}(\theta/2)}$$

$$\cot\phi = -\cot\theta/2$$

$$\phi_2 = -\theta/2$$

$$|\phi_2-\phi_1| = \left[-\frac{\theta}{2} - \left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right|$$

$$= \left|-\cancel{\frac{\theta}{2}} - \frac{\pi}{2} + \cancel{\theta/2}\right| = \left|-\frac{\pi}{2}\right| = \frac{\pi}{2}$$

5. $r = a\log\theta$, $r = \dfrac{a}{\log\theta}$

Sol:- $r = a\log\theta$

$$\log r = \log a + \log(\log\theta)$$

d.w.r.t. θ

$$\frac{1}{r}\cdot\frac{dr}{d\theta} = 0 + \frac{1}{\log\theta}\times\frac{1}{\theta}$$

$$\cot\phi = \frac{1}{\theta\log\theta}$$

$$\tan\phi_1 = \theta\log\theta$$

$r = \dfrac{a}{\log\theta}$

$$\log r = \log a - \log(\log\theta)$$

d.w.r.t. θ

$$\frac{1}{r}\cdot\frac{dr}{d\theta} = 0 - \frac{1}{\log\theta}\times\frac{1}{\theta}$$

$$\cot\phi = -\frac{1}{\theta\log\theta}$$

$$\tan\phi_2 = -\theta\log\theta$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1+\tan A.\tan B}$$

$$\tan(\phi_1-\phi_2) = \frac{\tan\phi_1 - \tan\phi_2}{1+\tan\phi_1.\tan\phi_2} = \frac{\theta\log\theta - (-\theta\log\theta)}{1+\theta\log\theta.(-\theta\log\theta)}$$

$$= \frac{2\theta\log\theta}{1-(\theta\log\theta)^2} \rightarrow (1)$$

$r = a\log\theta \quad , \quad r = \frac{a}{\log\theta}$

$$\cancel{a}\log\theta = \frac{a}{\log\theta}$$

$$(\log\theta)^2 = 1 \Rightarrow \log\theta = 1$$

$$\boxed{\theta = e}$$

$$\tan(\phi_1 - \phi_2) = \frac{2 \cdot e\log_e^e}{1-(e\log_e e)^2} = \frac{2e}{1-e^2}$$

$$\phi_1 - \phi_2 = \tan^{-1}\left(\frac{2e}{1-e^2}\right)$$

Curvature and Radius of Curvature:-

Consider a curve on XOY plane. Let A be a fixed point on it and P & Q be any two point which are neighbouring points on the curve such that $AP = s$, $AQ = s + \delta s$.

Draw a tangent at the point 'P' which makes angle 'ψ' with X-axis.

Draw another tangent at the point 'Q' which makes an angle $\psi + \delta\psi$.

Let ψ & $\psi + \delta\psi$ be the angles made by the tangents drawn at the points P & Q with X-axis respectively.

The angle $\delta\psi$ between tangents is called the bending of the curve which depends on δs.

$\frac{\delta\psi}{\delta s}$ is called as mean curvature of the curve.

The amount of the bending of the curve at P is called curvature of curve at P. and is defined mathematically as

$$\lim_{\delta s \to 0} \frac{\delta\psi}{\delta s} = \frac{d\psi}{ds} = K \quad \text{where } K \text{ is constant.}$$

$\therefore$ Curvature of the curve is a measure of rate of change of bentness.

Radius of Curvature:-

It is denoted by 'ρ' and it is reciprocal of the curvature.

$$\rho = \frac{1}{k} \text{ if } k \neq 0.$$

Note:- Curvature of straight line is zero.
Curvature of circle is constant.

i) Radius of Curvature in Cartesian form:-

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

where, y_1 & y_2 are 1st order & 2nd order differentiation respectively.

Note:- If y_2 is zero at some point then the ρ is infinity. So, we can represent ρ radius of curvature in another form

$$\rho = \frac{(1+x_1^2)^{3/2}}{x_2} \text{ where } x_1 \text{ is } \frac{dx}{dy} \text{ \& } x_2 = \frac{d^2x}{dy^2}$$

1. Find the radius of curvature for the curve $y = e^x$ at the point where it crosses Y-axis.

Sol:- Given,

Curve crosses Y-axis,

$\therefore x = 0$

$y = e^x \rightarrow (1)$

$\therefore$ eq-(1) becomes, $y = e^0 = 1$

points $(x,y) = (0,1)$

Diff. eq-(1) w.r.t. x

$y_1 = e^x \rightarrow (2)$

$y_1 = e^0 \Rightarrow 1$

D. eq-(2) w.r.t. x

$y_2 = e^x$ at $(0,1)$

$y_2 = e^0 = 1$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{(1+1)^{3/2}}{1} = 2^{3/2} = 2.2^{1/2} = 2\sqrt{2}$$

$$\boxed{\rho = 2\sqrt{2}}$$

2. Find the radius of curvature for the curve $y = ax^2+bx+c$ at the point $x = \frac{1}{2a}(\sqrt{a^2-1} - b)$

Sol:- $y = ax^2+bx+c \rightarrow (1)$

eq-(1) d. w. r. to x

$y_1 = a.2x + b + 0$ at the point $x \rightarrow (2)$

$y_1 = 2\cancel{a}.\frac{1}{2\cancel{a}}(\sqrt{a^2-1} - b) + b$

$y_1 = \sqrt{a^2-1} - \cancel{b} + \cancel{b} \Rightarrow y_1 = \sqrt{a^2-1}$

d (2) w. r. to x

$y_2 = 2a(1) + 0$

$y_2 = 2a$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{(1+(\sqrt{a^2-1})^2)^{3/2}}{2a}$$

$$= \frac{(\cancel{1}+a^2-\cancel{1})^{3/2}}{2a} = \frac{a^3}{2a}$$

$$\boxed{\rho = \frac{a^2}{2}}$$

3. Find the radius of curvature for the curve $y = a\log\sec\left(\frac{x}{a}\right)$

Sol:- $y = a\log\sec\left(\frac{x}{a}\right)$

D. w. r. t x

$$y_1 = \cancel{a}.\frac{1}{\cancel{\sec\left(\frac{x}{a}\right)}}\cancel{\sec\left(\frac{x}{a}\right)}.\tan\left(\frac{x}{a}\right).\frac{1}{\cancel{a}}$$

$y_1 = \tan\left(\frac{x}{a}\right)$ d. w. r. t. x

$$y_2 = \sec^2\left(\frac{x}{a}\right)\frac{1}{a}$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{\left(1+\left(\tan^2\left(\frac{x}{a}\right)\right)\right)^{3/2}}{\frac{1}{a}\sec^2\left(\frac{x}{a}\right)} = \frac{a\left(\sec^2\left(\frac{x}{a}\right)\right)^{3/2}}{\sec^2(x/a)}$$

$$= \frac{a\sec^3(x/a)}{\sec^2(x/a)}$$

$$\boxed{\rho = a\sec(x/a)}$$

4. $y = c\cosh\left(\frac{x}{c}\right)$

d.w.r.t x

$$y_1 = \cancel{c}\cdot\left(\sinh\left(\frac{x}{c}\right)\right)\cdot\frac{1}{\cancel{c}}$$

$$y_1 = \sinh\left(\frac{x}{c}\right)$$

d.w.r.t x

$$y_2 = \cosh\left(\frac{x}{c}\right)\cdot\frac{1}{c}$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$= \frac{\left(1+\left(\sinh^2\left(\frac{x}{c}\right)\right)\right)^{3/2}}{\frac{1}{c}\cdot\cosh\left(\frac{x}{c}\right)}$$

$$= \frac{c\cdot\left(\cosh^{\cancel{2}}\left(\frac{x}{c}\right)\right)^{3/\cancel{2}}}{\cosh\left(\frac{x}{c}\right)}$$

$$= \frac{c\left(\cosh^{\cancel{3}\,2}\left(\frac{x}{c}\right)\right)}{\cancel{\cosh}\left(\frac{x}{c}\right)}$$

$$\rho = c\cdot\cosh^2\left(\frac{x}{c}\right)$$

$$y = c\cdot\cosh\left(\frac{x}{c}\right)$$

$$\frac{y}{c} = \cosh\left(\frac{x}{c}\right) \Rightarrow \rho = \frac{y^2}{c^{\cancel{2}}}\cdot\cancel{c} \Rightarrow \boxed{\rho = \frac{y^2}{c}}$$

5. Define curvature & radius of curvature, find the radius of curvature of curve. $x^3+y^3=3axy$. $\left(\frac{3a}{2}, \frac{3a}{2}\right)$

Sol:- $x^3+y^3=3axy$

d.w.r.t.x

$$3x^2+3y^2y_1=3a(xy_1+y)$$

$$\cancel{3}(x^2+y^2y_1)=\cancel{3}(axy_1+ay)$$

$$x^2+y^2y_1=axy_1+ay$$

$$y^2y_1-axy_1=ay-x^2$$

$$y_1(y^2-ax)=ay-x^2$$

$$y_1=\frac{ay-x^2}{(y^2-ax)} \rightarrow (2)$$

at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$

$$y_1=\frac{a\left(\frac{3a}{2}\right)-\left(\frac{3a}{2}\right)^2}{\left(\frac{3a}{2}\right)^2-a\left(\frac{3a}{2}\right)} = \frac{-\left(\cancel{\left(\frac{3a}{2}\right)^2-a\left(\frac{3a}{2}\right)}\right)}{\cancel{\left(\frac{3a}{2}\right)^2-a\left(\frac{3a}{2}\right)}}$$

$$\boxed{y_1=-1}$$

d. eq-(2) w.r.t. x

$$y_2=\frac{(y^2-ax)(ay_1-2x)-(ay-x^2)(2y.y_1-a)}{(y^2-ax)^2}$$

at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$, $y_2=\dfrac{\left(\left(\frac{3a}{2}\right)^2-a\left(\frac{3a}{2}\right)\right).\left(a(-1)-2\left(\frac{3a}{2}\right)\right)-\left(a\left(\frac{3a}{2}\right)-\left(\frac{3a}{2}\right)^2\right).\left(\cancel{2}\left(\frac{3a}{\cancel{2}}\right)(-1)-a\right)}{(y^2-ax)^2}$

$$y_2 = \frac{\left(\left(\frac{3a}{2}\right)^2 - a\left(\frac{3a}{2}\right)\right)(-4a) - \left(a\left(\frac{3a}{2}\right) - \left(\frac{3a}{2}\right)^2\right)(-4a)}{\left(\left(\frac{3a}{2}\right)^2 - a\left(\frac{3a}{2}\right)\right)^2}$$

$$= \frac{\frac{3a}{2}\left(\frac{3a}{2} - a\right)(-4a) - \frac{3a}{2}\left(a - \frac{3a}{2}\right)(-4a)}{\left[\frac{9a^2}{4} - \frac{9a^2}{2}\right]^2}$$

$$= \frac{\frac{3a}{2}(-\cancel{4}^{2}a)\left[\frac{a}{2} - \left(-\frac{a}{2}\right)\right]}{\left[\frac{9a^2 - 6a^2}{4}\right]^2} = \frac{-6a^2(a)}{\left(\frac{3a^2}{4}\right)^2} = \frac{-6a^2 \cdot a}{\frac{9a^4}{16}}$$

$$= -\cancel{6}^{2}\cancel{a^3} \times \frac{16}{\cancel{9}_{3}a^{\cancel{4}}} = \frac{-32}{3a}$$

6. Find the radius of curvature for the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$.

Sol:- $\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$

$$\sqrt{x} + \sqrt{y} = \sqrt{a}$$

d.w.r. to x

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} \cdot y_1 = 0$$

$$\frac{1}{\cancel{2}\sqrt{y}} \cdot y_1 = -\frac{1}{\cancel{2}\sqrt{x}}$$

$$y_1 = \frac{-\sqrt{y}}{\sqrt{x}} \rightarrow (1)$$

d.w.r.t x

$$y_2 = -\left[\frac{\sqrt{x} \cdot \frac{1}{2\sqrt{y}} \cdot y_1 - \sqrt{y} \cdot \frac{1}{2\sqrt{x}}}{(\sqrt{x})^{\cancel{2}}}\right]$$

$$y_2 = -\left[\frac{\cancel{\sqrt{x}} \cdot \frac{1}{2\cancel{\sqrt{y}}}\left(\frac{-\cancel{\sqrt{y}}}{\cancel{\sqrt{x}}}\right) - \sqrt{y} \cdot \frac{1}{2\sqrt{x}}}{x}\right] = y_2 = -\left[\frac{-\frac{1}{2} - \frac{\sqrt{y}}{2\sqrt{x}}}{x}\right]$$

$$y_2 = \frac{1}{2}\left[\frac{1+\frac{\sqrt{y}}{\sqrt{x}}}{x}\right]$$

$$y_2 = \frac{1}{2}\left[\frac{\frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}}}{x}\right]$$

$$y_2 = \frac{1}{2}\left[\frac{\sqrt{a}}{x\sqrt{x}}\right]$$

$$\rho = \frac{\left(1+\left(-\frac{\sqrt{y}}{\sqrt{x}}\right)^2\right)^{3/2}}{\frac{1}{2}\left[\frac{\sqrt{a}}{x\sqrt{x}}\right]}$$

$$= \frac{2\left(1+\frac{y}{x}\right)^{3/2} \cdot x\sqrt{x}}{\sqrt{a}}$$

$$= \frac{2x\sqrt{x}}{\sqrt{a}}\left(\frac{x+y}{x}\right)^{3/2} = \frac{2\cancel{x\sqrt{x}}}{\sqrt{a}} \frac{(x+y)^{3/2}}{\cancel{x^{3/2}}}$$

$$\boxed{\rho = \frac{2}{\sqrt{a}}(x+y)^{3/2}}$$

7\. Find the radius of curvature for the curve $x^{2/3} + y^{2/3} = a^{2/3}$

Sol:- $x^{2/3} + y^{2/3} = a^{2/3}$

d.w.r.t x

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3} \cdot y_1 = 0$$

$$\frac{\cancel{2}}{\cancel{3}}y^{-1/3} \cdot y_1 = -\frac{\cancel{2}}{\cancel{3}}x^{-1/3}$$

$$y_1 = \frac{-x^{-1/3}}{y^{-1/3}} \Rightarrow y_1 = -x^{-1/3} \cdot y^{1/3}$$

d.w.r.to x

$$y_2 = -\left[x^{-1/3} \cdot \frac{1}{3}y^{1/3-1} \cdot y_1 + y^{1/3} \cdot \left(-\frac{1}{3}\right)x^{-1/3-1}\right]$$

$$y_2 = \frac{1}{3}\left[y^{1/3} \cdot x^{-4/3} - x^{-1/3}y^{-2/3}\left(-x^{-1/3} \cdot y^{1/3}\right)\right]$$

$$= \frac{1}{3}\left[y^{1/3}.x^{-4/3} + x^{-1/3-1/3}.y^{-2/3+1/3}\right]$$

$$= \frac{1}{3}\left[y^{1/3}.x^{-4/3} + x^{-2/3}.y^{-1/3}\right]$$

$$= \frac{1}{3}x^{-4/3}.y^{-1/3}\left(x^{2/3}+y^{2/3}\right)$$

$$y_2 = 1/3\, x^{-4/3}.y^{-1/3}\left(a^{2/3}\right)$$

$$\rho = \frac{\left(1+(y_1)^2\right)^{3/2}}{y_2} = \frac{\left(1+\left(-x^{-1/3}.y^{1/3}\right)^2\right)^{3/2}}{\frac{1}{3}.x^{-4/3}.y^{-1/3}.a^{2/3}}$$

$$\rho = 3.x^{4/3}.y^{1/3}.a^{-2/3}\left(1+x^{-2/3}.y^{2/3}\right)^{3/2}$$

$$= 3x^{4/3}.y^{1/3}.a^{-2/3}\left(1+\frac{y^{2/3}}{x^{2/3}}\right)^{3/2}$$

$$= 3x^{4/3}.y^{1/3}.a^{-2/3}\left(\frac{x^{2/3}+y^{2/3}}{x^{2/3}}\right)^{3/2}$$

$$= 3x^{\frac{4}{3}}.y^{\frac{1}{3}}.a^{-\frac{2}{3}}\ \frac{\left(a^{\frac{2}{3}}\right)^{\frac{3}{2}}}{\left(x^{2/3}\right)^{3/2}}$$

$$= 3x^{4/3}.y^{1/3}.a^{-2/3}.\frac{a}{x}$$

$$\rho = 3.x^{1/3}.y^{1/3}.a^{1/3}$$

$$\rho = 3(xya)^{1/3}$$

Cubing on both side

$$\rho^3 \propto xy$$

8. $y = \frac{ax}{a+x}$. PT $\left(\frac{2P}{a}\right)^{2/3} = \left(\frac{x}{y}\right)^2 + \left(\frac{y}{x}\right)^2$.

Sol:- $y = \frac{ax}{a+x} \rightarrow (1)$

d. w. r. t x

$$y_1 = a\left[\frac{(a+x)(1) - x(1)}{(a+x)^2}\right]$$

$$y_1 = a\left[\frac{a+\cancel{x}-\cancel{x}}{(a+x)^2}\right]$$

$$y_1 = a\left[\frac{a}{(a+x)^2}\right] \Rightarrow y_1 = \left[\frac{a}{a+x}\right]^2 \longrightarrow (2)$$

from (1) $\frac{a}{a+x} = \frac{y}{x}$

$$y_1 = \left(\frac{y}{x}\right)^2 = \frac{y^2}{x^2}$$

D eq-(2) w. r. t x

$$y_2 = a^2 \cdot \frac{(-2)}{(a+x)^3}$$

$$y_2 = \frac{-2}{(a+x)} \cdot \left(\frac{a}{a+x}\right)^2$$

$$y_2 = \frac{-2}{(a+x)} \cdot \frac{y^2}{x^2} \Rightarrow y_2 = -2\frac{y}{ax} \cdot \frac{y^2}{x^2} = \frac{-2y^3}{ax^3}$$

$$y_2 = \frac{-2y^3}{ax^3}$$

$$P = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$P = \frac{\left(1+\left(\frac{y^2}{x^2}\right)^2\right)^{3/2}}{\frac{-2y^3}{ax^3}}$$

$$P = \left(\frac{x^4+y^4}{x^4}\right)^{3/2} \times \frac{ax^3}{-2y^3}$$

$$\frac{-2P}{a} = \left(\frac{x^4+y^4}{x^4}\right)^{3/2} \times \frac{x^3}{y^3}$$

$$\frac{-2P}{a} = \frac{(x^4+y^4)^{3/2}}{(x^{4})^{3/\cancel{2}}} \times \frac{x^3}{y^3}$$

$$= \frac{(x^4+y^4)^{3/2}}{x^3 \cdot y^3} \Rightarrow \frac{-2P}{a} = \left(\frac{x^4+y^4}{x^3 \cdot y^3}\right)^{3/2}$$

$$\left(\frac{-2P}{a}\right)^{2/3} = \frac{x^4+y^4}{x^2 y^2} = \frac{x^4+y^4}{(xy)^2}$$

$$= \frac{x^{\cancel{4}2}}{\cancel{x^2}y^2} + \frac{y^{\cancel{4}2}}{x^2\cancel{y^2}}$$

$$\left(\frac{-2P}{a}\right)^{\frac{2}{3}} = \frac{x^2}{y^2} + \frac{y^2}{x^2}$$

$$\left(\frac{2P}{a}\right)^{2/3} = \left(\frac{x}{y}\right)^2 + \left(\frac{y}{x}\right)^2$$

9. Find the radius of curvature of the curve $x^2y = a(x^2+y^2)$ $(-2a, 2a)$

Sol:- $x^2y = a(x^2+y^2)$

d.w.r.t x

$$x^2y_1 + 2xy_1 = a[2x + 2yy_1]$$

$$x^2 \cdot y_1 + 2x \cdot y = 2ax + 2ayy_1$$

$$x^2 \cdot y_1 - 2ayy_1 = 2ax - 2xy$$

$$y_1(x^2 - 2ay) = 2x(a-y)$$

$$y_1 = \frac{2x(a-y)}{(x^2 - 2ay)}$$

at $(-2a, 2a)$

$$y_1 = \frac{2(-2a)(a-(2a))}{((-2a)^2 - 2a(2a))} = \frac{-4a(-a)}{4a^2-4a^2}$$

$$\Rightarrow y_1 = \infty$$

Since, $y_1 = \infty$, ρ becomes ∞

so, WKT, $\rho = \dfrac{(1+x_1^2)^{3/2}}{x_2}$

$$x^2y = a(x^2+y^2)$$

d.w.r.t y

$$x^2(1) + 2x.x_1y = a(2xx_1 + 2y)$$

$$x^2 + 2xx_1y = 2axx_1 + 2ay$$

$$2axx_1 - 2xx_1y = x^2 - 2ay$$

$$x_1(2ax - 2xy) = x^2 - 2ay$$

$$x_1 = \frac{x^2 - 2ay}{(2ax - 2xy)} \rightarrow (1) \quad \text{at } (-2a, 2a)$$

$$x_1 = \frac{(-2a)^2 - 2a(2a)}{(2a(-2a) - 2(-2a)(2a))} = \frac{4a^2 - 4a^2}{-4a^2 - 8a^2} = 0$$

$$\therefore x_1 = 0$$

d. eq-(1) w.r.t. y

$$x_2 = \frac{(2ax-2xy)(2xx_1 - 2a) - (x^2 - 2ay)(2ax_1 - 2x - 2x_1y)}{(2ax - 2xy)^2}$$

at $(-2a, 2a)$ & $x_1 = 0$

$$x_2 = \frac{4a^2(-2a) - 0}{(2a(-2a) - 2(-2a)(2a))^2} = \frac{-8a^3}{(-4a^2 + 8a^2)^2}$$

$$x_2 = \frac{-\cancel{8}^{1}\cancel{a^3}}{\cancel{16}_{2}a^{4}} = \frac{-1}{2a}$$

$x_1 = 0$ & $x_2 = \frac{-1}{2a}$

$$\rho = \frac{(1+x_1^2)^{3/2}}{x_2} = \frac{(1+0)^{3/2}}{-1/2a} = -2a$$

$$|\rho| = |-2a|$$

$$\boxed{\rho = 2a}$$

10. Find the radius of curvature of the curve $y^2 = \frac{4a^2(2a-x)}{x}$ where the curve meets X-axis.

Sol:- $$y^2 = \frac{4a^2(2a-x)}{x} \longrightarrow (1)$$

Given curve meets X-axis $\Rightarrow y=0$

$$\Rightarrow 0 = \frac{4a^2(2a-x)}{x}$$

$$\Rightarrow 2a - x = 0$$

$$x = 2a$$

$$\therefore (2a, 0)$$

D.w.r.t x eq -(1)

$$2y \cdot y_1 = 4a^2 \left[\frac{x(0-1) - (2a-x)(1)}{x^2}\right]$$

$$2y \cdot y_1 = 4a^2 \left[\frac{-x-(2a-x)}{x^2}\right]$$

$$y_1 = \frac{\overset{2}{\cancel{4}}a^2}{\cancel{2}y} \left[\frac{-\cancel{x} - 2a + \cancel{x}}{x^2}\right]$$

at $(2a, 0)$

$$y_1 = \frac{2a^2}{0} \left[\frac{-2a-2a+2a}{(2a)^2}\right]$$

$$y_1 = \infty$$

Since, $y_1 = \infty$ ---- ρ becomes ∞

Rewrite ρ in alternate formula.

$$P = \frac{(1+x_1^2)^{3/2}}{x_2}, \quad x_1 = \frac{dx}{dy}, \quad x_2 = \frac{d^2x}{dy^2}$$

$$y^2 = \frac{4a^2(2a-x)}{x}$$

$$y^2x = 8a^3 - 4a^2x$$

$$y^2x + 4a^2x = 8a^3$$

$$x(y^2+4a^2) = 8a^3$$

$$x = \frac{8a^3}{y^2+4a^2}$$

d.w.r.t. y

$$x_1 = 8a^3 \cdot \frac{-1}{(y^2+4a^2)^2}(2y)$$

at $(2a, 0)$ $x_1 = \frac{-16a^3 \cdot 0}{(0+4a^2)} = 0$

$$x_1 = 0$$

d.w.r. to y

$$x_2 = -16a^3\left[\frac{(y^2+4a^2)^2(1) - y \cdot 2(y^2+4a^2)(2y)}{((y^2+4a^2)^2)^2}\right]$$

at $(2a, 0)$

$$x_2 = -16a^3\left[\frac{(0+4a^2)^2 - 0}{(0+4a^2)^4}\right] = -\cancel{16}a^3\left[\frac{\cancel{16}a^4}{\cancel{16}\times\cancel{16}\times a^8}\right]$$

$$x_2 = \frac{-1}{a}$$

$$P = \frac{(1+x_1^2)^{3/2}}{x_2}$$

$$= \frac{(1+0)^{3/2}}{\frac{-1}{a}} \quad P = |-a|$$

$$\boxed{P = a}$$

Derive Radius of curvature in cartesian form:-

Note:- Derivative of arc length in cartesian form,

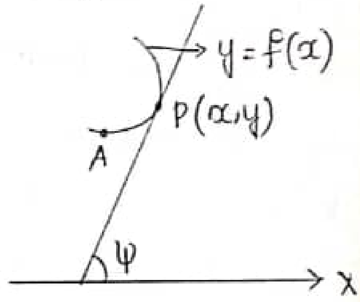
If the equation of the curve in the form $y = f(x)$ then derivative of arc length 's' with respect to x is given by

$$\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}, \quad \frac{dx}{ds} = \cos\psi, \quad \frac{dy}{ds} = \sin\psi$$

Radius of curvature in cartesian form:-

Let the equation of the curve be $y = f(x)$ & 'A' be a fixed point on it. Let $p(x,y)$ is any point on the curve such that $\widehat{AP} = s$.

Let ψ be the angle made by the tangent at 'p' with X-axis.



WKT, $\tan\psi = \frac{dy}{dx} = y_1$

d.w.r.t. s

$$\sec^2\psi \cdot \frac{d\psi}{ds} = \frac{d}{ds}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{dx}\right)\left(\frac{dx}{ds}\right)$$

$$(1+\tan^2\psi)\frac{d\psi}{ds} = \frac{d^2y}{dx^2} \cdot \frac{dx}{ds}$$

$$(1+y_1^2)\frac{d\psi}{ds} = y_2 \cdot \frac{1}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}} = \frac{y_2}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}} = \frac{y_2}{\sqrt{1+y_1^2}}$$

$$\frac{d\psi}{ds} = \frac{y_2}{\sqrt{1+y_1^2}}$$

$$\frac{1}{\rho} = \frac{y_2}{(1+y_1^2)^{3/2}} \Rightarrow \boxed{\rho = \frac{(1+y_1^2)^{3/2}}{y_2}}$$

Radius of curvature in polar form:-

Note:- Derivative of arc length in polar form.

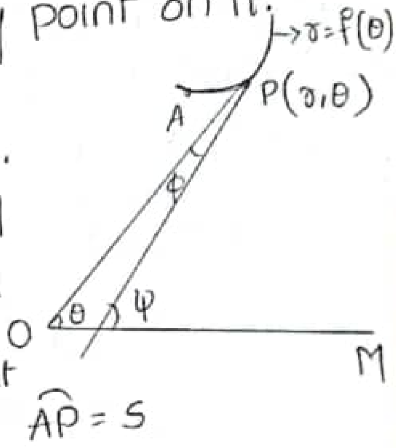
$$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{r^2 + r_1^2}$$

Prove that radius of curvature in polar form,

$$\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - r r_2}$$

Proof:-

Let $r = f(\theta)$ be any polar curve & $P(r,\theta)$ be any point on it. Let $OP = r$ be the radius vector. Draw a tangent at P which makes an angle ψ with the initial line. Let θ & ϕ be angle between radius vector and initial line also angle between radius vector and tangent respectively. Let 'A' be the fixed point



WKT, $\psi = \phi + \theta$

d.w.r.t. s

$$\frac{d\psi}{ds} = \frac{d\phi}{ds} + \frac{d\theta}{ds} = \frac{d\phi}{d\theta} \cdot \frac{d\theta}{ds} + \frac{d\theta}{ds}$$

$$\frac{d\psi}{ds} = \frac{d\theta}{ds}\left(1 + \frac{d\phi}{d\theta}\right)$$

$$\frac{1}{\rho} = \frac{d\theta}{ds}\left(1 + \frac{d\phi}{d\theta}\right)$$

$$\boxed{\rho = \frac{ds/d\theta}{1 + \frac{d\phi}{d\theta}}} \longrightarrow (1)$$

WKT $\tan\phi = r \cdot \frac{d\theta}{dr} = \frac{r}{\frac{dr}{d\theta}}$

$$\tan\phi = \frac{r}{r_1}$$

d.w.r.t. θ

$$\sec^2\phi \cdot \frac{d\phi}{d\theta} = \frac{r_1 \frac{dr}{d\theta} - r \cdot \frac{d}{d\theta}(r_1)}{r_1^2}$$

$$(1+\tan^2\phi).\frac{d\phi}{d\theta} = \frac{r_1^2 - rr_2}{r_1^2}$$

$$\frac{d\phi}{d\theta} = \frac{r_1^2 - rr_2}{r_1^2(1+\tan^2\phi)}$$

$$\frac{d\phi}{d\theta} = \frac{r_1^2 - rr_2}{r_1^2\left(1+\frac{r^2}{r_1^2}\right)} = \frac{r_1^2 - rr_2}{r_1^2\left(\frac{r_1^2+r^2}{r_1^2}\right)}$$

add 1 both sides

$$1+\frac{d\phi}{d\theta} = 1+\frac{r_1^2 - rr_2}{r_1^2+r^2}$$

$$1+\frac{d\phi}{d\theta} = \frac{r_1^2+r^2+r_1^2-rr_2}{r_1^2+r^2}$$

$$1+\frac{d\phi}{d\theta} = \frac{2r_1^2+r^2-rr_2}{r_1^2+r^2}$$

$$1+\frac{d\phi}{d\theta} = \frac{r^2+2r_1^2-rr_2}{r_1^2+r^2}$$

$\therefore$ (1) - becomes

$$\rho = \frac{\sqrt{r^2+r_1^2}}{\frac{r^2+2r_1^2-rr_2}{r_1^2+r^2}} = \frac{(r^2+r_1^2)^{1/2} \times (r_1^2+r^2)}{r^2+2r_1^2-rr_2}$$

$$\Rightarrow \boxed{\rho = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2}}$$

1. PT $\frac{\rho^2}{r}$ is constant for the cardiode $r = a(1+\cos\theta)$ where ρ is the radius of curvature.

Sol:- $$\rho = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2 - rr_2}$$

$$r = a(1+\cos\theta) \longrightarrow (1)$$

d.w.r.t.θ

$$r_1 = \frac{dr}{d\theta} = a(-\sin\theta) \longrightarrow (2)$$

d (2) w.r.t.θ

$$\frac{d^2r}{d\theta^2} = r_2 = a(-\cos\theta)$$

$$\rho = \frac{(r^2+(-a\sin\theta)^2)^{3/2}}{r^2+2(-a\sin\theta)^2 - r(-a\cos\theta)}$$

$$= \frac{(a^2(1+\cos\theta)^2+(a^2\sin^2\theta))^{3/2}}{a^2(1+\cos\theta)^2+2a^2\sin^2\theta + a(1+\cos\theta)(a\cos\theta)}$$

$$= \frac{(a^2)^{3/2}(1+\cos^2\theta+2\cos\theta+\sin^2\theta)^{3/2}}{a^2(1+\cos^2\theta+2\cos\theta)+2a^2\sin^2\theta + a^2\cos\theta + a^2\cos^2\theta}$$

$$= \frac{a^3(1+1+2\cos\theta)^{3/2}}{a^2(1+\cos^2\theta+2\cos\theta+2\sin^2\theta+\cos\theta+\cos^2\theta)}$$

$$= \frac{a^{\cancel{3}}(2+2\cos\theta)^{3/2}}{\cancel{a^2}(1+2\sin^2\theta+2\cos^2\theta+3\cos\theta)} = \frac{(2)^{3/2}a(1+\cos\theta)^{3/2}}{(1+2+3\cos\theta)}$$

$$= \frac{(2)^{3/2}a(1+\cos\theta)^{3/2}}{3(1+\cos\theta)}$$

$$= \frac{a2\sqrt{2}}{3}(1+\cos\theta)^{1/2}$$

from eq - (1)

$$= \frac{a2\sqrt{2}}{3}\left(\frac{r}{a}\right)^{1/2}$$

$$\rho = \frac{a2\sqrt{2}}{3}\frac{\sqrt{r}}{\sqrt{a}}$$

$$\rho = \sqrt{a}\cdot\cancel{\sqrt{a}}\cdot\frac{2\sqrt{2}}{3}\cdot\frac{\sqrt{r}}{\cancel{\sqrt{a}}}$$

$$\rho = \frac{\sqrt{a}\,2\sqrt{2}}{3}\cdot\sqrt{r}$$

Squaring o.b.s

$$\rho^2 = \frac{8a}{9}r$$

$$\frac{\rho^2}{r} = \frac{8a}{9}$$

$$\frac{\rho^2}{r} = \text{constant}$$

$$\rho^2 \propto r$$

2. PT for the curve $r(1-\cos\theta)=2a$, ρ^2 varies as r^3.

Sol:- $$\rho = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2}$$

$$r(1-\cos\theta) = 2a \quad -(1)$$

d.w.r.t.θ

$$r(1-\cos\theta) = 2a$$

$$r(0-(-\sin\theta)) + (1-\cos\theta)r_1 = 0$$

$$r(\sin\theta) + (1-\cos\theta)r_1 = 0$$

$$(1-\cos\theta)r_1 = -r\sin\theta \longrightarrow (2)$$

d.w.r.t.θ (2)

$$(1-\cos\theta)r_2 + (0-(-\sin\theta))r_1 = -\left[r(\cos\theta)+\sin\theta\, r_1\right]$$

$$(1-\cos\theta)r_2 + r_1\sin\theta = -r\cos\theta - \sin\theta\, r_1$$

$$\frac{2a}{r}\cdot r_2 = -r\cos\theta - r_1\sin\theta - r_1\sin\theta$$

from eq-(2)

$$(1-\cos\theta) r_1 = -r\sin\theta$$

$$r_1 = \frac{-r\sin\theta}{1-\cos\theta}$$

$$r_1 = \frac{-r\sin\theta}{2a/r}$$

$$\boxed{r_1 = \frac{-r^2}{2a}\sin\theta}$$

$$r_2 = (-r\cos\theta - 2r_1\sin\theta)\frac{r}{2a}$$

$$r_2 = \left(-r\cos\theta - 2\sin\theta\left(\frac{-r^2}{2a}\right)\sin\theta\right)\frac{r}{2a}$$

$$r_2 = \frac{-r^2}{2a}\cos\theta + \frac{r^3}{2a^2}\sin^2\theta$$

$$P = \frac{\left(r^2 + \left(\frac{-r^2}{2a}\sin\theta\right)^2\right)^{3/2}}{r^2 + 2\left(\frac{-r^2}{2a}\sin\theta\right)^2 - r\left(\frac{-r^2}{2a}\cos\theta + \frac{r^3}{2a^2}\sin^2\theta\right)}$$

———— X ————

2. $r(1-\cos\theta) = 2a$

$$\log r + \log(1-\cos\theta) = \log 2a$$

d.w.r.t. θ

$$\frac{1}{r}\cdot\frac{dr}{d\theta} + \frac{\sin\theta}{1-\cos\theta} = 0$$

$$\frac{1}{r}\cdot\frac{dr}{d\theta} + \frac{\cancel{2}\sin\theta/2\cdot\cos\theta/2}{\cancel{2}\sin^2\theta/2} = 0$$

$$\frac{1}{r}\cdot\frac{dr}{d\theta} = -\cot\theta/2$$

$$\frac{1}{r}\cdot r_1 = -\cot\theta/2$$

$$r_1 = -r\cdot\cot\theta/2 \quad \text{d.w.r. to } \theta$$

$$r_2 = -\left[r\left(-\operatorname{cosec}^2\frac{\theta}{2}\right)\frac{1}{2} + \cot\left(\frac{\theta}{2}\right) r_1\right]$$

$$= \frac{r}{2}\operatorname{cosec}^2(\theta/2) - \cot(\theta/2)\,(-r\cot\theta/2)$$

$$r_2 = \frac{r}{2}\operatorname{cosec}^2(\theta/2) + r\cdot\cot^2(\theta/2)$$

$$\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - r r_2}$$

$$= \frac{\left(r^2 + (-r\cot\theta/2)^2\right)^{3/2}}{r^2 + 2(-r\cot\theta/2)^2 - r\left(\frac{r}{2}\operatorname{cosec}^2(\theta/2) + r\cdot\cot^2\theta/2\right)}$$

$$= \frac{\left(r^2 + r^2\cot^2\theta/2\right)^{3/2}}{r^2 + 2r^2\cot^2\theta/2 - r^2\left(\frac{1}{2}\operatorname{cosec}^2\theta/2 + \cot^2\theta/2\right)}$$

$$= \frac{(r^{\cancel{2}})^{3/\cancel{2}}\left(1 + \cot^2\theta/2\right)^{3/2}}{r^{\cancel{2}}\left(1 + 2\cot^2\theta/2 - \frac{1}{2}\operatorname{cosec}^2\theta/2 - \cot^2\theta/2\right)}$$

$$= \frac{r\left(\operatorname{cosec}^{\cancel{2}}\theta/2\right)^{3/\cancel{2}}}{1 + \cot^2(\theta/2) - \frac{1}{2}\operatorname{cosec}^2\theta/2} = \frac{r\operatorname{cosec}^3\theta/2}{\operatorname{cosec}^2\theta/2 - \frac{1}{2}\operatorname{cosec}^2\theta/2}$$

$$= \frac{r\operatorname{cosec}^3(\theta/2)}{\operatorname{cosec}^2\theta/2\left(1 - \frac{1}{2}\right)}$$

$$= \frac{r\operatorname{cosec}(\theta/2)}{\frac{1}{2}} = 2r\operatorname{cose}(\theta/2)$$

WKT, $r(1-\cos\theta) = 2a$

$$r\cdot \cancel{2}\sin^2\theta/2 = \cancel{2}a$$

$$\sin^2\theta/2 = \frac{a}{r}$$

$$\sin\theta/2 = \sqrt{\frac{a}{r}}$$

$\operatorname{cosec}(\theta/2) = \sqrt{\frac{r}{a}}$

$$P = 2r\frac{\sqrt{r}}{\sqrt{a}}$$

Squaring o.b.s

$$p^2 = 4r^2 \cdot \frac{r}{a}$$

$$p^2 = \left(\frac{4}{a}\right)r^3$$

$$p^2 = cr^3 \quad \text{where } \frac{4}{a} = \text{constant}$$

$$\boxed{p^2 \propto r^3}$$

3. Find P for the curve $r = ae^{\theta\cot\alpha}$

Sol:- $r = ae^{\theta\cot\alpha} \longrightarrow (1)$

Take log

$$\log r = \log a + \log e^{\theta\cot\alpha}$$

d.w.r.t θ

$$\frac{1}{r}\cdot\frac{dr}{d\theta} = 0 + \cot\alpha(1)$$

$$r_1 = r\cot\alpha$$

d.w.r.t θ

$$r_2 = \cot\alpha \cdot r_1$$

$$= \cot\alpha(r\cot\alpha) = r\cot^2\alpha$$

$$P = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2}$$

$$= \frac{(r^2+(r\cot\alpha)^2)^{3/2}}{r^2+2(r\cot\alpha)^2-r(r\cot^2\alpha)}$$

$$= \frac{(r^2+r^2\cot^2\alpha)^{3/2}}{r^2+2r^2\cot^2\alpha-r^2\cot^2\alpha}$$

$$= \frac{(r^2)^{3/2}(1+\cot^2\alpha)^{3/2}}{r^2(1+2\cot^2\alpha-\cot^2\alpha)}$$

$$\frac{r(\operatorname{cosec}^2\alpha)^{3/2}}{(1+\cot^2\alpha)} = \frac{r\operatorname{cosec}^3\alpha}{\operatorname{cosec}^2\alpha} = \boxed{P = r\operatorname{cosec}\alpha}$$

4. Find P of the curve $r^n = a^n \cos n\theta$

Sol:-

$$r^n = a^n \cos n\theta \longrightarrow (1)$$

$$n\log r = n\log a + \log\cos n\theta$$

d.w.r.t. θ

$$\cancel{n}\cdot\frac{1}{r}\cdot r_1 = 0 + \frac{(-\sin n\theta)\cancel{n}}{\cos n\theta}$$

$$\frac{1}{r}\cdot r_1 = -\frac{\sin n\theta}{\cos n\theta}$$

$$r_1 = -r\cdot\tan n\theta$$

$$\boxed{r_1 = -r\cdot\tan\theta}$$

d.w.r.t. θ

$$r_2 = -\left[r_1\cdot\tan n\theta + r\cdot\sec^2 n\theta\cdot n\right]$$

$$= -\left[(-r\cdot\tan n\theta)\cdot\tan n\theta + nr\sec^2 n\theta\right]$$

$$= -\left[nr\sec^2 n\theta - r\cdot\tan^2 n\theta\right]$$

$$= -nr\sec^2 n\theta + r\cdot\tan^2 n\theta$$

$$P = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2} = \frac{\left(r^2+(-r\cdot\tan n\theta)^2\right)^{3/2}}{r^2+2(-r\cdot\tan n\theta)^2 - r(-nr\sec^2 n\theta + r\tan^2 n\theta)}$$

$$= \frac{(r^2+r^2\tan^2 n\theta)^{3/2}}{r^2+2r^2\tan^2 n\theta + nr^2\sec^2 n\theta - r^2\tan^2 n\theta}$$

$$= \frac{(r^2)^{\cancel{3/2}\,1}(1+\tan^2 n\theta)^{3/2}}{\cancel{r^2}(1+2\tan^2 n\theta + n\sec^2 n\theta - \tan^2 n\theta)}$$

$$= \frac{r(\sec^{2} n\theta)^{3/2}}{(1+\tan^{2} n\theta + n\sec^{2} n\theta)} = \frac{r\sec^{3} n\theta}{(\sec^{2} n\theta + n\sec^{2} n\theta)} = \frac{r\sec^{3} n\theta}{\sec^{2} n\theta(1+n)}$$

$$P = \frac{r\sec n\theta}{(1+n)}$$

Given, $r^{n} = a^{n}\cos n\theta$

$$\cos n\theta = \frac{a^{n}}{r^{n}}$$

$$\frac{1}{\cos n\theta} = \frac{a^{n}}{r^{n}}$$

$$\sec n\theta = \frac{a^{n}}{r^{n}} \Rightarrow P = \frac{r}{(1+n)} \cdot \frac{a^{n}}{r^{n}}$$

$$P = C \cdot \frac{1}{r^{n-1}} \quad \text{where } C = \frac{a^{n}}{(1+n)}$$

$$P \propto \frac{1}{r^{n-1}}$$

5. $r^{n} = a\sin n\theta$

Sol:- $r^{n} = a\sin n\theta \longrightarrow (1)$

$$n\log r = \log a + \log\sin n\theta$$

d.w.r.t. θ

$$\cancel{n} \cdot \frac{1}{r} \cdot \frac{dr}{d\theta} = 0 + \frac{\cos n\theta}{\sin n\theta} \cdot \cancel{n}$$

$$\boxed{r_1 = r \cdot \cot n\theta}$$

d.w.r.t. θ

$$r_2 = (r \cdot (-\operatorname{cosec}^{2} n\theta) n + r_1 \cot n\theta)$$

$$r_2 = (-nr\operatorname{cosec}^{2} n\theta + (r \cdot \cot n\theta)(\cot n\theta))$$

$$r_2 = (r \cdot \cot^{2} n\theta - nr\operatorname{cosec}^{2} n\theta)$$

$$p = \frac{(r^{2}+r_1^{2})^{3/2}}{r^{2}+2r_1^{2}-rr_2} = \frac{(r^{2}+(r\cot n\theta)^{2})^{3/2}}{r^{2}+2(r\cdot\cot n\theta)^{2}-r(r\cdot\cot^{2} n\theta - nr\operatorname{cosec}^{2} n\theta)}$$

$$= \frac{(r^2 + r^2 \cdot \cot^2 n\theta)^{3/2}}{r^2 + 2r^2 \cot^2 n\theta - r^2 \cot^2 n\theta + nr^2 \operatorname{cosec}^2 n\theta}$$

$$= \frac{(\cancel{r^2})^{3/\cancel{2}} (1+\cot^2 n\theta)^{3/2}}{\cancel{r^2} (1+ 2\cot^2 n\theta - \cot^2 n\theta + n\operatorname{cosec}^2 n\theta)}$$

$$= \frac{r(\operatorname{cosec}^{\cancel{2}} n\theta)^{3/\cancel{2}}}{(1+\cot^2 n\theta + n\operatorname{cosec}^2 n\theta)} = \frac{r\operatorname{cosec}^3 n\theta}{(\operatorname{cosec}^2 n\theta + n\operatorname{cosec}^2 n\theta)}$$

$$= \frac{r\operatorname{cosec}^{\cancel{3}\,1} n\theta}{\operatorname{cosec}^{\cancel{2}} n\theta (1+n)}$$

$$\rho = \frac{r\operatorname{cosec} n\theta}{(n+1)}$$

$$\Rightarrow r^n = a \sin n\theta$$

$$\sin n\theta = \frac{r^n}{a}$$

$$\frac{1}{\sin n\theta} = \frac{a}{r^n}$$

$$\operatorname{cosec} n\theta = \frac{a}{r^n}$$

$$\rho = \frac{r}{(1+n)} \cdot \frac{a}{r^n}$$

$$\rho = C \cdot \frac{1}{r^{n-1}}$$

$$\boxed{\rho \propto \frac{1}{r^{n-1}}} \quad \text{where, } C = \frac{a}{(1+n)}$$

6. $r^n = a^n \sin n\theta$

Take log on both sides

$n\log r = n\log a + \log \sin n\theta$

d.w.r.t. θ

$$n \cdot \frac{1}{r} \cdot \frac{dr}{d\theta} = 0 + \frac{\cos n\theta \cdot n}{\sin n\theta}$$

$$\cancel{n} \cdot r_1 = r\cot n\theta \cdot \cancel{n}$$

$$\boxed{r_1 = r \cdot \cot n\theta}$$

d.w.r.t. θ

$$r_2 = (r(-\operatorname{cosec}^2 n\theta) \cdot n + r_1 \cot n\theta)$$

$$= (-nr\operatorname{cosec}^2 n\theta + (r\cot n\theta)\cot n\theta)$$

$$r_2 = (r\cot^2 n\theta - nr\operatorname{cosec}^2 n\theta)$$

$$\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2} = \frac{\left(r^2 + (r\cot n\theta)^2\right)^{3/2}}{r^2 + 2(r.\cot n\theta)^2 - r(r\cot^2 n\theta - nr\operatorname{cosec}^2 n\theta)}$$

$$= \frac{(r^2 + r^2\cot^2 n\theta)^{3/2}}{r^2 + 2r^2\cot^2 n\theta - r^2\cot^2 n\theta + nr^2\operatorname{cosec}^2 n\theta)}$$

$$= \frac{(r^2)^{3/2}(1 + \cot^2 n\theta)^{3/2}}{r^2(1 + 2\cot^2 n\theta - \cot^2 n\theta + n\operatorname{cosec}^2 n\theta)}$$

$$= \frac{r(\operatorname{cosec}^2 n\theta)^{3/2}}{(1 + \cot^2 n\theta + n\operatorname{cosec}^2 n\theta)}$$

$$= \frac{r\operatorname{cosec}^3 n\theta}{(\operatorname{cosec}^2 n\theta + n\operatorname{cosec}^2 n\theta)} = \frac{r\operatorname{cosec}^3 n\theta}{\operatorname{cosec}^2 n\theta(1+n)}$$

$$\rho = \frac{r\operatorname{cosec} n\theta}{(1+n)}$$

Given, $r^n = a^n \sin n\theta$

$$\sin n\theta = \frac{r^n}{a^n}$$

$$\frac{1}{\sin n\theta} = \frac{a^n}{r^n}$$

$$\operatorname{cosec} n\theta = \frac{a^n}{r^n}$$

$$P = \frac{r}{(1+n)} \cdot \frac{a^n}{r^n} \Rightarrow P = \frac{1}{r^{n-1}} C \text{ where } C = \frac{a^n}{(1+n)}$$

$$\boxed{P \propto \frac{1}{r^{n-1}}}$$

Radius of Curvature in Pedal form:-

Radius of curvature of in pedal form is given by.

$$\rho = r \cdot \frac{dr}{dp}$$

1. Find the pedal form of radius of curvature. $r(1+\cos\theta) = a$

Sol:- $r(1+\cos\theta) = a \rightarrow (1)$

Take log o.b.s

$\log r + \log(1+\cos\theta) = \log a$

d.w.r.t. θ

$$\frac{1}{r} \cdot \frac{dr}{d\theta} + \frac{0+(-\sin\theta)}{1+\cos\theta} = 0$$

$$\frac{1}{r} \cdot \frac{dr}{d\theta} - \frac{\cancel{2}\sin\theta/2 \cdot \cancel{\cos\theta/2}}{\cancel{2}\cos^{\cancel{2}}\theta/2} = 0$$

$$\cot\phi - \tan\theta/2 = 0$$

$$\cot\phi = \tan\theta/2$$

$$\cot\phi = \cot\left(\frac{\pi}{2} - \theta/2\right)$$

$$\boxed{\phi = \frac{\pi}{2} - \frac{\theta}{2}}$$

$$p = r\sin\phi$$

$$p = r\sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right)$$

$$p = r\cos(\theta/2)$$

from eq-(1)

$$r(1+\cos\theta) = a$$

$$r \cdot 2\cos^2\theta/2 = a$$

$$\cos^2\theta/2 = \frac{a}{2r}$$

$$\cos\theta/2 = \sqrt{\frac{a}{2r}}$$

$$p = r \cdot \frac{\sqrt{a}}{\sqrt{2}.\sqrt{r}}$$

$$p = \frac{\sqrt{r}.\sqrt{a}}{\sqrt{2}}$$

$$p = \frac{\sqrt{a}}{\sqrt{2}} . \sqrt{r}$$

d.w.r.to r

$$\frac{dp}{dr} = \sqrt{\frac{a}{2}} \cdot \frac{1}{2\sqrt{r}}$$

WKT Radius of curvature in pedal form is given by

$$\rho = r.\frac{dr}{dp}$$

$$\rho = \frac{r.2\sqrt{2}.\sqrt{r}}{\sqrt{a}}$$

$$\rho = C.r\sqrt{r} \qquad \text{where } C = \frac{2\sqrt{2}}{\sqrt{a}} \text{(constant)}$$

squaring obs

$$\rho^2 = C^2.r^2.r$$

$$\boxed{\rho^2 \propto r^3}$$

2. PT p is $\frac{a^2}{3r}$ for the curve $r^2 \sec 2\theta = a^2$ in pedal form. 22

Sol:- $r^2 \sec 2\theta = a^2 \longrightarrow (1)$

$2\log r + \log \sec 2\theta = 2\log a$

d.w.r.t. θ

$2 \cdot \frac{1}{r} \cdot \frac{dr}{d\theta} + \frac{(\cancel{\sec 2\theta} \cdot \tan 2\theta)\, 2}{\cancel{\sec 2\theta}} = 0$

$2\cot\phi + \tan 2\theta \cdot 2 = 0$

$\cancel{2}\cot\phi = -\tan 2\theta \cdot \cancel{2}$

$\cot\phi = \cot\left(\frac{\pi}{2} + 2\theta\right)$

$\boxed{\phi = \frac{\pi}{2} + 2\theta}$

$p = r\sin\phi$

$p = r\sin\left(\frac{\pi}{2} + 2\theta\right)$

$p = r\cos 2\theta$

from eq-(1) $r^2 \sec 2\theta = a^2$

$r^2 \cdot \frac{1}{\cos 2\theta} = a^2$

$\cos 2\theta = \frac{r^2}{a^2}$

$p = r \cdot \frac{r^2}{a^2} = p = \frac{r^3}{a^2}$

d.w.r.t. 'r'

$\frac{dp}{dr} = \frac{1}{a^2} 3r^2$

$p = r \cdot \frac{dr}{dp}$

$p = \cancel{r} \cdot \frac{a^2}{3r^{\cancel{2}}} \Rightarrow \boxed{p = \frac{a^2}{3r}}$

3. Find the radius of curvature for the curvature $r = ae^{\theta\cot\alpha}$ in pedal form.

Sol:- $r = ae^{\theta\cot\alpha} \longrightarrow (1)$

$$\log r = \log a + \log e^{\theta\cot\alpha}$$

$$\log r = \log a + \theta\cot\alpha \cdot \log e. \quad (\because \log e = 1)$$

d.w.r.t.θ

$$\frac{1}{r}\cdot\frac{dr}{d\theta} = 0 + \cot\alpha\,(1).$$

$$\cot\phi = \cot\alpha$$

$$\phi = \alpha$$

$$P = r\sin\phi$$

$$P = r\sin\alpha$$

d.w.r.t r

$$\frac{dP}{dr} = \sin\alpha \cdot 1$$

$$\frac{dr}{dp} = \frac{1}{\sin\alpha}$$

$$\rho = r\cdot\frac{dr}{dp}$$

$$\rho = r\cdot\frac{1}{\sin\alpha} \Rightarrow \rho = r.C$$

$$\boxed{\rho \propto r}$$

4. Find the radius of curvature for the curve $r = a(1-\cos\theta)$ in pedal form.

Sol:- $r = a(1-\cos\theta) \longrightarrow (1)$

$$\log r = \log a + \log(1-\cos\theta)$$

d.w.r.t. θ

$$\frac{1}{r}\cdot\frac{dr}{d\theta} = 0 + \frac{0-(-\sin\theta)}{(1-\cos\theta)}$$

$$\cot\phi = \frac{\cancel{2}\sin\cancel{\theta/2}\cos\theta/2}{\cancel{2}\sin^{2}\theta/2}$$

$\cot\phi = \cot \theta/2$

$$\boxed{\phi = \theta/2}$$

$P = r\sin\phi$

$P = r\sin \theta/2$

from eq-(1) , $r = a(1-\cos\theta)$

$r = a\, 2\sin^2 \theta/2$

$\sin \theta/2 = \sqrt{\frac{r}{2a}}$

$P = \frac{r.\sqrt{r}}{\sqrt{2a}}$

d. w.r.t r

$\frac{dP}{dr} = \frac{1}{\sqrt{2a}} \cdot \frac{1}{2\sqrt{r}}$

$\frac{dr}{dP} = 2\sqrt{2}\ \sqrt{a}.\sqrt{r}$

$\rho = r.\frac{dr}{dP}$

$\rho = r.\ 2\sqrt{2}.\sqrt{a}.\sqrt{r}$

sq. o.b.s

$\rho^2 = r^2.\ 8ar$.

$\rho^2 = 8ar^3 \Rightarrow \rho^2 = Cr^3$

$$\boxed{\rho^2 \propto r^3}$$

5. $r^n = a^n \cos n\theta$

Sol:- $r^n = a^n \cos n\theta \longrightarrow (1)$

$n\log r = n\log a + \log\cos n\theta$

d.w.r.t. θ

$n.\frac{1}{r}.\frac{dr}{d\theta} = 0 + \frac{(-\sin n\theta)n}{\cos n\theta}$

$$\cancel{r}\cot\phi = -\cancel{r}\cdot\frac{\sin n\theta}{\cos n\theta}$$

$$\cot\phi = \cot\left(\frac{\pi}{2}+n\theta\right)$$

$$\phi = \frac{\pi}{2}+n\theta$$

$$p = r\sin\phi$$

$$p = r\sin\left(\frac{\pi}{2}+n\theta\right)$$

$$p = r\cos n\theta$$

from eq -(1) $\quad r^n = a^n\cos n\theta$

$$\cos n\theta = \frac{r^n}{a^n}$$

$$p = r.\frac{r^n}{a^n} \Rightarrow p = \frac{1}{a^n}\cdot r^{n+1}$$

d.w.r.t 'r'

$$\frac{dp}{dr} = \frac{1}{a^n}(n+1)r^{n+\cancel{1}-\cancel{1}}$$

$$\frac{dp}{dr} = \frac{1}{a^n}(n+1)r^n$$

$$\frac{dr}{dp} = \frac{a^n}{r^n(n+1)}$$

$$\rho = r.\frac{dr}{dp}$$

$$\rho = r.\frac{a^n}{(n+1)}\cdot\frac{1}{r^n}$$

$$\rho = \frac{1}{r^{n-1}}\cdot\frac{a^n}{(n+1)}$$

$$\boxed{\rho \propto \frac{1}{r^{n-1}}}$$

6. $\frac{2a}{r} = 1+\cos\theta$

Sol:- $\frac{2a}{r} = 1+\cos\theta \rightarrow (1)$

$\log 2a - \log r = \log(1+\cos\theta)$

d.w.r.t.θ

$0 - \frac{1}{r}\cdot\frac{dr}{d\theta} = \frac{1}{(1+\cos\theta)} \times (0+(-\sin\theta))$

$+\cot\phi = \frac{+\cancel{2}\sin\theta/2 . \cos\theta/2}{\cancel{2}\cos^2\theta/2}$

$\cot\phi = \tan\theta/2$

$\cot\phi = \cot\left(\frac{\pi}{2}-\frac{\theta}{2}\right)$

$\phi = \frac{\pi}{2}-\frac{\theta}{2}$

$p = r\sin\phi$

$p = r\sin\left(\frac{\pi}{2}-\frac{\theta}{2}\right)$

$p = r\cos\left(\frac{\theta}{2}\right)$

from eq -(1)

$\frac{2a}{r} = 1+\cos\theta \Rightarrow \frac{\cancel{2}a}{r} = \cancel{2}\cos^2\theta/2$

$\cos\theta/2 = \sqrt{\frac{a}{r}}$

$p = r.\frac{\sqrt{a}}{\sqrt{r}} = p = \sqrt{r}.\sqrt{a}$

d.w.r.t r

$\frac{dp}{dr} = \sqrt{a}\cdot\frac{1}{2\sqrt{r}}$

$\frac{dr}{dp} = \frac{2\sqrt{r}}{\sqrt{a}}$

$\rho = r.\ \frac{2\sqrt{r}}{\sqrt{a}}$

$\rho = C.r\sqrt{r}.$ where $C = \frac{2}{\sqrt{a}}$

s.o.b.s

$\rho^2 = r^2.r.C^2$

$\boxed{\rho^2 \propto r^3}$

7. $r^m = a^m(\cos m\theta + \sin m\theta)$

Sol:- $r^m = a^m(\cos m\theta + \sin m\theta) \longrightarrow (1)$

$$m\log r = m\log a + \log(\cos m\theta + \sin m\theta)$$

d.w.r.t. θ

$$m \cdot \frac{1}{r} \cdot \frac{dr}{d\theta} = 0 + \frac{(-\sin m\theta)m + (\cos m\theta)m}{(\cos m\theta + \sin m\theta)}$$

$$\cancel{m} \cdot \cot\phi = \cancel{m} \cdot \frac{(\cos m\theta - \sin m\theta)}{(\cos m\theta + \sin m\theta)}$$

divide $\cos m\theta$ in nume & deno.

$$\cot\phi = \frac{1 - \tan m\theta}{1 + \tan m\theta}$$

$$\cot\phi = \tan\left(\frac{\pi}{4} - m\theta\right)$$

$$\cot\phi = \cot\left(\frac{\pi}{2} - \left(\frac{\pi}{4} - m\theta\right)\right)$$

$$\phi = \frac{\pi}{4} + m\theta$$

$$P = r\sin\phi$$

$$P = r\sin\left(\frac{\pi}{4} + m\theta\right)$$

$$= r\left(\sin\frac{\pi}{4} \cdot \cos m\theta + \cos\frac{\pi}{4} \cdot \sin m\theta\right)$$

$$= r\left(\frac{1}{\sqrt{2}}\cos m\theta + \frac{1}{\sqrt{2}}\sin m\theta\right)$$

$$P = \frac{1}{\sqrt{2}}(\cos m\theta + \sin m\theta)$$

from eq -(1) $r^m = a^m(\cos m\theta + \sin m\theta)$

$$\cos m\theta + \sin m\theta = \frac{r^m}{a^m}$$

$$P = \frac{r}{\sqrt{2}} \cdot \frac{r^m}{a^m} \Rightarrow P = \frac{1}{a^m\sqrt{2}} \cdot r^{m+1}$$

d.w.r.t r

$$\frac{dp}{dr} = \frac{1}{\sqrt{2}\,a^m}(m+1)\,r^{m+\cancel{1}-\cancel{1}}$$

$$\frac{dp}{dr} = \frac{1}{\sqrt{2}\,a^m}(m+1)r^m$$

$$\frac{dr}{dp} = \frac{\sqrt{2}\,a^m}{(m+1)} \cdot \frac{1}{r^m}$$

$$\rho = r \cdot \frac{dr}{dp}$$

$$\rho = r \cdot \frac{\sqrt{2}\,a^m}{(m+1)} \cdot \frac{1}{r^m}$$

$$\rho = \frac{1}{r^{m-1}} \cdot C$$

$$\boxed{\rho \propto \frac{1}{r^{m-1}}}$$

Radius of Curvature in parametric form:-

if $x = x(t)$, $y = y(t)$

The radius of curvature in parametric form is given by

$$\rho = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - x''y'}$$

1. Find radius of curvature for the cycloid at any point given

$x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$

Sol:- $x = a(\theta + \sin\theta)$ $\quad y = a(1 - \cos\theta)$

WKT radius of curvature of in parametric form is given by

$$\rho = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - x''y'}$$

$x = a(\theta + \sin\theta)$

d.w.r.t. θ

$x' = a(1 + \cos\theta)$

d.w.r.t θ

$x'' = a(-\sin\theta)$

$y = a(1 - \cos\theta)$

d.w.r.t. θ

$y' = a(0 - (-\sin\theta))$

$y' = a\sin\theta$

d.w.r.t θ

$y'' = a\cos\theta$

Consider, $x'^2 + y'^2 = (a(1+\cos\theta))^2 + (a\sin\theta)^2$

$$= a^2\left[1 + \cos^2\theta + 2\cos\theta + \sin^2\theta\right]$$

$$= a^2\left[2 + 2\cos\theta\right]$$

$$= 2a^2\left[1 + \cos\theta\right]$$

Consider, $x'y'' - x''y'$

$$= a(1+\cos\theta)(a\cos\theta) - (-a\sin\theta)\,a\sin\theta$$

$$= a^2(\cos\theta + \cos^2\theta) + a^2\sin^2\theta$$

$$= a^2(\cos\theta + \cos^2\theta + \sin^2\theta) \quad = a^2(1+\cos\theta)$$

$$\rho = \frac{(2a^2[1+\cos\theta])^{3/2}}{a^2(1+\cos\theta)}$$

$$= \frac{(2a^2)^{3/2}\,\cancel{(1+\cos\theta)}^{3/2}}{a^2\,\cancel{(1+\cos\theta)}}$$

$$= \frac{2\sqrt{2}\,\cancel{a^3}\,(1+\cos\theta)^{1/2}}{\cancel{a^2}}$$

$$= 2\sqrt{2}\,a\,(1+\cos\theta)^{1/2}$$

$$= 2\sqrt{2}\,a\,\sqrt{2\cos^2\theta/2}$$

$$= 2\sqrt{2}\,a\,\sqrt{2}\,\cos\theta/2$$

$$\boxed{\rho = 4a\cos\theta/2}$$

2. Find the radius of curvature at any point ellipse
$x = a\cos\theta$, $y = b\sin\theta$

Sol:- $x = a\cos\theta$ $\quad$ $y = b\sin\theta$

d.w.r.t θ $\quad$ d.w.r.t θ

$x' = a(-\sin\theta)$ $\quad$ $y' = b\cos\theta$

d.w.r.t θ $\quad$ d.w.r.t. θ

$x'' = -a\cos\theta$ $\quad$ $y'' = -b\sin\theta$

$$\rho = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - x''y'}$$

Consider, $x'^2 + y'^2 = (-a\sin\theta)^2 + (b\cos\theta)^2 = a^2\sin^2\theta + b^2\cos^2\theta$

Consider, $x'y'' - x''y' = -a\sin\theta(-b\sin\theta) - (-a\cos\theta)(b\cos\theta)$

$= ab\sin^2\theta + ab\cos^2\theta$

$= ab(\sin^2\theta + \cos^2\theta)$

$= ab$

$$\rho = \frac{a^2\sin^2\theta + b^2\cos^2\theta}{ab}$$

$$= \frac{a^2\sin^2\theta}{ab} + \frac{b^2\cos^2\theta}{ab}$$

$$= \frac{a\sin^2\theta}{b} + \frac{b\cos^2\theta}{a}$$

$$\rho = \frac{a}{b}\sin^2\theta + \frac{b}{a}\cos^2\theta$$

3. PT the radius of curvature at any point of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ is times the length of perpendicular from the origin to the tangent at that point.

Sol:- Write the equation.

Consider $x^{2/3} + y^{2/3} = a^{2/3} \longrightarrow (1)$

eq-(1) in parametric form

$x = a\cos^3 t$

d.w.r. to t

$x' = -a3\cos^2 t \sin t$

d.w.r. to 't'

$x'' = -a3[2\cos t . \sin t(-\sin t) + \cos t . \cos^2 t]$

$x'' = -3a(-2\cos t \sin^2 t + \cos^3 t)$

$x'' = -3a\cos t(-2\sin^2 t + \cos^2 t)$

$y = a\sin^3 t$

d.w.r. to 't'

$y' = 3a\sin^2 t . \cos t$

d.w.r. to 't'

$y'' = 3a(2\sin t . \cos^2 t - \sin^2 t . \sin t)$

$= 3a(2\sin t \cos^2 t - \sin^3 t)$

$y'' = 3a\sin t(2\cos^2 t - \sin^2 t)$

Consider, $x'^2 + y'^2 = (-3a\cos^2 t \sin t)^2 + (3a\sin^2 t . \cos t)^2$

$= 9a^2(\cos^4 t . \sin^2 t + \sin^4 t . \cos^2 t)$

$= 9a^2 \cos^2 t . \sin^2 t(\cos^2 t + \sin^2 t)$

$= 9a^2 \cos^2 t . \sin^2 t \Rightarrow (3a . \cos t . \sin t)^2$

Consider, $x'y'' - x''y'$

$= (-3a\cos^2 t \sin t)[3a(2\sin t \cos^2 t - \sin^3 t)] -$

$[-3a(-2\cos t \sin^2 t + \cos^3 t)](3a\sin^2 t . \cos t)$

$= -9a^2[(\cos^2 t \sin t)(2\sin t \cos^2 t - \sin^3 t) - (-2\cos t . \sin^2 t + \cos^3 t)(\sin^2 t . \cos t)]$

$= -9a^2 \cos t . \sin t[\cos t(2\sin t \cos^2 t - \sin^3 t) - (-2\cos t . \sin^2 t + \cos^3 t)\sin t]$

$-9a^2\cos t.\sin t\left[\cos t.\sin t\,(2\cos^2 t-\sin^2 t)-(-2\sin^2 t+\cos^2 t)\cos t.\sin t\right]$ 2 7

$= -9a^2\cos^2 t.\sin^2 t\left[2\cos^2 t-\sin^2 t+2\sin^2 t-\cos^2 t\right]$

$= -9a^2\cos^2 t.\sin^2 t\left[\sin^2 t+\cos^2 t\right]$

$= -9a^2\cos^2 t\sin^2 t = x'y''-x''y'$

$$P = \frac{\left((3a.\cos t.\sin t)^{\cancel{2}}\right)^{\cancel{3}/\cancel{2}}}{-(3a\cos t.\cancel{\sin t})^{\cancel{2}}}$$

$P = |-3a\cos t.\sin t|$

$P = 3a\cos t.\sin t$

WKT equation of tangent at (x_1,y_1)

$$y-y_1 = m(x-x_1)$$

at $(x_1,y_1) = (a\cos^3 t,\ a\sin^3 t)$

$$\text{slope} = m = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y_1}{x_1}$$

$$= \frac{\cancel{3a}\sin^{\cancel{2}} t.\cancel{\cos t}}{-\cancel{3a}\cos^{\cancel{2}} t.\cancel{\sin t}} = -\tan t$$

$\Rightarrow y-a\sin^3 t = -\tan t(x-a\cos^3 t)$

$y-a\sin^3 t = -x\tan t + a.\dfrac{\sin t}{\cancel{\cos t}}.\cos^{\cancel{3}2} t$

$y-a\sin^3 t = -x\tan t + a\sin t.\cos^2 t$

$y = -x\tan t + a\sin t.\cos^2 t + a\sin^3 t$

$= -x\tan t + a\sin t(\cos^2 t+\sin^2 t)$

$y = -x\tan t + a\sin t$